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Plane Waves in Linear Viscoelastic Media

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Summary

A class of plane inhomogeneous waves which propagate in linear viscoelastic media is considered. A theoretical description of the physical properties and energy associated with these waves is given. Attention is restricted to harmonic *P*- and *SV*-waves.

1. Introduction

The problem of reflection and refraction of plane harmonic waves in linear viscoelastic solids has been investigated previously by Lockett (1962), Cooper & Reiss (1966) and Cooper (1967). It transpired that a general class of inhomogeneous waves had to be considered. For example, even for an incident homogeneous wave (i.e. a wave with amplitude constant on lines of constant phase), the reflected and refracted waves are in general inhomogeneous (i.e. with a component of attenuation along the lines of constant phase). This paper gives a detailed description of the physical properties and energy associated with these inhomogeneous waves.

2. Equations of linear viscoelasticity

At time t and position $\mathbf{x}$ relative to a Cartesian reference frame, let $\sigma_{ij}(\mathbf{x}, t)$, $\varepsilon_{ij}(\mathbf{x}, t)$ and $u_i(\mathbf{x}, t)$ be respectively the stress tensor, strain tensor and displacement vector of an isotropic, linear viscoelastic solid. (i, j run over the values 1, 2, 3, and where repeated subscripts occur the usual summation convention applies). The corresponding deviatoric stress and strain tensors are defined by

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{1}$$

and

$$e_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{2}$$

where δ_{ij} is the Kronecker delta.

The constitutive relations can be written in the form (Gurtin & Sternberg 1961)

$$s_{ij} = 2\int_{-\infty}^{t} g(t-\tau)\, de_{ij}(\tau) = 2g * de_{ij} \tag{3}$$

$$\sigma_{kk} = 3\int_{-\infty}^{t} f(t-\tau)\, d\varepsilon_{kk}(\tau) = 3f * d\varepsilon_{kk} \tag{4}$$

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where $g(t)$ and $f(t)$ are stress relaxation functions in shear and dilatation respectively.

For sufficiently small strains, we have

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) \tag{5}$$

where $u_{i,j}$ is the usual notation for $\partial u_i/\partial x_j$. Thus with (1) to (5) we obtain

$$\sigma_{ij} = g * d(u_{i,j}+u_{j,i})+h * du_{k,k}\delta_{ij} \tag{6}$$

where

$$h(t) = f(t)-\tfrac{2}{3}g(t). \tag{7}$$

The equations of motion in the absence of any body forces are

$$\rho\ddot{u}_i = \sigma_{ij,j} \tag{8}$$

where ρ is the density assumed independent of $\mathbf{x}$ and t. In the usual manner we introduce the scalar and vector potentials $\Phi(\mathbf{x}, t)$ and $\mathbf{\Psi}(\mathbf{x}, t)$ such that

$$\mathbf{u} = \nabla\Phi+\nabla\wedge\mathbf{\Psi} \tag{9}$$

with the subsidiary condition

$$\nabla . \mathbf{\Psi} = 0. \tag{10}$$

Equations (6) and (8) then lead to

$$\rho\ddot{\Phi} = (h+2g) * d(\nabla^2 \Phi) \tag{11}$$

and

$$\rho\ddot{\mathbf{\Psi}} = g * d(\nabla^2 \mathbf{\Psi}). \tag{12}$$

We refer to solutions of (11) and (12) as P- and S-waves respectively.

Introducing the real, positive radial frequency ω, we define the complex shear modulus as

$$G(i\omega) = G_1-iG_2 = -i\omega \int_0^\infty g(t)\, e^{i\omega t}\, dt \tag{13}$$

with similar definitions for $F(i\omega)$ and $H(i\omega)$.

We further introduce the complex quantities defined by

$$\Omega(i\omega) = \Omega_1+i\Omega_2 = \frac{\rho\omega^2}{H+2G} \tag{14}$$

and

$$\Gamma(i\omega) = \Gamma_1+i\Gamma_2 = \frac{\rho\omega^2}{G}. \tag{15}$$

Finally we remark that the limiting case $H_2 = G_2 = 0$ is that of perfect elasticity when $H_1 = \lambda$, $G_1 = \mu$ are the Lamé parameters.

3. The inhomogeneous P-wave

We seek harmonic plane wave solutions of (11) and thus consider the general form

$$\Phi = \mathscr{Re}\{A \exp i(\mathbf{k}\cdot\mathbf{x}-\omega t)\} \tag{16}$$

where $\mathscr{Re}\{\ \}$ denotes the real part, A is a complex constant and $\mathbf{k}$ is the 'complex vector'

$$\mathbf{k} = \boldsymbol{\kappa} + i\boldsymbol{\alpha} \tag{17}$$

which expresses the directions of the normals to both lines of constant phase and lines of constant amplitude. as well as the wave number κ and attenuation coefficient α.

