

Reeb Spaces of Smooth Functions on Manifolds

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The Reeb space of a continuous function is the space of connected components of the level sets. In this paper, we first prove that the Reeb space of a smooth function on a closed manifold with finitely many critical values has the structure of a finite graph without loops. We also show that an arbitrary finite graph without loops can be realized as the Reeb space of a certain smooth function on a closed manifold with finitely many critical values, where the corresponding level sets can also be preassigned. Finally, we show that a continuous map of a smooth closed connected manifold to a finite connected graph without loops that induces an epimorphism between the fundamental groups is identified with the natural quotient map to the Reeb space of a certain smooth function with finitely many critical values, up to homotopy.

Dedicated to Professor Toshizumi Fukui on the occasion of his 60th birthday.

1 Introduction

In this article, we prove three theorems on Reeb spaces of smooth functions on compact manifolds of dimension $m \geq 2$ that have finitely many *critical values*. Here, the *Reeb space* is the space of connected components of the level sets, endowed with the quotient topology (for a precise definition, see Section 2).

The 1st theorem (Theorem 3.1) states that the Reeb space of such a function always has the structure of a finite (multi-)graph without loops. Many of the results in

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the literature concern Morse functions or smooth functions with finitely many *critical points* [7–11, 26, 29], and our theorem generalizes them. Note that the graph structure of the Reeb space of such a smooth function with finitely many critical values satisfies that the vertices correspond exactly to the connected components of level sets that contain critical points. In the literature, a Reeb space is often called a *Reeb graph* and our theorem justifies the terminology. We note that the same result also holds for every smooth function on an arbitrary compact manifold of dimension $m \geq 2$ possibly with boundary provided that the function and its restriction to the boundary both have finitely many critical values.

The 2nd result of this paper is a realization theorem (Theorem 5.3). We show that, for each $m \geq 2$, an arbitrary finite (multi-)graph without loops can be realized as the Reeb space of a smooth function on a closed m -dimensional manifold with finitely many critical values. Our result is even stronger: we can preassign the diffeomorphism types of the components of level sets for points in the graph. More precisely, to each edge, we assign a closed connected $(m - 1)$ -dimensional manifold, and to each vertex, a compact connected m -dimensional manifold so that certain consistency conditions are satisfied; then, a graph with such a preassignment, called an m -decorated graph, can always be realized as the Reeb space of a smooth function with finitely many critical values on a closed m -dimensional manifold in such a way that a point on an edge corresponds to the preassigned $(m - 1)$ -dimensional manifold and a vertex corresponds to the preassigned m -dimensional manifold.

The 3rd result of this paper also concerns the realization of a graph as the Reeb space (Theorem 6.1). In our 2nd result, the source manifold on which the function is constructed is not preassigned. On the other hand, in our 3rd theorem, we fix the source closed connected manifold M of dimension $m \geq 2$ and first consider a continuous map of M into a connected (multi-)graph without loops that induces an epimorphism between the fundamental groups. Then, we show that such a map is homotopic to the quotient map to the Reeb space of a smooth function on M with finitely many critical values, where the Reeb space is identified with the given graph. Note that the condition on the fundamental group is known to be necessary. A similar result has been obtained by Michalak [22, 23] (see also Gelbukh [4] and Marzantowicz and Michalak [17]): for $m \geq 3$, one can realize a given graph as the Reeb space of a Morse function on a closed m -dimensional manifold up to homeomorphism. Our theorem is slightly different from such results in that we not only realize the topological structure of a given graph but we also realize the given graph structure. We construct smooth functions with

finitely many critical values such that the images by the quotient map of the level set components containing critical points exactly coincide with the vertices of the given graph.

The paper is organized as follows. In Section 2, we review the definition and certain properties of Reeb spaces and present the problems addressed in this paper. In Section 3, we prove that the Reeb space of a smooth function on a closed manifold with finitely many critical values is a graph. The key to the proof is Lemma 3.8, which guarantees that the number of connected components of a level set corresponding to an isolated critical value is always finite. We use some general topology techniques to prove this lemma. In Section 4, we introduce the notion of a path Reeb space, which is the space of all path-components of level sets. We show that for the path Reeb space, our 1st theorem does not hold in general; in fact, even for a smooth function on a closed manifold with finitely many critical values, the path Reeb space may not be a T_1 -space and hence may not have the structure of a graph. However, we also show that the path Reeb space and the usual Reeb space coincide with each other for an arbitrary smooth function on a closed manifold with finitely many critical points. In Section 5, we prove the 2nd theorem. In order to construct a desired function on a manifold, we first construct non-singular functions corresponding to edges and constant functions corresponding to the vertices. Then, we glue them together “smoothly” using our consistency condition. In Section 6, we prove the 3rd theorem. By using the assumption on the fundamental groups, we first find $(m - 1)$ -dimensional submanifolds in the source manifold M that correspond to the edges using surgery techniques. Then, we use the techniques employed in the proof of the 2nd theorem.

Some of the results in this paper have been announced in [27].

Throughout the paper, all manifolds and maps between them are smooth of class C^∞ unless otherwise specified. The symbol “ \cong ” denotes a diffeomorphism between smooth manifolds.

2 Reeb Space

Let $f : X \rightarrow Y$ be a continuous map between topological spaces. For two points $x_0, x_1 \in X$, we write $x_0 \sim x_1$ if $f(x_0) = f(x_1)$ and x_0, x_1 lie on the same connected component of $f^{-1}(f(x_0)) = f^{-1}(f(x_1))$. Let $W_f = X / \sim$ be the quotient space with respect to this equivalence relation, that is, W_f is a topological space endowed with the quotient topology. Let $q_f : X \rightarrow W_f$ denote the quotient map. Then, there exists a unique map

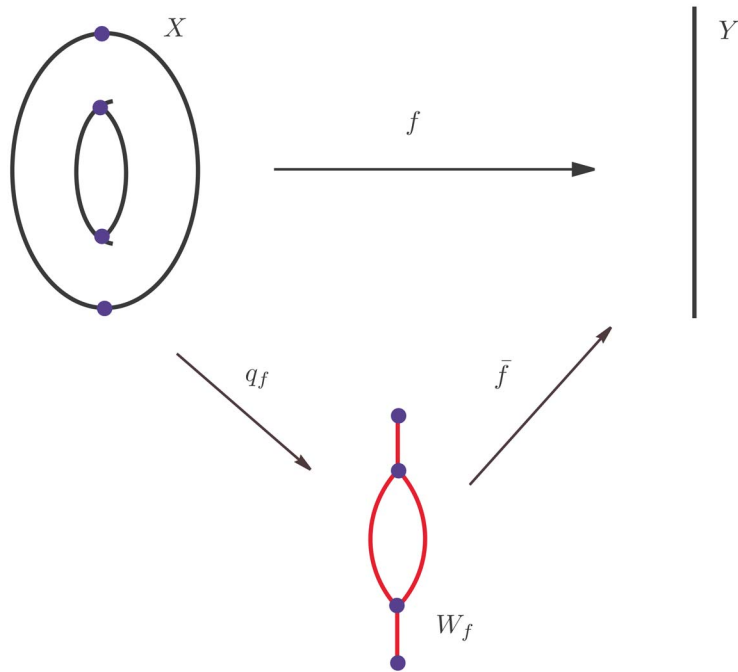


Fig. 1. Example of a Reeb space and a Stein factorization

$\bar{f} : W_f \rightarrow Y$ that is continuous and makes the following diagram commutative:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 q_f \searrow & & \nearrow \bar{f} \\
 & W_f &
 \end{array}$$

The space W_f is called the *Reeb space* of f and the map $\bar{f} : W_f \rightarrow Y$ is called the *Reeb map* of f . The decomposition of f as $\bar{f} \circ q_f$ as in the above commutative diagram is called the *Stein factorization* of f [16]. For a schematic example, see Figure 1.

