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ANALYSIS OF DISPERSION ON A SPHERE

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Summary

The probability density on a sphere, exp ($\kappa \cos \theta'$), where θ' is the angle between the polar and observation vectors, has recently been suggested by Fisher for the analysis of palaeo-magnetic data. In this paper, a series of approximate tests of significance is given for hypotheses about the precision constant κ and the polar vector

1. *Introduction.*—In palaeomagnetism a sample of rock specimens is collected from a site and the direction of remanent magnetization of each specimen measured. Thus the sample may be represented by points on a unit sphere or by unit vectors. The points may be closely or loosely grouped about a pole.

Fisher (1) has suggested that the appropriate probability density on the sphere is given by

$$C \exp(\kappa \cos \theta') \tag{1.1}$$

where κ is a precision parameter and θ' is the angle between the polar and observation vectors. The density is thus symmetrical about the polar axis, and attains a maximum at the pole and a minimum at the anti-pole. When $\kappa = 0$, the density is uniform over the sphere, i.e., the points are randomly distributed on the sphere. When κ is large, the density is confined to the region about the pole and has approximately a circular normal distribution in this region. The constant C is easily seen to be equal to

$$\frac{\kappa}{4\pi \sinh \kappa}. \tag{1.2}$$

If the direction cosines of the polar and observation vectors are (λ, μ, ν) and (l, m, n) respectively, the density is given by

$$\frac{\kappa}{4\pi \sinh \kappa} \exp \{\kappa(\lambda l + \mu m + \nu n)\}. \tag{1.3}$$

Fisher has derived the fundamental estimation and distributional results required for studying the statistics of samples from this distribution. He has also given the fiducial test of the hypothesis that the polar axis of the population, from which a sample has been drawn, has a prescribed direction. The first approximation to this test admits a simple interpretation in terms of analysis of variance or dispersion. This interpretation has been used for other testing problems. Thus an approximate analysis of dispersion on the sphere is obtained which is entirely analogous to univariate analysis of variance.

1.1. *Basic results.*—In this section we give the basic estimation and distribution results derived by Fisher. If (l_i, m_i, n_i) are the direction cosines of the i-th $(i=1, \ldots, N)$ observation, and (l, m, n) are the direction cosines of the maximum likelihood estimate of the polar vector (λ, μ, ν), then

$$l=\frac{\Sigma l_i}{R}, \quad m=\frac{\Sigma m_i}{R}, \quad n=\frac{\Sigma n_i}{R} \tag{1.11}$$

where $R^2=(\Sigma l_i)^2+(\Sigma m_i)^2+(\Sigma n_i)^2$. Thus R is the length of the vector-resultant of all the vector observations. The maximum likelihood estimate k of κ is the solution of the equation

$$\coth k-\frac{1}{k}=\frac{R}{N}. \tag{1.12}$$

The left-hand side of (1.12) is a monotonic increasing function of k and its value changes from 0 to 1 as k runs from 0 to ∞. When R/N is near unity, (1.12) has the approximate solution

$$k=\frac{N}{N-R}. \tag{1.13}$$

The accuracy of (1.13) is sufficient for practical purposes, e.g. when $R/N=0{\cdot}7$, (1.13) gives $k=3\frac{1}{3}$ whereas (1.12) has the solution $3{\cdot}304$.

When the polar vector is known, the maximum likelihood estimate k of κ satisfies the equation

$$\coth k-\frac{1}{k}=\frac{\Sigma \cos\theta_i'}{N}=\frac{Rc}{N} \tag{1.14}$$

where θ_i' is the angle between the i-th observed vector and the polar vector, R is the length of the resultant vector, and c is the cosine of the angle between the resultant and the polar vector. Thus here k is given approximately by

$$k=\frac{N}{N-Rc}. \tag{1.15}$$

The probability density function of R and c is given by

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^N e^{\kappa Rc} R\phi_N(R) \tag{1.16}$$

where

$$\phi_N(R)=\frac{1}{(N-2)!}\sum_{r=0}^{\infty}(-1)^r\binom{N}{r}\langle N-R-2r\rangle^{N-2}, \tag{1.17}$$

with the notation $\langle x\rangle=x$ if $x\geqslant 0$, $\langle x\rangle=0$ if $x\leqslant 0$. Thus the probability density of $z=Rc$ is