It is shown in Section 6, that the angle ϑ between these two directions must be less than $\pi/2$. When $\vartheta = 0$ we call the wave homogeneous, otherwise (16) represents an inhomogeneous P-wave propagating with phase velocity

$$\mathbf{V} = \frac{\omega\boldsymbol{\kappa}}{\kappa^2} \tag{18}$$

normal to the lines of constant phase.

From (14), the equation of motion (11) is satisfied when

$$k^2 = \mathbf{k}\cdot\mathbf{k} = \Omega \tag{19}$$

or in real form, when

$$\left.\begin{aligned} \kappa^2 - \alpha^2 &= \Omega_1 \\ 2\boldsymbol{\kappa}.\boldsymbol{\alpha} = 2\kappa\alpha\cos\vartheta &= \Omega_2 \end{aligned}\right\}. \tag{20}$$

We can solve (20) for κ and α in terms of ϑ, giving

$$\left.\begin{aligned} 2\kappa^2 &= \Omega_1 + \sqrt{(\Omega_1{}^2 + \Omega_2{}^2 \sec^2\vartheta)} \\ 2\alpha^2 &= -\Omega_1 + \sqrt{(\Omega_1{}^2 + \Omega_2{}^2 \sec^2\vartheta)} \end{aligned}\right\}. \tag{21}$$

It follows at once that provided $\Omega_2 \neq 0$, ϑ cannot be equal to $\pi/2$. When $\Omega_2 = 0$ (perfectly elastic case), then from (20) either,

(i) $\alpha = 0$; the undamped homogeneous wave or;

(ii) $\alpha \neq 0$, $\vartheta = \pi/2$; the inhomogeneous wave damped normal to the direction of propagation.

Thus there is a distinct difference between the inhomogeneous wave of elastic media and the inhomogeneous wave of viscoelastic media. This difference is further investigated in the following sections.

4. The displacement and particle motion

For the plane P-wave given by (16) the displacement vector is

$$\mathbf{u} = \nabla\Phi = \mathscr{Re}\{iA\mathbf{k}\exp i(\mathbf{k}.\mathbf{x} - \omega t)\}, \tag{22}$$

which lies in the plane of $\boldsymbol{\kappa}$ and $\boldsymbol{\alpha}$. To determine the particle motion, let us define two new vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ through

$$\boldsymbol{\xi}_1 + i\boldsymbol{\xi}_2 = \mathbf{k}/k \tag{23}$$

where $k = \Omega^{\frac{1}{2}} = \kappa_0 + i\alpha_0$ can be obtained from (21) with $\vartheta = 0$. Evidently

$$\boldsymbol{\xi}_1 = \frac{\kappa_0\,\boldsymbol{\kappa} + \alpha_0\,\boldsymbol{\alpha}}{\kappa_0{}^2 + \alpha_0{}^2},\ \boldsymbol{\xi}_2 = \frac{\kappa_0\,\boldsymbol{\alpha} - \alpha_0\,\boldsymbol{\kappa}}{\kappa_0{}^2 + \alpha_0{}^2}. \tag{24}$$

We also note that from (19),

$$\boldsymbol{\xi}_1.\boldsymbol{\xi}_2 = 0,\ \xi_1{}^2 - \xi_2{}^2 = 1 \tag{25}$$

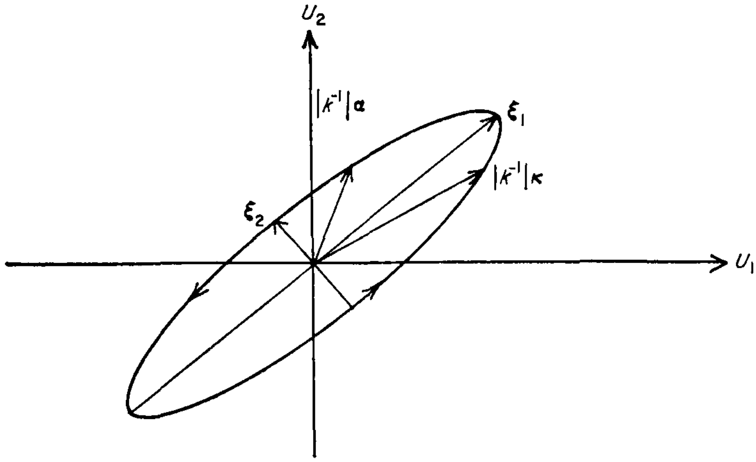

FIG. 1. The P-wave particle motion. Also shown is the relation between the vectors $\boldsymbol{\kappa}$, $\boldsymbol{\alpha}$, $\boldsymbol{\xi}_1$, and $\boldsymbol{\xi}_2$. For the homogeneous wave the ellipse degenerates into a straight line giving pure longitudinal motion.

so $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are perpendicular and $\xi_1 > \xi_2 \geqslant 0$. Thus writing $C = |Ak| \exp(-\boldsymbol{\alpha}.\mathbf{x})$ and $\phi = \boldsymbol{\kappa}.\mathbf{x} - \omega t + \arg(Ak)$ we have

$$\mathbf{u} = -C[\boldsymbol{\xi}_1 \sin\phi + \boldsymbol{\xi}_2 \cos\phi]. \tag{26}$$