In this article, a smooth real-valued function on a manifold is called a *Morse function* if its critical points are all non-degenerate. Such a Morse function is *simple* if its restriction to the set of critical points is injective.

Furthermore, in the following, a *graph* means a finite “multi-graph” that may contain multi-edges and/or loops. When considered as a topological space, it is a compact one-dimensional CW complex. A manifold is *closed* if it is compact and has no boundary.

It is known that the Reeb space of a Morse function on a smooth closed manifold has the structure of a graph, which is often called a *Reeb graph* [26] or is sometimes called a *Kronrod–Reeb graph* [29]. This fact has been first stated in [26] without proof. A proof for simple Morse functions can be found in [9, Teorema 2.1] (see also [7]–[11]).

If we use several known results, this fact for Morse functions can also be proved as follows, for example. First, Morse functions on closed manifolds are triangulable (e.g., by a result of Shiota [29]). Then, by [6], its Stein factorization is triangulable and consequently the quotient space, that is, the Reeb space, is a one-dimensional polyhedron. Therefore, it has the structure of a graph.

Furthermore, the following properties are known for Morse functions:

1. the vertices of a Reeb graph correspond to the components of level sets that contain critical points;
2. the restriction of the Reeb map on each edge is an embedding into \mathbb{R} . In particular, no edge is a loop.

Then, the following natural problems arise.

Problem 2.1. Given a smooth function on a closed manifold, does the Reeb space always have the structure of a graph?

Problem 2.2. Given a graph, is it realized as the Reeb space of a certain smooth function?

In this article, we consider these problems and give some answers.

3 Reeb Graph Theorem

Let M be a smooth closed manifold of dimension $m \geq 2$ and $f : M \rightarrow \mathbb{R}$ a smooth function. Then, we have the following.

Theorem 3.1. If f has at most finitely many critical values, then the Reeb space W_f has the structure of a graph. Furthermore, $\bar{f} : W_f \rightarrow \mathbb{R}$ is an embedding on each edge.

Remark 3.2. The above theorem has been known for smooth functions with finitely many critical *points*. A proof can be found in [29].

Remark 3.3. (1) If f has infinitely many critical values, then the above theorem does not hold in general. Some examples will be presented later.

(2) The same result also holds for smooth functions $f : M \rightarrow \mathbb{R}$ on compact manifolds of dimension $m \geq 2$ with $\partial M \neq \emptyset$, provided that both f and $f|_{\partial M}$ have at most finitely many critical values. This can be proved by the same argument as in the proof of Theorem 3.1 given below.

Remark 3.4. It is known that if $f : M \rightarrow N$ with $m = \dim M > \dim N = n \geq 1$ is a smooth map between manifolds with M being compact, and if f is triangulable, then its Reeb space W_f has the structure of an n -dimensional finite simplicial complex (or a compact polyhedron) in such a way that \tilde{f} is an embedding on each simplex [6]. In particular, if f is C^0 -stable, then the Reeb space W_f is an n -dimensional polyhedron. Therefore, if a smooth function $f : M \rightarrow \mathbb{R}$ on a compact manifold is triangulable (e.g., if f is a Morse function), then it follows that W_f is a graph.

As we will see later, there exist smooth functions with finitely many critical values that are not triangulable.

For the proof of Theorem 3.1, we need the following. In the following, for a subset S of a manifold, \bar{S} denotes the closure in the ambient manifold.

Lemma 3.5. Let $f : M \rightarrow \mathbb{R}$ be a smooth function on a closed manifold. Suppose $c \in \mathbb{R}$ is a critical value such that $[c, c + 2\varepsilon)$ does not contain any critical value other than c for some $\varepsilon > 0$. Then, the number of connected components of

$$\overline{f^{-1}((c, c + \varepsilon])} \setminus f^{-1}((c, c + \varepsilon))$$

does not exceed that of $f^{-1}(c + \varepsilon)$, which has finitely many connected components.

Proof. Let us first consider the case where $f^{-1}(c + \varepsilon)$ is non-empty and connected. For each $n \geq 1$, as

$$f^{-1}\left(\left(c, c + \frac{\varepsilon}{n}\right]\right) \cong f^{-1}(c + \varepsilon) \times \left(c, c + \frac{\varepsilon}{n}\right]$$

is connected, its closure,

$$X_n = \overline{f^{-1}\left(\left(c, c + \frac{\varepsilon}{n}\right]\right)},$$

is also connected. Note that $X_1 \supset X_2 \supset X_3 \supset \dots$. We see easily that

$$\bigcap_{n=1}^{\infty} X_n = \overline{f^{-1}((c, c + \varepsilon])} \setminus f^{-1}((c, c + \varepsilon]).$$

Note that this is compact.

Suppose that $\bigcap_{n=1}^{\infty} X_n$ is not connected. Then, it is decomposed into the disjoint union $A \cup B$ of non-empty closed sets A and B . As A and B are compact and M is Hausdorff, there exist disjoint open sets U and V of M such that $A \subset U$ and $B \subset V$. Note that then, we have $\bigcap_{n=1}^{\infty} X_n \subset U \cup V$.

Suppose for all n , we have $X_n \not\subset U \cup V$. Set $\tilde{X} = X_1 \setminus (U \cup V)$, which is non-empty and compact. As $F_n = X_n \setminus (U \cup V)$, $n \geq 1$, is a family of closed sets of \tilde{X} , which has a finite intersection property, we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, which contradicts the fact that $\bigcap_{n=1}^{\infty} X_n \subset U \cup V$. This shows that we have $X_{n_0} \subset U \cup V$ for some $n_0 \geq 1$.

Then, we have $X_{n_0} \cap U \supset A \neq \emptyset$ and $X_{n_0} \cap V \supset B \neq \emptyset$, which contradicts the connectedness of X_{n_0} . Consequently, $\bigcap_{n=1}^{\infty} X_n$ is connected.

If $f^{-1}(c + \varepsilon)$ is empty, then the consequence of the lemma trivially holds. Let us now consider the general case where $f^{-1}(c + \varepsilon) \neq \emptyset$ may not be connected. In this case, $f^{-1}((c, c + \varepsilon])$ has finitely many connected components, say K_1, K_2, \dots, K_k . By the same argument as above, we can show that $\overline{K_i} \setminus K_i$ is connected for each $1 \leq i \leq k$. Therefore, the number of connected components of

$$\overline{f^{-1}((c, c + \varepsilon])} \setminus f^{-1}((c, c + \varepsilon]) = \bigcup_{i=1}^k (\overline{K_i} \setminus K_i)$$

is at most k . This completes the proof of Lemma 3.5. \blacksquare

Remark 3.6. Note that in Lemma 3.5, we can similarly show that if $(c - 2\varepsilon, c]$ does not have any critical value other than c for some $\varepsilon > 0$, then the number of connected components of

$$\overline{f^{-1}([c - \varepsilon, c])} \setminus f^{-1}([c - \varepsilon, c])$$

does not exceed that of $f^{-1}(c - \varepsilon)$, which has finitely many connected components.