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^N \frac{e^{\kappa z}}{(N-1)!}\sum_{r=0}^{\infty}(-1)^r\binom{N}{r}\langle N-z-2r\rangle^{N-1}, \tag{1.18}$$

and the density of R is

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^N \frac{2\sinh\kappa R}{\kappa}\phi_N(R). \tag{1.19}$$

Finally, Fisher has shown that, if two samples of N_1 and N_2 are taken from the same population, the joint probability density of R_1 and R_2, the resultants of the two samples, and R, the resultant of the two samples combined, is given by

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^N \frac{2\sinh\kappa R}{\kappa}\phi_{N_1}(R_1)\phi_{N_2}(R_2) \tag{1.110}$$

where $N=N_1+N_2$.

2. *Tests of significance.*—Practical applications of Fisher's distribution on a sphere require a series of significance tests entirely analogous to those in current use for the normal distribution. Fisher has already derived the analogue of

"Student's" t-test for a single sample. The analysis of variance approach is used below to derive a series of approximate tests which are of a form familiar to statisticians and which are sufficiently accurate for most practical situations.

2.1. *Tests for* κ.—When the true polar vector is known, it is easy to make exact significance tests for κ because $z = Rc$ is then a sufficient statistic for κ. The density (1.18) will be used. Usually the true polar vector is not known. From the estimation equation (1.12), R is seen to be the natural statistic, and the density (1.16) will therefore be used.

Often a test of $\kappa = 0$ will be required. Bruckshaw and Vincenz (**2**) suggested that the value of the density of R, with $\kappa = 0$, computed for the observed value, be compared with the modal value of the density of R (with $\kappa = 0$). A significance test at level α will be defined if the null-hypothesis $\kappa = 0$ is rejected whenever R is greater than R_0 where

$$\int_{R_0}^{N} R\phi_N(R)dR = \alpha 2^{N-1}. \tag{2.11}$$

A significance test of $\kappa = \kappa_0$ may be made in the same way by using the density (1.19) with $\kappa = \kappa_0$ when the alternative is a value of κ greater than κ_0. When κ is large, R will be near N with high probability so that (1.19) has the approximate form

$$\kappa^{N-1} e^{-\kappa(N-R)}(N-R)^{N-2}/(N-2)!. \tag{2.12}$$

This suggests that $2\kappa(N-R)$ will be distributed approximately as χ^2 with $2(N-1)$ degrees of freedom.

To determine the values of κ and N for which this approximation is satisfactory, we note first that $(2 \sinh \kappa / e^{\kappa})^N = (1 - e^{-2\kappa})^N$ does not differ from unity by more than 0·0045 if $N \leqslant 100$ and $\kappa \geqslant 5$. This range covers most samples analysed by the writer. Thus, except near $R = 0$, (1.19) is accurately represented by

$$\kappa^{N-1} e^{-\kappa(N-R)}/(N-2)! \sum_{r=0}^{\infty} (-1)^r \binom{N}{r} \langle N-R-2r \rangle^{N-2}, \tag{2.13}$$

the first term of which is (2.12). Now, for any integer r,

$$\begin{aligned} P(N-R \leqslant 2r) = {} & \frac{1}{(N-2)!} \int_0^{2r} \kappa^{N-1} e^{-\kappa(N-R)} (N-R)^{N-2} d(N-R) \\ & - \frac{\binom{N}{1}}{(N-2)!} \int_2^{2r} \kappa^{N-1} e^{-\kappa(N-R)} (N-R-2)^{N-2} d(N-R) \\ & + \dots \\ & + (-1)^r \frac{\binom{N}{r}}{(N-2)!} \int_{2(r-1)}^{2r} \kappa^{N-1} e^{-\kappa(N-R)} (N-R-2\overline{r-1})^{N-2} d(N-R) \\ = {} & \frac{1}{(N-2)!} \int_0^{2r\kappa} e^{-t} t^{N-2} dt - \frac{\binom{N}{1}}{(N-2)!} e^{-2\kappa} \int_0^{2(r-1)\kappa} e^{-t} t^{N-2} dt \\ & + \dots + (-1)^{r-1} \frac{\binom{N}{r-1}}{(N-2)!} e^{-2(r-1)\kappa} \int_0^{2\kappa} e^{-t} t^{N-2} dt. \end{aligned} \tag{2.14}$$