If we now set

$$U_1 = \frac{\mathbf{u}.\boldsymbol{\xi}_1}{\xi_1 C}$$

and

$$U_2 = \frac{-\mathbf{u}.\boldsymbol{\xi}_2}{\xi_2 C}$$

and eliminate ϕ from (26) we obtain

$$\frac{U_1^2}{\xi_1^2} + \frac{U_2^2}{\xi_2^2} = 1.$$

Thus the particle motion is an ellipse centred on the rest position of the particle ($\mathbf{u} = 0$), with major and minor axes given by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ respectively, and of eccentricity $(1 - \xi_2^2/\xi_1^2)^{\frac{1}{2}} = 1/\xi_1$. The sense of rotation is from $\boldsymbol{\kappa}$ to $\boldsymbol{\alpha}$. (See Fig. 1). For the homogeneous wave, $\vartheta = 0$ implies $\xi_2 = 0$ so that the ellipse degenerates into a straight line giving pure longitudinal motion.

5. Energy flux

The mean rate of working of the stresses given by

$$e_j = -\langle \sigma_{ij} \dot{u}_i \rangle \tag{28}$$

expresses the mean energy flux in the wave. From (22) and (6) we find for the P-wave that

$$\dot{u}_i = \mathcal{Re}\{\omega A k_i \exp i(\mathbf{k}.\mathbf{x} - \omega t)\} \tag{29}$$

$$\sigma_{ij} = -\mathcal{Re}\{A(2Gk_i k_j + Hk^2 \delta_{ij}) \exp i(\mathbf{k}.\mathbf{x} - \omega t)\}. \tag{30}$$

Averaging their product over one cycle, we obtain using (14) and (19)

$$e_j = \tfrac{1}{2}\omega|A|^2 \exp(-2\boldsymbol{\alpha}.\mathbf{x})\, \mathcal{R}e\,\{\rho\omega^2 \bar{k}_j + 2G(k_i k_j - k^2 \delta_{ij})\bar{k}_i\} \tag{31}$$

where the bar denotes the complex conjugate. With (17) we can show that $(k_i k_j - k^2 \delta_{ij})\bar{k}_i$ is the j-component of the vector $2(\boldsymbol{\alpha} \wedge \boldsymbol{\kappa}) \wedge (\boldsymbol{\alpha} - i\boldsymbol{\kappa})$, whence

$$\mathbf{e} = \tfrac{1}{2}\omega|A|^2 \exp(-2\boldsymbol{\alpha}.\mathbf{x})\{\rho\omega^2 \boldsymbol{\kappa} + 4(\boldsymbol{\kappa} \wedge \boldsymbol{\alpha}) \wedge (G_2 \boldsymbol{\kappa} - G_1 \boldsymbol{\alpha})\}. \tag{32}$$

We conclude that for the inhomogeneous P-wave energy propagates in the plane of $\boldsymbol{\kappa}$ and $\boldsymbol{\alpha}$ but in a direction different from the wave front normal. This phenomenon does not occur for the homogeneous wave when $\boldsymbol{\kappa} \wedge \boldsymbol{\alpha} = 0$.

6. Rate of dissipation

Let S be a closed surface of the medium with unit inward normal υ, enclosing a volume V. Let $\mathcal{D}$ be the rate of dissipation by the medium per unit volume and suppose the existence of the function $\mathcal{W}$, the energy density per unit volume (see Section 7). Then the following energy balance equation holds:

$$\left\langle \int_V \mathcal{D}\, dV + \frac{\partial}{\partial t} \int_V \mathcal{W}\, dV \right\rangle = \int_S \mathbf{e}.\upsilon dS = -\int_V \nabla.\mathbf{e} dV \tag{33}$$

For steady harmonic waves the mean value over one cycle of the term in $\mathcal{W}$ vanishes. Thus with (32) we obtain the mean rate of dissipation for the inhomogeneous P-wave as

$$\langle \mathcal{D} \rangle = -\nabla.\mathbf{e} = 2\mathbf{e}.\boldsymbol{\alpha} \tag{34}$$

$$= \tfrac{1}{2}\omega\Omega_2|A|^2 \exp(-2\boldsymbol{\alpha}.\mathbf{x})(\rho\omega^2 + 2G_2\,\Omega_2 \tan^2 \vartheta). \tag{35}$$

Since $\mathcal{D}$ must be non-negative for all ϑ, it follows that $\Omega_2(\omega) \geqslant 0$ whenever $\omega > 0$. Hence from (20), $0 \leqslant \vartheta \leqslant \pi/2$ as asserted in Section 3.

In Appendix A we show the equivalence of our result with a formulation obtained from a spring-dashpot model of linear viscoelasticity. Note, however, that (34) is independent of any model.