Remark 3.7. In Lemma 3.6, the number of connected components of

$$\overline{f^{-1}((c, c + \varepsilon])} \setminus f^{-1}((c, c + \varepsilon])$$

can be strictly smaller than that of $f^{-1}(c + \varepsilon)$. For an example, see Figure 2.

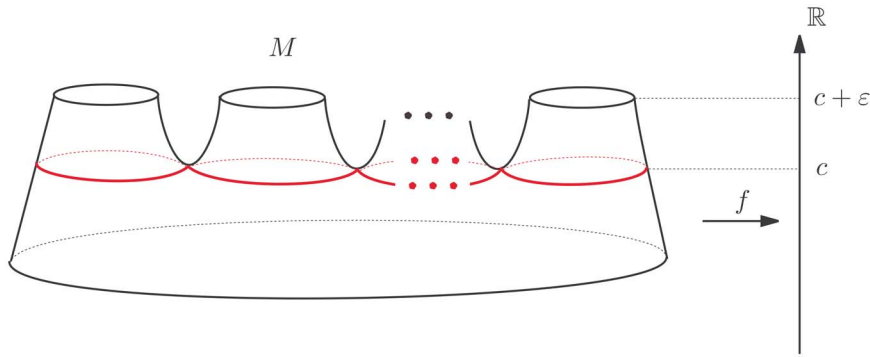


Fig. 2. Number of components of $f^{-1}(c)$ can be strictly smaller than that of $f^{-1}(c + \epsilon)$.

Lemma 3.8. Let $f : M \rightarrow \mathbb{R}$ be a smooth function on a closed manifold. Suppose $c \in \mathbb{R}$ is an isolated critical value of f . Then, $f^{-1}(c)$ has at most finitely many connected components.

Proof. Let $A_\lambda, \lambda \in \Lambda$, be the connected components of $f^{-1}(c)$. Note that A_λ are closed in M . Then, for a sufficiently small $\epsilon > 0$, the cardinality of λ such that A_λ intersects $\overline{f^{-1}([c - \epsilon, c])} \cup \overline{f^{-1}((c, c + \epsilon])}$ is at most finite by virtue of Lemma 3.5 and Remark 3.6.

On the other hand, suppose A_λ does not intersect $\overline{f^{-1}([c - \epsilon, c])} \cup \overline{f^{-1}((c, c + \epsilon])}$ for some $\lambda \in \Lambda$. Then, we have

$$A_\lambda \subset f^{-1}(c) \setminus (\overline{f^{-1}([c - \epsilon, c])} \cup \overline{f^{-1}((c, c + \epsilon])}).$$

For every $x \in A_\lambda$, there exists a small open disk neighborhood U_x of M such that $U_x \cap (f^{-1}([c - \epsilon, c]) \cup f^{-1}((c, c + \epsilon))) = \emptyset$. This implies that U is completely contained in $f^{-1}(c)$ by the continuity of f . Then, we have $U_x \subset A_\lambda$. This shows that A_λ is an open codimension zero submanifold of M . As it is also closed and connected, A_λ is a component of M . Thus, the cardinality of such λ s is at most finite.

This completes the proof of Lemma 3.8. ■

Proof of Theorem 3.1. Let $C_f \subset \mathbb{R}$ be the set of critical values of f , which is finite by our assumption. Let G_f be the graph constructed as follows. The vertices correspond bijectively to the connected components of $f^{-1}(C_f)$ that contain critical points. By our assumption and Lemma 3.8, the number of vertices is finite. Let $L_f \subset f^{-1}(C_f)$ denote the union of such connected components. Then, the edges correspond bijectively to the connected components of $M \setminus L_f$. Note that each such connected component is

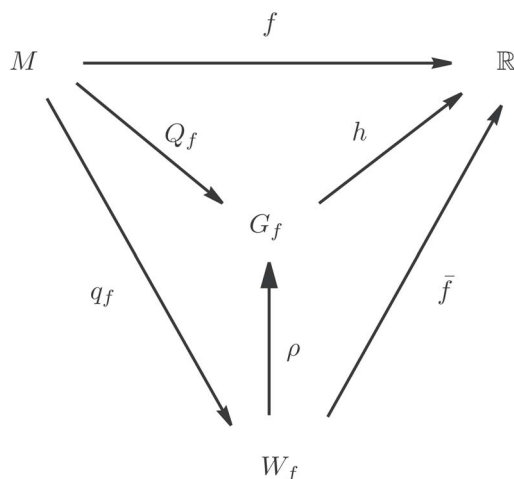


Fig. 3. A commutative diagram.

diffeomorphic to the product of a closed connected $(m-1)$ -dimensional manifold and an open interval. For each vertex v (or edge e), let us denote by V_v (resp. E_e) the component of L_f (resp. $M \setminus L_f$) corresponding to v (resp. e). Then, an edge e is incident to a vertex v if and only if the closure of E_e intersects V_v . More precisely, each edge e is oriented and its initial vertex (or the terminal vertex) is given by v if and only if $x > f(V_v)$ (resp. $x < f(V_v)$) for all $x \in f(E_e)$ and the closure of E_e intersects V_v . Note that by the proof of Lemma 3.5, this is well defined and we get a finite graph G_f .

In the following, the terminology edge often refers to the corresponding open 1-cell of the graph. For each edge e , $f(E_e)$ is an open interval, say (a_e, b_e) for $a_e < b_e$. Then, we have a canonical orientation preserving embedding $h_e : e \rightarrow (a_e, b_e) \subset \mathbb{R}$. These functions for all e can naturally be extended to a continuous function $h : G_f \rightarrow \mathbb{R}$, where $h(v) = f(V_v)$ for each vertex v . (The reader is referred to the commutative diagram in Figure 3 for various spaces and maps defined here and in the following.)

Let $p_e : E_e \rightarrow e$ be the composition of $f|_{E_e} : E_e \rightarrow (a_e, b_e)$ and h_e^{-1} . Then, these maps p_e for all the edges e can naturally be extended to a map $Q_f : M \rightarrow G_f$ in such a way that $Q_f(V_v) = v$ for all vertices v of G_f . By our definition of G_f , this is well defined and continuous.

Let us define the map $\rho : W_f \rightarrow G_f$ as follows. For a point $y \in W_f$, $q_f^{-1}(y)$ is contained in a unique E_e or V_v . Then, $\rho(y)$ is defined to be the point in e corresponding to $p_e(q_f^{-1}(y))$ in the former case and is defined to be v in the latter case. Since $Q_f = \rho \circ q_f$ is continuous, we see that ρ is continuous. Furthermore, we see easily that ρ is bijective.