The terms on the right-hand side of (2.14) are majorized by the terms of $(1 + e^{-2\kappa})^N$, and so the use of only the first term is permissible when this expression is sufficiently close to unity. Thus over a wide range of practical cases it may be assumed that $2\kappa(N-R)$ is distributed like χ^2 with $2(N-1)$ degrees of freedom. A similar argument shows that $2\kappa(N-Rc)$ is distributed like χ^2 with $2N$ degrees of freedom.

Hence the test of $\kappa=\kappa_0$ may usually be made by referring $2\kappa_0(N-R)$ or $2\kappa_0(N-Rc)$ to the χ^2 tables with $2(N-1)$ or $2N$ degrees of freedom. For the comparison of two κ's, an F-test may be made. To test a set of κ's for homogeneity, Bartlett's test or Hartley's maximum F-ratio (3) test is available. This approximation also suggests that the approximate maximum likelihood estimator of κ given by (1.13) might be changed to $(N-2)/(N-R)$, which would then be an unbiassed estimator of κ. As will be seen in the example of Section 3, it is more convenient to use the numerator $N-1$ for testing purposes.

2.2. *Test of a given polar vector.*—If a polar vector is prescribed by hypothesis, the estimate of κ obtained by assuming that the data has been drawn from a population with this polar vector is approximately (using 1.15)

$$k_0=\frac{N}{N-Rc}. \tag{2.21}$$

Using an estimated polar vector, the estimate of κ is (using the modified (1.13))

$$k=\frac{N-1}{N-R}. \tag{2.22}$$

It is reasonable to make the identification

$$2\kappa(N-Rc)=\text{dispersion of the sample about the assumed axis,}$$

$$2\kappa(N-R)=\text{dispersion of the sample about the estimated axis.}$$

Analogy with the familiar identity in normal samples,

$$\Sigma(x_i-\mu)^2=\Sigma(x_i-\bar{x})^2+n(\bar{x}-\mu)^2,$$

leads us to write

$$2\kappa(N-Rc)=2\kappa(N-R)+(\text{dispersion of estimated mean vector about assumed mean vector}), \tag{2.23}$$

i.e. $\chi^2_{2N}=\chi^2_{2(N-1)}+\chi_2{}^2$, and therefore to suggest that

$$(N-1)\frac{R(1-c)}{N-R}\simeq F_{2,\,2(N-1)}. \tag{2.24}$$

Fisher found this result as a first, and very good, approximation to the fiducial test for this situation.

To find the error in (2.24), we use the slightly approximated form of (1.16),

$$\kappa^N e^{-\kappa(N-Rc)}R\phi_N(R). \tag{2.25}$$

Putting $u=\kappa(N-R)$, $s=R(1-c)/(N-R)$, the joint density of u and s is found to be

$$\frac{e^{-u(1+s)}u^{N-1}}{(N-2)!}\left\{1-\binom{N}{1}\left(1-\frac{2\kappa}{u}\right)^{N-2}+\binom{N}{2}\left(1-\frac{4\kappa}{u}\right)^{N-2}-\dots\right\}.$$

Hence the density of s is given by

$$\frac{N-1}{(1+s)^N}\left\{\int_0^{\kappa N(1+s)}\frac{e^{-v}v^{N-1}}{(N-1)!}dv-\binom{N}{1}\int_{2\kappa(1+s)}^{\kappa N(1+s)}\frac{e^{-v}v^{N-1}}{(N-1)!}\left(1-\frac{2\kappa(1+s)}{v}\right)^{N-2}dv+\dots\right\}. \tag{2.26}$$