7. Energy density

To determine the mean energy density, reference must be made to a particular model. For this purpose the linear spring-dashpot models are best suited. In these, it is assumed that energy is 'stored' in the springs and 'dissipated' in the dashpots. The extension of the model to three dimensions has been effected by Bland (1960). Thus using his results (p. 40) and including a term for the kinetic energy density, we have

$$\mathcal{W} = \tfrac{1}{2}\rho\dot{u}_i^{\,2} + \int_{-\infty}^{t}\int_{-\infty}^{t} g(2t-\tau-\theta)\, de_{ij}(\tau)\, de_{ij}(\theta)$$

$$+\tfrac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} f(2t-\tau-\theta)\, d\varepsilon_{ii}(\tau)\, d\varepsilon_{jj}(\theta). \tag{36}$$

The mean value of the integrals in this expression can be obtained as in Appendix A, to give for the P-wave,

$$\tfrac{1}{4}|A|^2 \exp(-2\boldsymbol{\alpha}.\mathbf{x})\{\rho\omega^2\,\Omega_1 + 2G_1\,\Omega_2^{\,2} \tan^2 \vartheta\}.$$

Since this must be a positive definite quantity for all ϑ, it follows that $G_1 > 0$, $\Omega_1 > 0$. Averaging the kinetic energy term, we find

$$\langle \tfrac{1}{2}\rho \dot{u}_i^2 \rangle = \tfrac{1}{4}\rho\omega^2 |A|^2 \exp(-2\boldsymbol{\alpha}.\mathbf{x})\surd(\Omega_1^2 + \Omega_2^2 \sec^2 \vartheta). \tag{37}$$

Hence putting the two terms together and using (21) and (32) we find

$$\langle \mathscr{W} \rangle = \omega^{-1}\,\mathbf{e}.\boldsymbol{\kappa}. \tag{38}$$

Thus the mean energy density depends on the component of energy flux normal to the lines of constant phase, while the mean rate of dissipation depends on the component of energy flux normal to the lines of constant amplitude. For the homogeneous wave the two directions coincide and the corresponding results are obtained with $\vartheta = 0$.

Note that only when the medium is perfectly elastic ($\Omega_2 = 0$) are the kinetic and potential energy densities the same.

8. Velocity of energy transport

The mean velocity of energy transport is defined as the ratio of the mean energy flux to the mean energy density. Denoting this vector by $\mathbf{U}$, we have for the inhomogeneous P-wave,

$$\mathbf{U} = \mathbf{e}/\langle \mathscr{W} \rangle. \tag{39}$$

In general this is a complicated function of the material parameters, the frequency and the angle ϑ. However, from (18) and (38), we obtain the relation

$$\mathbf{U}.\mathbf{V} = V^2 \tag{40}$$

where V is the phase velocity normal to the lines of constant phase. This has a simple geometrical interpretation. Instead of considering the lines of constant phase to propagate with speed V in the direction of $\boldsymbol{\kappa}$, we can consider them to propagate with speed U in the direction of $\mathbf{e}$. For the homogeneous wave, $\vartheta = 0$ implies $\mathbf{U} = \mathbf{V}$.

9. Dissipation factor Q^{-1}

The attenuation of seismic body waves is often expressed in terms of a dimensionless parameter $2\pi Q^{-1}$, defined to be the fractional energy loss per cycle. Thus, for a cycle of period $T = 2\pi/\omega$.

$$2\pi Q^{-1} = \frac{\langle \mathscr{D} \rangle T}{\langle \mathscr{W} \rangle}. \tag{41}$$

For the inhomogeneous P-wave, using (34) and (38),

$$Q^{-1} = 2\left(\frac{\mathbf{e}.\boldsymbol{\alpha}}{\mathbf{e}.\boldsymbol{\kappa}}\right) = \frac{2\mathbf{U}.\boldsymbol{\alpha}}{\omega}. \tag{42}$$

For the homogeneous wave (42) reduces to $Q^{-1} = 2\alpha V/\omega$ which is the usual expression given to body waves.

10. The inhomogeneous SV-Wave

The harmonic plane wave solution of (12) which represents the inhomogeneous SV-wave has the form

$$\boldsymbol{\Psi} = \mathscr{R}\mathrm{e}\,\{B\mathbf{n} \exp i(\mathbf{l}.\mathbf{x} - \omega t)\} \tag{43}$$

where B is a complex constant, $\mathbf{n}$ is a real unit vector and

$$\mathbf{l} = \boldsymbol{\gamma} + i\boldsymbol{\beta} \tag{44}$$

where $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ can be interpreted as in Section 3. The auxiliary equation (10) leads to $\mathbf{n}.\boldsymbol{\gamma} = \mathbf{n}.\boldsymbol{\beta} = 0$ so that $\mathbf{n}$ is a unit vector normal to the plane of $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$. The equation of motion (12) is satisfied when

$$l^2 = \mathbf{l}.\mathbf{l} = \Gamma \tag{45}$$

and this can also be written in the real forms analogous to (20) and (21). From (9), the displacement vector has the form

$$\mathbf{u} = \nabla \wedge \boldsymbol{\Psi} = \mathscr{R}e\,\{iB(\mathbf{n} \wedge \mathbf{l}) \exp i(\mathbf{l}.\mathbf{x} - \omega t)\} \tag{46}$$

and this lies in the plane of $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$.