Since W_f is compact and G_f is Hausdorff, we conclude that ρ is a homeomorphism. Thus, W_f has the structure of a graph.

Finally, we see easily that $\bar{f} = h \circ \rho : W_f \rightarrow \mathbb{R}$. As h is an embedding on each edge, so is \bar{f} . This completes the proof. ■

Remark 3.9. By the above proof, for the graph structure of W_f , the set of vertices of W_f corresponds bijectively to the set of connected components of level sets containing critical points.

Let us give an example of a smooth function with *infinitely many critical values* for which the consequence of Theorem 3.1 does not hold.

Example 3.10. Let M be an arbitrary smooth closed manifold of dimension $m \geq 2$. Then, by [30], there always exists a smooth function $f : M \rightarrow [0, \infty)$ such that $f^{-1}(0)$ is a Cantor set embedded in M . In particular, $f^{-1}(0)$ has uncountably many connected components. Thus, the consequence of Theorem 3.1 does not hold for such an f . In this example, we can show that f has infinitely many critical values.

Let us give a more explicit example. Let $\varphi_1 : \mathbb{R}^2 \rightarrow [0, 1]$ be a smooth function as follows (Figure 4).

- (1) The level set $\varphi_1^{-1}(0)$ coincides with the complement of the open unit disk.
- (2) The level set $\varphi_1^{-1}(1)$ coincides with the disjoint union of two disks centered at $a_1 = (-1/2, 0)$ and $a_2 = (1/2, 0)$ with radius $(1/4) - \varepsilon$ for a sufficiently small $\varepsilon > 0$.
- (3) On $\varphi_1^{-1}((0, 1))$, it has a unique critical point at the origin, which is non-degenerate of index 1 and whose φ_1 -value is equal to $1/2$.
- (4) The level set $\varphi_1^{-1}(t)$ is homeomorphic to a circle, which is connected, for all $t \in (0, 1/2)$, and is homeomorphic to the union of two circles for all $t \in (1/2, 1)$. In particular, for t sufficiently close to 0 or 1, the components of the level sets are circles whose centers are the origin and the points a_1 and a_2 , respectively.

Then, we define the smooth function $\varphi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\varphi_2(x) = \varphi_1(4(x - a_1)) + \varphi_1(4(x - a_2)), \quad x \in \mathbb{R}^2.$$

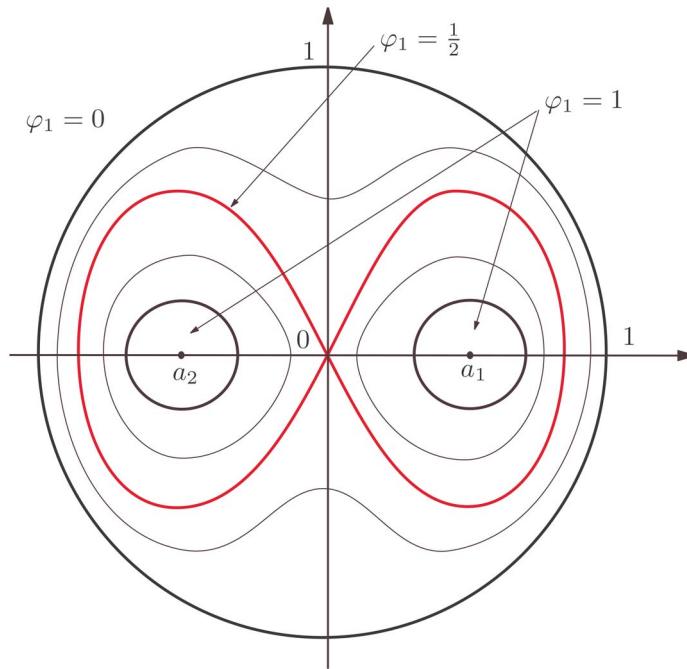


Fig. 4. Level sets of the smooth function φ_1 .

Note that $\varphi_2^{-1}(1)$ consists of disjoint four disks of radius slightly smaller than $1/16$. Let b_1, b_2, b_3 , and b_4 be the centers of the disks. Then, we define $\varphi_3 : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\varphi_3(x) = \sum_{i=1}^4 \varphi_1(16(x - b_i)), \quad x \in \mathbb{R}^2.$$

Repeating this procedure inductively, we can construct a sequence of smooth functions φ_n , $n \geq 1$. Then, consider the series $\psi = \sum_{n=1}^{\infty} c_n \varphi_n$ for a rapidly decreasing sequence $\{c_n\}_{n=1}^{\infty}$ of positive real numbers. We can show that this series converges uniformly and that ψ defines a smooth function on \mathbb{R}^2 . Furthermore, we see that it has the following properties:

- (1) the level set $\psi^{-1}(0)$ coincides with the complement of the open unit disk;
- (2) the level set $\psi^{-1}(c)$ is a Cantor set, where $c = \sum_{n=1}^{\infty} c_n$;
- (3) the critical value set of ψ consists of countably many real numbers $\{r_k\}_{k=0}^{\infty}$ with $0 = r_0 < r_1 < r_2 < \dots$ converging to c together with $r_{\infty} = c$ itself, where $r_k = \sum_{n=1}^k c_n$ for $k \geq 1$;
- (4) for $t \in (r_k, r_{k+1})$, the level set $\psi^{-1}(t)$ is homeomorphic to the disjoint union of 2^k circles for each $k \geq 0$.

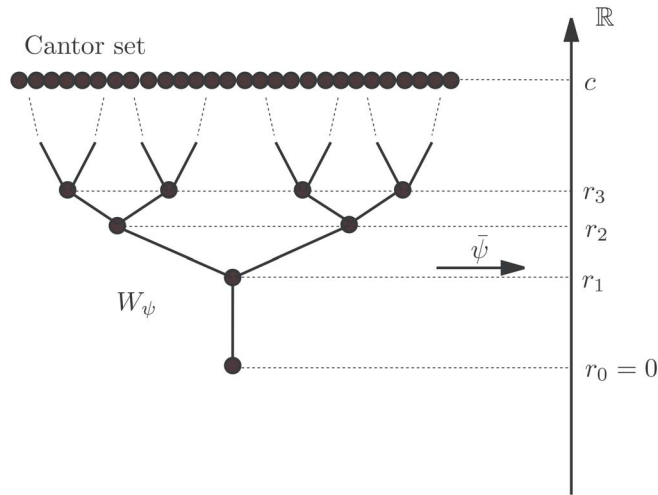


Fig. 5. Reeb space of ψ .

Then, we see that the Reeb space of ψ is as depicted in Figure 5. It consists of countably many “edges” and uncountably many “vertices”. However, this is not a cell complex, as every point of $\bar{\psi}^{-1}(c)$ is an accumulation point. (In fact, one can show that the Reeb space W_ψ can be embedded in \mathbb{R}^2 as in Figure 5.)

Note that, for an arbitrary closed connected surface M , by embedding \mathbb{R}^2 into M and by extending the function ψ by the zero function on the complement, we can construct a smooth function $f : M \rightarrow \mathbb{R}$ whose Reeb space has the same properties.

Let us give another example of a smooth function whose Reeb space does not have the structure of a graph.