Since $s>0$, the first term in the brackets of (2.26) is very close to unity for practical values of κ and N, and leads to the approximation (2.24). The second and higher terms may be shown to be small except when κ is small and N is large. As an example of a suitable pair of values of κ and N, let us consider $\kappa=5$ and $N=20$. The second and third terms have the crude upper bounds

$$20\left(1-\frac{2}{5}\right)^{18}+20\int_{5\kappa}^{\infty}\frac{e^{-v}v^{19}}{19!}\,dv=0\cdot018,$$

and

$$\frac{20\cdot19}{2}\left(1-\frac{4}{9}\right)^{18}+\frac{20\cdot19}{2}\int_{9\kappa}^{\infty}\frac{e^{-v}v^{19}}{19!}\,dv=0\cdot007.$$

The higher terms will be smaller still, so that (2.24) holds in this case with good approximation.

2.3. *Comparison of two polar vectors.*—Suppose that samples of N_1 and N_2 are drawn from two populations and that a test is to be made that their polar directions are identical. Assuming that both populations have equal values of κ, we may write in the notation of (1.110):

Dispersion of both samples about a common estimated mean vector = Sum of the dispersions of each sample about its estimated vector + Dispersion of the two estimated mean vectors,

i.e. $$2\kappa(N-R)=2\kappa(N_1-R_1)+2\kappa(N_2-R_2)+2\kappa(R_1+R_2-R) \qquad (2.31)$$

i.e. $$\chi^2_{2(N-1)}=\chi^2_{2(N_1-1)}+\chi^2_{2(N_2-1)}+\chi_2{}^2.$$

This suggests that

$$(N-2)\frac{(R_1+R_2-R)}{N-R_1-R_2}\simeq F_{2,\,2(N-2)}. \qquad (2.32)$$

The statistic in (2.32) has an immediate intuitive interpretation; if the mean vectors are very different, R_1+R_2 will be much greater than R so that large F shows significance.

To verify the approximation (2.32), we take the density (1.110) in the form appropriate to κ large; this is

$$\kappa^{N-1}e^{-\kappa(R_1+R_2-R)}e^{-\kappa(N-R_1-R_2)}\frac{(N_1-R_1)^{N_1-2}}{(N_1-2)!}\,\frac{(N_2-R_2)^{N_2-2}}{(N_2-2)!}, \qquad (2.33)$$

and make the transformation $u=\kappa(R_1+R_2-R)$, $v=\kappa(N-R_1-R_2)$, $w=(N_2-R_2)\kappa$. The joint density of u, v, and w is

$$e^{-(u+v)}\frac{(v-w)^{N_1-2}}{(N_1-2)!}\,\frac{w^{N_2-2}}{(N_2-2)!} \qquad (2.34)$$

where $0\leqslant w\leqslant v$, $0\leqslant u\leqslant\infty$, $0\leqslant v\leqslant\infty$.

Integrating out w, the joint density of u and v is

$$\frac{1}{(N-3)!}e^{-(u+v)}v^{N-3}.$$

Thus $u/2$ and $v/2$ are independent and respectively have χ^2's with 2 and $2(N-2)$ degrees of freedom as required by the approximation (2.32).

The range of values of κ, N_1 and N_2, for which (2.32) holds with good approximation, could be investigated in the manner of Section 2.2 or, alternatively, the analogue of the Fisher–Behrens test derived for this situation.

2.4. *Further tests.*—The comparison of several polar vectors will follow in the same way as the test of Section 2.3, leading to the analysis of variance of a one-way classification, provided that all the populations can be assumed to have the same κ. In more complex examples, such as testing whether three polar vectors are coplanar (i.e. that the three poles lie on the same great circle), the method applies but the estimation equations are no longer directly soluble.

The discriminant function for deciding whether a given observation (l, m, n) belongs to one of two populations with the same κ but polar vectors $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$ will simply be

$$(\lambda_1 - \lambda_2)l + (\mu_1 - \mu_2)m + (\nu_1 - \nu_2)n.$$

3. *Numerical examples.*—The tests of significance, proposed in Section 2, will now be applied to some unpublished data, collected by Mr E. Irving of the Australian National University. The measurements relate to three sites in the Torridonian Sandstone Series. The first sample comes from the base of the Aultbea group, and the second and third samples from the base of the Applecross group.