To determine the particle motion we proceed as in Section 4, by defining two new vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ by

$$\boldsymbol{\eta}_1 + i\boldsymbol{\eta}_2 = \mathbf{l}/l \tag{47}$$

where $l = \gamma_0 + i\beta_0$ and

$$\boldsymbol{\eta}_1 = \frac{\gamma_0 \boldsymbol{\gamma} + \beta_0 \boldsymbol{\beta}}{\gamma_0^2 + \beta_0^2}, \boldsymbol{\eta}_2 = \frac{\gamma_0 \boldsymbol{\beta} - \beta_0 \boldsymbol{\gamma}}{\gamma_0^2 + \beta_0^2} \tag{48}$$

and

$$\boldsymbol{\eta}_1 . \boldsymbol{\eta}_2 = 0, \; \eta_1^2 - \eta_2^2 = 1. \tag{49}$$

Since $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ lie in the plane of $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ we can write

$$\mathbf{n} \wedge \mathbf{l}/l = \boldsymbol{\zeta}_1 + i\boldsymbol{\zeta}_2 \tag{50}$$

where

$$\boldsymbol{\zeta}_1 = \left(\frac{\eta_1}{\eta_2}\right) \boldsymbol{\eta}_2, \; \boldsymbol{\zeta}_2 = -\left(\frac{\eta_2}{\eta_1}\right) \boldsymbol{\eta}_1. \tag{51}$$

Thus writing $D = |Bl| \exp(-\boldsymbol{\beta}.\mathbf{x})$ and $\psi = \boldsymbol{\gamma}.\mathbf{x} - \omega t + \arg(Bl)$ we have for the inhomogeneous SV-wave,

$$\mathbf{u} = -D(\boldsymbol{\zeta}_1 \sin\psi + \boldsymbol{\zeta}_2 \cos\psi). \tag{52}$$

Writing

$$U_1 = \frac{-\mathbf{u}.\boldsymbol{\zeta}_1}{\zeta_1 D}$$

and

$$U_2 = \frac{-\mathbf{u}.\boldsymbol{\zeta}_2}{\zeta_2 D}$$

and eliminating ψ from (52),

$$\frac{U_1^2}{\zeta_1^2} + \frac{U_2^2}{\zeta_2^2} = 1. \tag{53}$$

Thus the particle motion is again an ellipse with centre $\mathbf{u} = 0$, major and minor axes given by ζ_1 and ζ_2 respectively and of eccentricity $(1 - \zeta_2^2/\zeta_1^2)^{\frac{1}{2}} = 1/\eta_1$. The sense of rotation is from $\boldsymbol{\gamma}$ to $\boldsymbol{\beta}$. (See Fig. 2). For the case of the homogeneous wave we get $\eta_2 = \zeta_2 = 0$ and the ellipse degenerates into a straight line giving pure transverse motion.

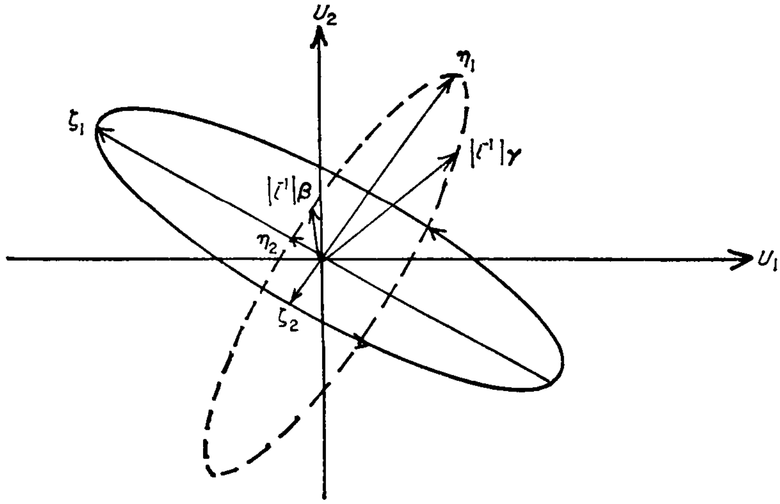

FIG. 2. The SV-wave particle motion. Also shown is the relation between the vectors γ, β, η_1, η_2, ζ_1, and ζ_2. For the homogeneous wave the ellipse degenerates into a straight line giving pure transverse motion.