Example 3.11. Let us consider the smooth function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows. For $n \geq 1$, let D_n be the closed disk in \mathbb{R}^2 centered at the point $(1/n, 0)$ with radius $1/2n(n+1)$. Note that the disks $\{D_n\}_{n=1}^\infty$ are disjoint. Let $g : D^2 \rightarrow [0, 1]$ be a smooth function on the unit disk in \mathbb{R}^2 with the following properties:

- (1) the function g restricted to a small collar neighborhood $C(\partial D^2)$ of ∂D^2 is constantly zero;
- (2) the restriction $g|_{D^2 \setminus C(\partial D^2)}$ has the unique critical point at the origin, which is the maximum point and takes the value 1;
- (3) each level set $g^{-1}(t)$ is a circle centered at the origin for $t \in (0, 1)$.

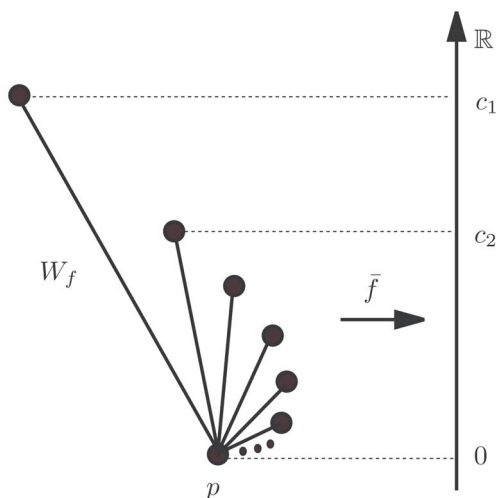


Fig. 6. Reeb space of f .

Note that the Reeb space of g can be identified with $[0, 1]$. Let f_n be the smooth function on D_n defined by

$$f_n(x) = g\left(2n(n+1)\left(x - \frac{1}{n}\right)\right).$$

Then, we define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x) = c_n f_n(x)$ if $x \in D_n$ and $f(x) = 0$ otherwise for a rapidly decreasing sequence $\{c_n\}_{n=1}^{\infty}$ of positive real numbers.

We can show that f is a smooth function. Furthermore, we can also show that the Reeb space W_f of f is homeomorphic to the union of line segments I_n in \mathbb{R}^2 , $n \geq 1$, where

$$I_n = \left\{ t \left(\cos \frac{\pi}{n}, \sin \frac{\pi}{n} \right) \in \mathbb{R}^2 \mid 0 \leq t \leq c_n \right\}.$$

So, W_f is a union of infinitely many intervals where exactly one end point of each interval is glued to a fixed point, say p (Figure 6). In this sense, it seems to have the structure of a cell complex. However, its possible vertex set V does not have the discrete topology. In fact, p is in the closure of $V \setminus \{p\}$. Therefore, W_f does not have the topology of a graph.

Note that, in this example, the critical value set is equal to the infinite set

$$\{c_n \mid n = 1, 2, 3, \dots\} \cup \{0\}.$$

Note also that by embedding \mathbb{R}^2 into an arbitrary closed surface M and by extending the function by the zero map in the complement, one can construct a smooth function $M \rightarrow \mathbb{R}$ with Reeb space having the same property. Note also that for this example, an arbitrary level set has at most finitely many connected components; however, the number of connected components is not uniformly bounded from above.

Let us give an example of a non-triangulable smooth function with finitely many critical values.

Example 3.12. Given an arbitrary smooth closed manifold M of dimension $m \geq 2$, there always exists a smooth function $f : M \rightarrow \mathbb{R}$ with finitely many critical values that is not triangulable. Nevertheless, even in such a situation, W_f is a graph.

In fact, we can construct such a function so that for a critical value c , the closed set $f^{-1}(c)$ cannot be triangulated as follows. First, take any smooth function $g : M \rightarrow \mathbb{R}$ with finitely many critical values and a regular value c . Then, modify M along the submanifold $L = g^{-1}(c)$ as follows. Since the set of critical values is closed in \mathbb{R} , the closed interval $I = [c - \varepsilon, c + \varepsilon]$ contains no critical values for some $\varepsilon > 0$. Note that then $g^{-1}(I) \cong L \times I$. In the following, we fix such a diffeomorphism and identify $g^{-1}(I)$ with $L \times I$.

On the other hand, we consider smooth functions $h_i : L \rightarrow I$, $i = 1, 2$, such that

- (1) $h_1(x) \leq h_2(x)$ for all $x \in L$ and
- (2) the set

$$\tilde{L} = \{(x, t) \in L \times I \mid x \in L, h_1(x) \leq t \leq h_2(x)\}$$

is not triangulable.

(For example, construct such functions in such a way that the interior of \tilde{L} in the compact set $L \times I$ has infinitely many connected components.)

Then, we consider the compact manifold $M \setminus g^{-1}((c - \varepsilon, c + \varepsilon))$ and glue \tilde{L} along the boundary in such a way that $(x, h_1(x))$ (resp. $(x, h_2(x))$) in \tilde{L} is identified with $(x, c - \varepsilon)$ (resp. $(x, c + \varepsilon)$) in $g^{-1}(c - \varepsilon) \cong L \times \{c - \varepsilon\}$ (resp. $g^{-1}(c + \varepsilon) \cong L \times \{c + \varepsilon\}$). Then, the resulting space \tilde{M} is easily seen to be a smooth manifold naturally diffeomorphic to M . Furthermore, we can modify g slightly on a neighborhood of $g^{-1}(I)$ by using bump functions to get a smooth function $f : \tilde{M} \rightarrow \mathbb{R}$ such that $f^{-1}(c) = \tilde{L}$ and c is the unique

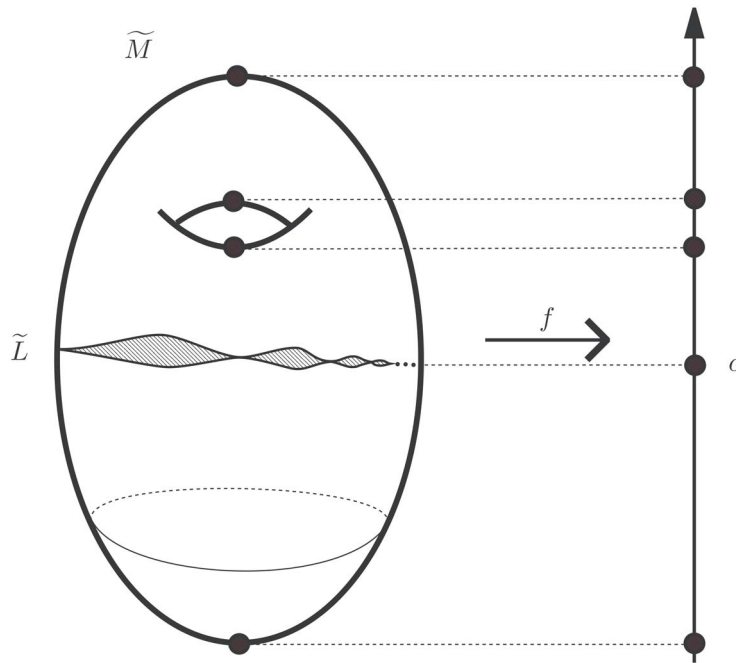


Fig. 7. Non-triangulable function with finitely many critical values.

critical value of f in I . Then, the smooth function f has the desired properties. See Figure 7.