To show the details of the reduction of the observations, it is sufficient to give the actual measurements of only the first sample. The direction of remanent magnetization of a specimen is specified by the azimuthal angle, measured in degrees east of geographical north, and the dip angle, measured in degrees downwards from the bedding plane.

		Direction cosines		
Azimuth	Dip	Down	*N*	*E*
154	53	0·7986	−0·5409	0·2638
207	82	·9903	− ·1240	− ·0632
130	27	·4540	− ·5727	·6825
173	58	·8480	− ·5259	·0646
248	47	·7314	− ·2555	− ·6324
184	27	·4540	− ·8889	− ·0622
81	39	·6293	·1215	·7675
7	−17	− ·2924	·9491	·1166
212	60	·8660	− ·4240	− ·2650
162	63	0·8910	−0·4318	0·1403
	Totals	6·3702	−2·6931	1·0124

Thus the square of the length of the resultant of the first sample is

$$R_1^2 = (6{\cdot}3702)^2 + (-2{\cdot}6931)^2 + (1{\cdot}0124)^2,$$

$$\therefore \quad R_1 = 6{\cdot}990. \quad \text{Since } N = 10,\ k_1 = 2{\cdot}990.$$

The second and third samples yielded respectively

$$R_2^2 = (7{\cdot}1335)^2 + (-1{\cdot}6893)^2 + (3{\cdot}7011)^2,$$

$$\therefore \quad R_2 = 8{\cdot}212, \quad N = 11, \quad k_2 = 3{\cdot}587.$$

Also
$$R_3^2 = (11{\cdot}9296)^2 + (-2{\cdot}3687)^2 + (0{\cdot}8782)^2,$$

$$\therefore \quad R_3 = 12{\cdot}194, \quad N = 15, \quad k_3 = 4{\cdot}989.$$

To test the hypothesis that the three samples come from populations with the same κ, we use the result of Section 2.1 that

$$\frac{2\kappa(N_i - R_i)}{2(N_i - 1)} \simeq \frac{\chi^2_{2(N_i-1)}}{2(N_i - 1)}. \tag{3.1}$$

Thus, if we define k by $(N-1)/(N-R)$ or $N/(N-Rc)$, instead of by (1.13) and (1.15), we have the convenient result that

$$\kappa/k = \text{variance estimate with } 2(N-1) \text{ or } 2N \text{ d.f.} \tag{3.2}$$

This formula has been used to calculate k_1, k_2, and k_3 above. The test may now be carried out simply by Hartley's maximum F-ratio test

$$\frac{\max k}{\min k} = \frac{4{\cdot}989}{2{\cdot}990} = 1{\cdot}67.$$

For three variances of equal degrees of freedom, 22, the 5 per cent significance point lies between 2·46 and 2·07. Hence the data give no grounds for rejecting the null-hypothesis that the three populations have equal κ's.

To test the hypothesis that the polar vectors of the three populations are identical, we use the extension of equation (2.31) i.e.

$$2\kappa(N-R) = 2\kappa\Sigma(N_i - R_i) + 2\kappa(\Sigma R_i - R). \tag{3·3}$$

Thus, instead of (2.32), we have

$$F_{4,\,2(N-3)} = \frac{\Sigma R_i - R}{N - \Sigma R_i}\,\frac{2(N-3)}{4}. \tag{3·4}$$

The components of the resultant of the combined sample are the sums of the components of the sample resultants. Hence

$$R^2 = (25{\cdot}4333)^2 + (-6{\cdot}7511)^2 + (5{\cdot}5917)^2,$$

$$\therefore \quad R = 26{\cdot}902.$$

Hence
$$F_{4,\,66} = \frac{0{\cdot}494}{8{\cdot}604}\,\frac{66}{4} = 0{\cdot}95,$$

which is not significant. Thus the data give us no reason for asserting that the polar vectors are different.

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