The derivation of the mean energy flux vector for the inhomogeneous SV-wave is relegated to Appendix B where it is shown that

$$\mathbf{e} = \tfrac{1}{2}\omega|B|^2 \exp(-2\boldsymbol{\beta}.\mathbf{x})\{\rho\omega^2 \boldsymbol{\gamma} + 4(\boldsymbol{\gamma} \wedge \boldsymbol{\beta}) \wedge (G_2 \boldsymbol{\gamma} - G_1 \boldsymbol{\beta})\}. \tag{54}$$

This has precisely the same form as (32) for the P-wave so that results analogous to (34), (38), (40) and (42) also apply to the SV-wave.

11. Conclusions

We have presented here a detailed description of the physical properties and energy associated with plane inhomogeneous waves in linear viscoelastic media. Several interesting features have arisen, in particular that the energy in such an inhomogeneous wave does not propagate normal to the wave fronts. This may have an important effect on seismic body waves, especially those which have been more highly attenuated. As can be seen from (32), the effect increases with increasing angle ϑ between $\boldsymbol{\kappa}$ and $\boldsymbol{\alpha}$. Since ϑ approaching $\pi/2$ corresponds to supercritical reflection (or refraction) we expect the effect to be greatest in diffracted waves.

The situation is further complicated by energy conversion and interaction at a discontinuity. For example, consider a plane homogeneous SV-wave incident on the plane stress-free boundary $z = 0$ of an isotropic linear viscoelastic solid $z > 0$. The situation is described by the two dimensional potentials

$$\left.\begin{aligned} \Phi_1 &= \exp il(x\cos f - z\sin f) && \text{Incident homogeneous } SV \\ \Phi_2 &= B\exp il(x\cos f + z\sin f) && \text{Reflected homogeneous } SV \\ \Phi_3 &= A\exp ik(x\cos e + z\sin e) && \text{Reflected inhomogeneous } P \end{aligned}\right\} \tag{55}$$

where f is real and e is complex, satisfying $k\cos e = l\cos f$. A and B are complex reflection coefficients whose values are not important for this discussion.

Let us calculate the normal flux of energy across $z = 0$. Obviously the net result must be zero, but each wave produces its own displacement field and corresponding stress field, so that the total flux consists of nine terms: 3 'main' terms and 6 'interacting' terms. Denote these by E_{ij} where the subscripts correspond to those of (55). Thus E_{23}, for example, is the normal 'flux' across $z = 0$ due to the displacement field of the reflected SV-wave 'interacting' with the stress field of the reflected P-wave.

A lengthy calculation results in:

$$\left.\begin{aligned} E_{11} &= -\chi \sin f\, \mathscr{R}e\{l\} \\ E_{12} &= \chi \sin 3f\, \mathscr{R}e\{l\bar{B}\} \\ E_{13} &= -E_{11} - E_{12} \\ E_{21} &= -\chi \sin 3f\, \mathscr{R}e\{lB\} \\ E_{22} &= \chi \sin f |B|^2\, \mathscr{R}e\{l\} \\ E_{23} &= -E_{21} - E_{22} \\ E_{31} &= -\chi \sin f\, \mathscr{R}e\{l\} + \chi \sin 3f\, \mathscr{R}e\{lB\} \\ E_{32} &= \chi \sin f |B|^2\, \mathscr{R}e\{l\} - \chi \sin 3f\, \mathscr{R}e\{l\bar{B}\} \\ E_{33} &= -E_{31} - E_{32} \end{aligned}\right\} \tag{56}$$

where $\chi = \frac{1}{2}\rho\omega^3 \exp(-2\beta_0 x)$. We note that

$$E_{11} + E_{12} + E_{13} = E_{21} + E_{22} + E_{23} = E_{31} + E_{32} + E_{33} = 0$$

as expected, and in addition,

$$\left.\begin{aligned} E_{11} + E_{22} + E_{33} &= \varepsilon \\ E_{12} + E_{21} &= \varepsilon \\ E_{13} + E_{31} &= -\varepsilon \\ E_{23} + E_{32} &= -\varepsilon \end{aligned}\right\} \tag{57}$$

where $\varepsilon = 2\beta_0 B_2 \chi \sin 3f$ and $l = \gamma_0 + i\beta_0$, $B = B_1 + iB_2$. Thus the interacting terms $E_{12} + E_{21}$ etc. do not cancel as in the perfectly elastic case ($\beta_0 = 0$). We must conclude therefore, that energy can be transported to or from the boundary by the interacting waves. Again, this effect is most enhanced for supercritical reflection when B_2 need not be small, even for small dissipation.