4 Path Reeb Spaces

In the definition of a Reeb space, we usually use connected components of level sets. If we use *path-components* instead, then the resulting space is called the *path Reeb space*.

Definition 4.1. (1) Let X be a topological space. A continuous map $\lambda : [0, 1] \rightarrow X$ is called a *path* connecting $\lambda(0)$ and $\lambda(1) \in X$. We say that two points $x_0, x_1 \in X$ are *connected by a path* if there exists a path connecting x_0 and x_1 . This defines an equivalence relation on X and each equivalence class is called a *path-component* of X . A topological space X is said to be *path-connected* if it consists of a unique path-component.

(2) Let $f : X \rightarrow Y$ be a continuous map between topological spaces. For two points $x_0, x_1 \in X$, we define $x_0 \sim_p x_1$ if $f(x_0) = f(x_1)$ and x_0, x_1 lie on the same

path-component of $f^{-1}(f(x_0)) = f^{-1}(f(x_1))$. Let $W_f^p = X / \sim_p$ be the quotient space with respect to this equivalence relation, that is, W_f^p is a topological space endowed with the quotient topology. Let $q_f^p : X \rightarrow W_f^p$ denote the quotient map. Then, there exists a unique map $\tilde{f}^p : W_f^p \rightarrow Y$ that is continuous and makes the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q_f^p \searrow & & \nearrow \tilde{f}^p \\ & W_f^p & \end{array}$$

The space W_f^p is called the *path Reeb space* of f and the map $\tilde{f}^p : W_f^p \rightarrow Y$ is called the *path Reeb map* of f . The decomposition of f as $\tilde{f}^p \circ q_f^p$ as in the above commutative diagram is called the *path Stein factorization* of f .

The path Reeb space and the usual Reeb space are not homeomorphic to each other in general. To see this, let us first observe the following.

Lemma 4.2. Let $f : X \rightarrow Y$ be a continuous map between topological spaces. If Y is a T_1 -space, then so is the Reeb space W_f .

Proof. Take a point $c \in W_f$. Then, $q_f^{-1}(c)$ is a connected component of a level set $f^{-1}(\tilde{f}(c))$. Since Y is a T_1 -space and f is continuous, $f^{-1}(\tilde{f}(c))$ is closed in X . Therefore, each of its connected components is closed in X , and in particular, $q_f^{-1}(c)$ is closed in X . Hence, $\{c\}$ is closed in W_f . ■

For example, for an arbitrary smooth closed manifold M of dimension $m \geq 2$, we can easily construct a compact set K in M , which is connected but is not path-connected. There exists a smooth function $f : M \rightarrow [0, 1]$ such that $f^{-1}(0) = K$. Then, the Reeb space W_f and the path Reeb space W_f^p are not the same. Furthermore, we can find a compact connected set K as above containing a path-component that is not compact. Then, the path Reeb space of the resulting smooth function f is not a T_1 -space, since the point in the path Reeb space W_f^p corresponding to that path-component is not closed. So, the Reeb space W_f and the path Reeb space W_f^p are not homeomorphic to each other.

Let us give an example of a smooth function on a closed manifold with finitely many critical values whose path Reeb space is not a T_1 -space.

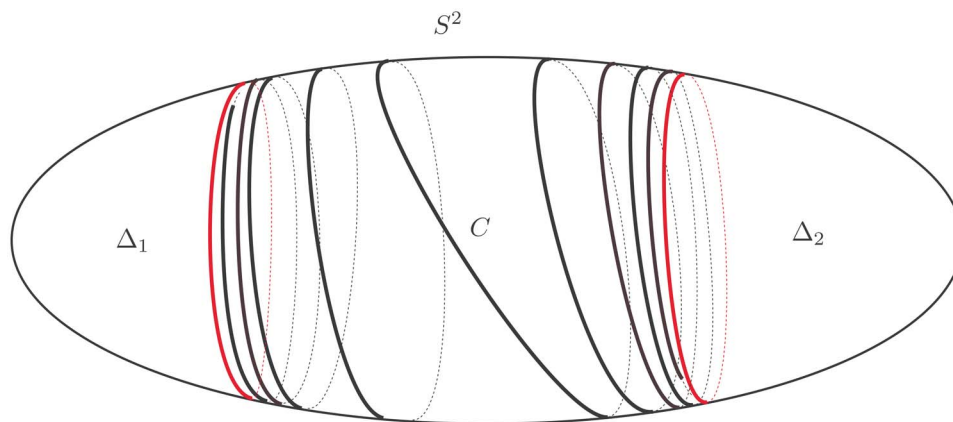


Fig. 8. Zero level set of $f : S^2 \rightarrow \mathbb{R}$.

Example 4.3. Let Δ_1 and Δ_2 be closed 2-disks disjointly embedded in the 2-sphere S^2 . Let C be a curve homeomorphic to the real line \mathbb{R} embedded in the annular region $S^2 \setminus (\Delta_1 \cup \Delta_2)$ whose ends wind around $\partial\Delta_1$ and $\partial\Delta_2$ as in Figure 8. Note that C is not closed in S^2 and the closure \bar{C} coincides with $C \cup \Delta_1 \cup \Delta_2$. Note also that $K = C \cup \Delta_1 \cup \Delta_2$ is a compact connected set, which is not path-connected.

We see easily that $S^2 \setminus K$ is diffeomorphic to an open 2-disk. Let us fix a diffeomorphism $\varphi : \text{Int} D^2 \rightarrow S^2 \setminus K$, where $\text{Int} D^2$ is the open unit disk in \mathbb{R}^2 . For $n \geq 1$, let $g_n : \text{Int} D^2 \rightarrow [0, 1]$ be a smooth function with the following properties:

- (1) the zero level set $g_n^{-1}(0)$ coincides with the complement of the open disk D_n of radius $1 - (n+1)^{-1}$ centered at the origin;
- (2) $g_n|_{D_n \setminus \{0\}}$ is a smooth function of $r = \sqrt{x^2 + y^2} \in (0, 1 - (n+1)^{-1})$, say $g_n(x, y) = h_n(r)$, with $h'_n(r) < 0$;
- (3) $g_n|_{D_n}$ has a unique critical point at the origin, which is the maximum point of g_n , with $g_n(0) = 1$.

Let $f_n : S^2 \rightarrow \mathbb{R}$ be the function defined by $f_n(z) = g_n \circ \varphi^{-1}(z)$ for $z \in S^2 \setminus K$, and $f_n(z) = 0$ otherwise. We see easily that f_n defines a smooth function. Then, we set $f = \sum_{n=1}^{\infty} c_n f_n$ for a rapidly decreasing sequence $\{c_n\}_{n=1}^{\infty}$ of positive real numbers, so that f defines a smooth function $f : S^2 \rightarrow \mathbb{R}$.