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Appendix A

Here we show the equivalence of our result (35) for the mean rate of dissipation in an inhomogeneous plane P-wave and the result given by Bland (1960, p. 40) for a linear spring-dashpot model. The latter gives

$$\mathscr{D} = -\int_{-\infty}^{t}\int_{-\infty}^{t} 2g'(2t-\tau-\theta)\, de_{ij}(\tau)\, de_{ij}(\theta) - \int_{-\infty}^{t}\int_{-\infty}^{t} f'(2t-\tau-\theta)\, d\varepsilon_{ii}(\tau)\, d\varepsilon_{jj}(\theta) \quad \text{(A1)}$$

where the dash indicates differentiation with respect to the argument. Now from (2) and (16), we find

$$\dot{e}_{ij} = \mathscr{R}e\,\{i\omega A(k_i k_j - \tfrac{1}{3}k^2\,\delta_{ij})\exp i(\mathbf{k}.\mathbf{x}-\omega t)\} \quad \text{(A2)}$$

and

$$\dot{\varepsilon}_{ii} = \mathscr{R}e\,\{i\omega Ak^2 \exp i(\mathbf{k}.\mathbf{x}-\omega t)\}. \quad \text{(A3)}$$

Substitution into (A1) requires the evaluation of the mean value of integrals of the form

$$I = \int_{-\infty}^{t}\int_{-\infty}^{t} g'(2t-\tau-\theta)\, \mathscr{R}e\,\{a\exp(-i\omega\tau)\}\, \mathscr{R}e\,\{a\exp(-i\omega\theta)\}\,d\tau\, d\theta \quad \text{(A4)}$$

where a is a complex quantity independent of t. With the change of variables $t_1 = 2t-\tau-\theta$, $t_2 = \theta-\tau$ the integral becomes

$$I = \tfrac{1}{2}\iint_{S(t_1,t_2)} g'(t_1)\, \mathscr{R}e\left\{a\exp\left[-i\omega\left(t-\frac{t_1+t_2}{2}\right)\right]\right\} \times \mathscr{R}e\left\{a\exp\left[-i\omega\left(t-\frac{t_1-t_2}{2}\right)\right]\right\} dt_1\, dt_2 \quad \text{(A5)}$$

where $S(t_1, t_2)$ is the transformed region of the $t_1\, t_2$-plane shown in Fig. 3.

Averaging with respect to t over one cycle, we get

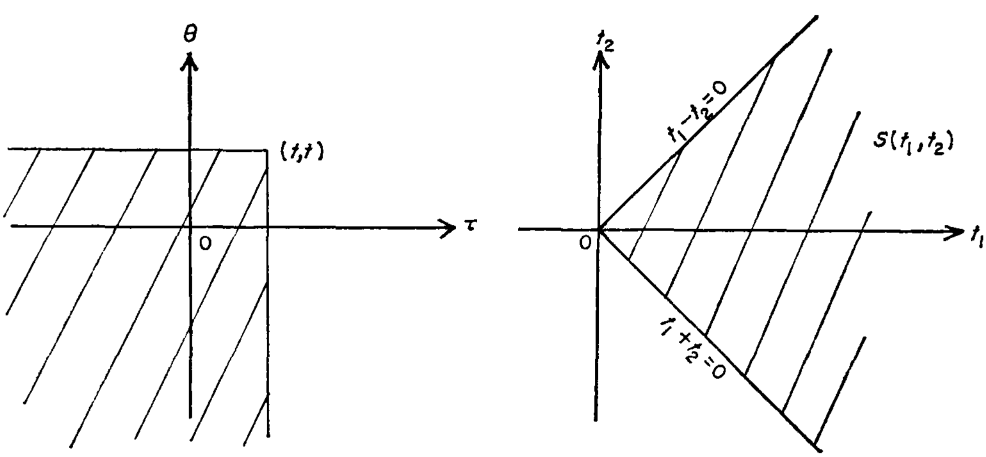

FIG. 3. Transformation from the $\tau\theta$-plane to the $t_1\, t_2$-plane under $t_1 = 2t-\tau-\theta$, $t_2 = \theta-\tau$.

$$\langle I\rangle = \tfrac{1}{4}|a|^2 \iint_S g'(t_1)\cos\omega t_2\,dt_1\,dt_2$$

$$= \tfrac{1}{4}|a|^2 \int_0^\infty dt_1\, g'(t_1) \int_{-t_1}^{t_1} \cos\omega t_2\,dt_2$$

$$= \frac{1}{2\omega}|a|^2 \int_0^\infty g'(t_1)\sin\omega t_1\,dt_1$$

$$= -\frac{1}{2\omega}|a|^2 G_2. \tag{A6}$$

The last line was obtained using the definition (13). Thus, substitution into (A1) leads to, after averaging,

$$\langle \mathscr{D}\rangle = \tfrac{1}{2}\omega|A|^2 \exp\,(-2\boldsymbol{\alpha}.\mathbf{x})\{2G_2|k_i k_j - \tfrac{1}{3}k^2\,\delta_{ij}|^2 + F_2\,|k|^4\} \tag{A7}$$

where

$$\begin{aligned}|k_i k_j - \tfrac{1}{3}k^2\,\delta_{ij}|^2 &= (k_i k_j - \tfrac{1}{3}k^2\,\delta_{ij})(\bar{k}_i \bar{k}_j - \tfrac{1}{3}\bar{k}^2\,\delta_{ij})\\ &= (\mathbf{k}.\bar{\mathbf{k}})^2 - \tfrac{1}{3}k^2\,\bar{k}^2\\ &= (\kappa^2+\alpha^2) - \tfrac{1}{3}|\Omega|^2\\ &= \tfrac{2}{3}(\Omega_1{}^2+\Omega_2{}^2) + \Omega_2{}^2\tan^2\vartheta\end{aligned}$$

and we have used (19) and (21). We now have

$$\langle \mathscr{D}\rangle = \tfrac{1}{2}\omega|A|^2 \exp\,(-2\boldsymbol{\alpha}.\mathbf{x})\{(F_2+\tfrac{4}{3}G_2)(\Omega_1{}^2+\Omega_2{}^2)+2G_2\,\Omega_2\tan^2\vartheta\}. \tag{A8}$$