By construction, we see that $f^{-1}(0) = K$. Since f takes the minimum value 0 on K , all the points in K are critical points. Furthermore, f has a unique critical point, $\varphi(0)$, on $S^2 \setminus K$, which is the maximum point of f . Hence, the critical value set consists of two values: 0 and $f(\varphi(0)) = \sum_{n=1}^{\infty} c_n$.

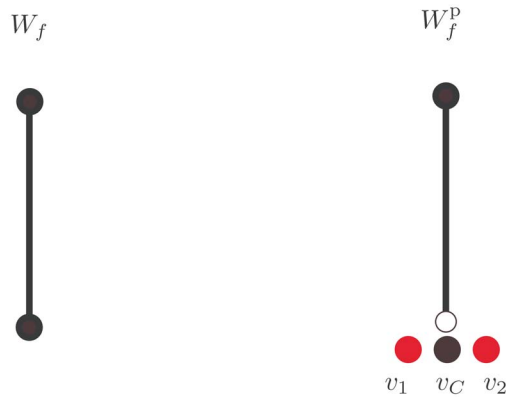


Fig. 9. The Reeb space and the path Reeb space of f .

The Reeb space W_f is easily seen to be homeomorphic to a closed interval, which has the structure of a graph (Figure 9). On the other hand, the path Reeb space W_f^p is not a T_1 -space. More precisely, it consists of a half open interval together with three points v_1, v_2 , and v_C that correspond to Δ_1, Δ_2 , and C , respectively: $\{v_1\}$ and $\{v_2\}$ are closed, while $\{v_C\}$ is not closed and its closure coincides with $\{v_C, v_1, v_2\}$. Furthermore, the closure of the half open interval contains these three points (Figure 9).

This example shows that our Theorem 3.1 does not hold in general for path Reeb spaces. In fact, Lemma 3.5 does not hold for the numbers of path-components for this example.

However, for smooth functions with finitely many *critical points*, we have the following.

Proposition 4.4. Let $f : M \rightarrow \mathbb{R}$ be a smooth function on a closed manifold with finitely many critical points. Then, we have $W_f = W_f^p$.

Proof. Take $y \in \mathbb{R}$. If y is a regular value of f , then $f^{-1}(y)$ is a smooth manifold and its connected components and path-components coincide. If y is a critical value, then take a sufficiently small positive real number ε such that y is the unique critical value of f in $[y - \varepsilon, y + \varepsilon]$. Then, by an argument using integral curves of a gradient vector field of f together with our assumption that f has only finitely many critical points, we can show that $f^{-1}(y)$ is a deformation retract of $V = f^{-1}([y - \varepsilon, y + \varepsilon])$. As V is a compact manifold with boundary and is locally path-connected, we see that

the connected components and the path-components coincide with each other and the numbers are finite. As $f^{-1}(y)$ is homotopy equivalent to V , the number of connected components (or the number of path-components) coincides with that of V . This implies that the number of connected components and that of path-components also coincide with each other for $f^{-1}(y)$. As the numbers are finite, we see that the connected components and the path-components coincide with each other for $f^{-1}(y)$. This completes the proof. ■

The above example shows that this proposition is no longer true for smooth functions with finitely many critical values in general.

5 Realization I

Let G be a graph without loops. In this section, we prove that G is always realized as the Reeb space of a certain smooth function on a closed manifold with finitely many critical values. In fact, our theorem is stronger: (fat) level sets can also be preassigned.

Definition 5.1. For $m \geq 2$, consider maps

$$\begin{aligned} \Phi : \{\text{edges of } G\} &\rightarrow \{\text{diffeomorphism types of closed connected} \\ &\quad (m-1)\text{-dimensional manifolds}\}, \\ \Gamma : \{\text{vertices of } G\} &\rightarrow \{\text{diffeomorphism types of compact connected} \\ &\quad m\text{-dimensional manifolds}\}, \end{aligned}$$

such that for each vertex v of G , we have

$$\partial(\Gamma(v)) \cong \sqcup_{v \in e} \Phi(e), \quad (5.1)$$

that is, the boundary of $\Gamma(v)$ is diffeomorphic to the disjoint union of the finitely many manifolds $\Phi(e)$, where e runs over all edges incident to v . The triple (G, Φ, Γ) is called an *m-decorated graph*.

Definition 5.2. We say that an m -decorated graph (G, Φ, Γ) is *Reeb realizable* if there exists a smooth function $f : M \rightarrow \mathbb{R}$ on a closed m -dimensional manifold M with finitely many critical values such that

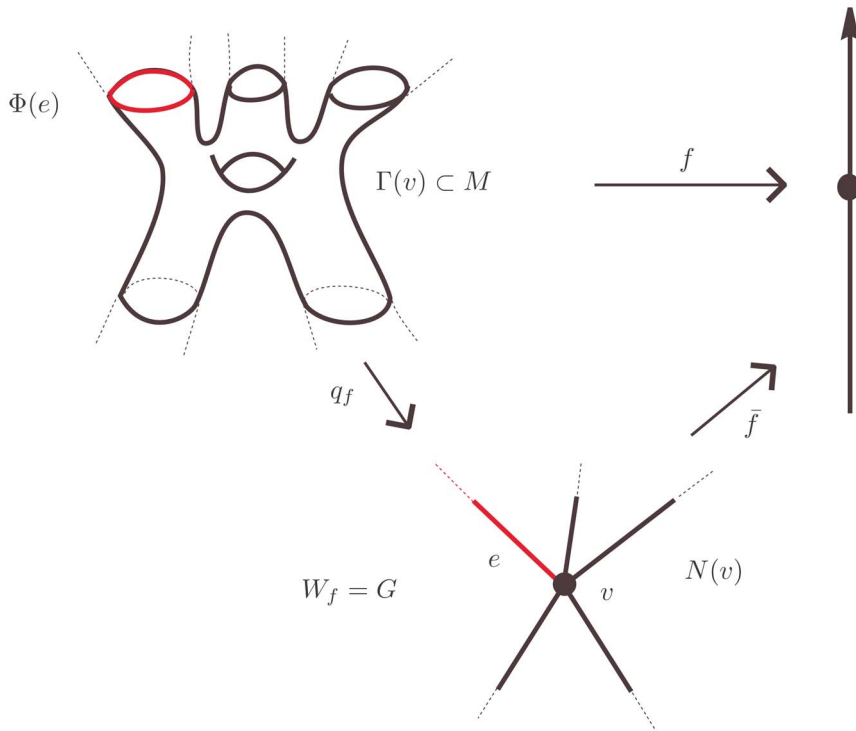


Fig. 10. Realizing an m -decorated graph (G, Φ, Γ) , where $\dim M = m$.

- (1) G is identified with the Reeb space W_f ;
- (2) for each edge e of $G = W_f$, we have $q_f^{-1}(x) \cong \Phi(e), \forall x \in \text{Int } e$;
- (3) for each vertex v of $G = W_f$, we have $q_f^{-1}(N(v)) \cong \Gamma(v)$, where $N(v)$ is a small regular neighborhood of v in G .

See Figure 10.

Then, we have the following realization theorem.

Theorem 5.3. For $m \geq 2$, every m -decorated graph (G, Φ, Γ) is Reeb realizable.