But from (14),

$$F_2+\tfrac{4}{3}G_2 = -\mathscr{I}m\{H+2G\} = \frac{\rho\omega^2\,\Omega_2}{\Omega_1{}^2+\Omega_2{}^2}$$

whence (A8) reduces to

$$\langle \mathscr{D}\rangle = \tfrac{1}{2}\omega|A|^2 \exp\,(-2\boldsymbol{\alpha}.\mathbf{x})\,\Omega_2(\rho\omega^2+2G_2\,\Omega_2\tan^2\vartheta)$$

which is in complete agreement with out result (35).

Appendix B

Here we derive the expression (54) for the mean energy flux vector for the inhomogeneous SV-wave. From (46) and (6) we obtain

$$\dot{u}_i = \mathscr{R}e\{\omega B m_i \exp i(\mathbf{l}.\mathbf{x}-\omega t)\} \tag{B1}$$

and

$$\sigma_{ij} = -\mathscr{R}e\{BG(m_i l_j + m_j l_i)\exp i(\mathbf{l}.\mathbf{x}-\omega t)\} \tag{B2}$$

where

$$\mathbf{m} = \mathbf{n}\wedge\mathbf{l}. \tag{B3}$$

Multiplying these together and averaging over a cycle,

$$e_j = \tfrac{1}{2}\omega\,|B|^2\exp(-2\boldsymbol{\beta}.\mathbf{x})\,\mathscr{R}e\{\,G(m_i,\,l_j\,+\,m_j\,l_i)\,\bar{m}_i\,\}. \tag{B4}$$

Now $l_i\,\bar{m}_i\,m_j$ is the j-component of the vector:

$$\begin{aligned}(\bar{\mathbf{m}}.\mathbf{l})\,\mathbf{m} &= [(\mathbf{n}\wedge\bar{\mathbf{l}}).\mathbf{l}](\mathbf{n}\wedge\mathbf{l})\\ &= 2(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})\wedge(\boldsymbol{\beta}-i\boldsymbol{\gamma}).\end{aligned} \tag{B5}$$

Also $\bar{m}_i m_i l_j$ is the j-component of the vector:

$$\begin{aligned}(m.\bar{\mathbf{m}})\mathbf{l} &= [(\mathbf{n}\wedge\mathbf{l}).(\mathbf{n}\wedge\bar{\mathbf{l}})]\mathbf{l} \\ &= (\mathbf{l}.\bar{\mathbf{l}})\mathbf{l} \\ &= (\gamma^2+\beta^2)(\boldsymbol{\gamma}+i\boldsymbol{\beta}).\end{aligned} \tag{B6}$$

Therefore,

$$\mathscr{R}\mathrm{e}\,\{Gm_i\bar{m}_i l_j\} = (\gamma^2+\beta^2)(G_1\boldsymbol{\gamma}+G_2\,\boldsymbol{\beta}). \tag{B7}$$

But since

$$\mathbf{l}.\mathbf{l} = l^2 \overset{\cdots}{=} \Gamma = \frac{\rho\omega^2}{G}$$

we can easily show that

$$\left.\begin{aligned}G_1(\gamma^2+\beta^2) &= \rho\omega^2+2(G_1\,\beta^2-G_2\,\boldsymbol{\gamma}.\boldsymbol{\beta}) \\ G_2(\gamma^2+\beta^2) &= \qquad\quad 2(G_2\,\gamma^2-G_1\,\boldsymbol{\gamma}.\boldsymbol{\beta})\end{aligned}\right\} \tag{B8}$$

Thus (B7) becomes

$$\mathscr{R}\mathrm{e}\,\{Gm_i\bar{m}_i l_j\} = \rho\omega^2\boldsymbol{\gamma}+2(\boldsymbol{\gamma}\wedge\boldsymbol{\beta})\wedge(G_2\boldsymbol{\gamma}-G_1\,\boldsymbol{\beta}). \tag{B9}$$

Finally, with (B5) we have

$$\mathbf{e} = \tfrac{1}{2}\omega|B|^2\exp(-2\boldsymbol{\beta}.\mathbf{x})\{\rho\omega^2\boldsymbol{\gamma}+4(\boldsymbol{\gamma}\wedge\boldsymbol{\beta})\wedge(G_2\boldsymbol{\gamma}-G_1\,\boldsymbol{\beta})\}. \tag{B10}$$