Proof. Let us first construct a continuous function $h : G \rightarrow \mathbb{R}$ such that it is an embedding on each edge. Such a function can easily be constructed by first defining h on the set of vertices so that it is injective, and then, by extending it on the closure of each edge “linearly”.

By h , we can identify each edge of G with an open interval of \mathbb{R} .

Now, for each vertex v , define $f_v : \Gamma(v) \rightarrow \mathbb{R}$ as a constant function by $f_v(\Gamma(v)) = h(v)$. For each edge e , identified with a bounded open interval (a, b) , consider a smooth function $\varphi_e : e \rightarrow e$ such that

- (1) φ_e is a monotone increasing diffeomorphism,
- (2) φ_e can be extended to a smooth homeomorphism $[a, b] \rightarrow [a, b]$ such that the r -th derivatives of f at a and b all vanish for all $r \geq 1$.

Such a smooth function can be constructed, for example, by integrating a bump function. Then, for each vertex e , define the smooth function $f_e : \Phi(e) \times e \rightarrow \mathbb{R}$ by $h \circ \varphi_e \circ p_e$, where $p_e : \Phi(e) \times e \rightarrow e$ is the projection to the 2nd factor.

Then, by condition (5.1), we can glue the m -dimensional manifolds $\Gamma(v)$ and $\Phi(e) \times e$ over all vertices v and all edges e of G in such a way that $\partial(\Gamma(v))$ and $\sqcup_{v \in e} \Phi(e)$ are identified by diffeomorphisms. Let us denote by M the resulting closed m -dimensional manifold. Then, we can also glue the smooth functions f_v and f_e over all vertices v and all edges e of G in order to obtain a continuous function $f : M \rightarrow \mathbb{R}$. This function f is smooth by construction.

Then, by construction and by our assumption that $\Gamma(v)$ and $\Phi(e)$ are connected, the Reeb space W_f of f can be identified with G . Furthermore, we see easily that for each edge e of $G = W_f$, we have $q_f^{-1}(x) \cong \Phi(e)$, $\forall x \in \text{Int } e$, and for each vertex v of $G = W_f$, we have $q_f^{-1}(N(v)) \cong \Gamma(v)$ for a small regular neighborhood $N(v)$ of v in G . This completes the proof. ■

Remark 5.4. We have a similar theorem for functions on compact manifolds with boundary as well. In this case, Φ associates to each edge of G a compact connected $(m - 1)$ -dimensional manifold possibly with boundary and Γ associates to each vertex of G a compact connected m -dimensional manifold with corners. Condition (5.1) should appropriately be modified.

Corollary 5.5. For all $m \geq 2$, every graph without loops is the Reeb space of a smooth function on a closed m -dimensional manifold with finitely many critical values.

Remark 5.6. In the above corollary, we can even construct such a smooth function in such a way that

- (1) every regular level set is a finite disjoint union of standard $(m - 1)$ -spheres and

- (2) the source manifold is diffeomorphic to S^m or a connected sum of a finite number of copies of $S^1 \times S^{m-1}$.

This can be achieved by associating S^{m-1} to each edge, $S^m \setminus (\cup_{k=1}^d \text{Int } D_k^m)$ to each vertex with degree d , and by choosing the attaching diffeomorphisms appropriately, where D_k^m , $k = 1, 2, \dots, d$, are disjoint m -dimensional disks embedded in S^m .

Remark 5.7. Some results similar to Theorem 5.3 and Remark 5.6 are presented in [13, 14].

6 Realization II

Let $f : M \rightarrow \mathbb{R}$ be a smooth function on a closed manifold of dimension $m \geq 2$ with finitely many critical values. Then, it is easy to show that the homomorphism $(q_f)_* : \pi_1(M) \rightarrow \pi_1(W_f)$ induced by the quotient map is surjective. See [26, Théorème 6] for a related statement. (In fact, this is true for an arbitrary continuous function f as long as W_f is semilocally simply connected. See [1].)

We have the following theorem, which corresponds to the converse of this fact.

Theorem 6.1. Let M be a smooth closed connected manifold of dimension $m \geq 2$, G a connected graph without loops, and $Q : M \rightarrow G$ a continuous map such that $Q_* : \pi_1(M) \rightarrow \pi_1(G)$ is surjective. Then, there exists a smooth function $f : M \rightarrow \mathbb{R}$ with finitely many critical values such that

- (1) G can be identified with W_f in such a way that the vertices are identified with the q_f -images of the level set components containing critical points and
- (2) $q_f : M \rightarrow W_f = G$ is homotopic to Q .

Remark 6.2. A similar result also holds for functions on compact manifolds with non-empty boundary.

Remark 6.3. A similar result has been obtained by Michalak [22, 23] (see also Gelbukh [4] and Marzantowicz and Michalak [17]). For $m \geq 3$, one can realize a given graph as the Reeb space of a Morse function on a closed manifold of dimension m up to homeomorphism. Our theorem is slightly different from such results in that we not only realize the topological structure of a given graph but we also realize the given graph structure. We construct smooth functions with finitely many critical values such that

the images by the quotient map of the level set components containing critical points exactly coincide with the vertices of the graph.

For example, if G is a connected graph consisting of two vertices and a unique edge connecting them, then for any closed connected manifold M of dimension $m \geq 2$, there exists a smooth function $f : M \rightarrow \mathbb{R}$ with exactly two critical values such that W_f has the graph structure equivalent to G . Note that according to Michalak [22, 23], there exists a Morse function $g : M \rightarrow \mathbb{R}$ such that W_f is homeomorphic to G ; however, the graph structure of W_f may not be equivalent to that of G in general. In fact, if M admits a smooth function $f : M \rightarrow \mathbb{R}$ with exactly two (possibly degenerate) critical points, then M must be homeomorphic to the sphere S^n [24, 25].

Proof of Theorem 6.1. Let r denote the 1st Betti number of G . Then, we can find points x_1, x_2, \dots, x_r on edges of G such that $G \setminus \{x_1, x_2, \dots, x_r\}$ is connected and contractible. Let $h : G \rightarrow \mathbb{R}$ be a continuous map such that h is an embedding on each edge. By using h , we induce a differentiable structure on each edge of G . By perturbing Q by homotopy if necessary, we may assume that Q is transverse to x_1, x_2, \dots, x_r . Then, each $Q^{-1}(x_i)$ is a compact $(m-1)$ -dimensional submanifold of M , which might not be connected. Let \tilde{r} denote the total number of connected components of $\cup_{i=1}^r Q^{-1}(x_i)$.

Let us show that we may arrange, by homotopy of Q , so that $\tilde{r} = r$, that is, each $Q^{-1}(x_i)$ is connected.

Suppose that $\tilde{r} > r$. Then, by re-ordering the points, we may assume that $E_1 = Q^{-1}(x_1)$ is not connected. As M is connected and $m = \dim M \geq 2$, there is a smooth embedded curve γ in M whose end points lie in distinct components of E_1 . Note that $Q|_\gamma$ is a loop in G based at x_1 . Since Q induces an epimorphism $Q_* : \pi_1(M, \tilde{x}_1) \rightarrow \pi_1(G, x_1)$, where \tilde{x}_1 is the initial point of γ , we may assume that $Q|_\gamma$ represents the neutral element of $\pi_1(G, x_1)$ by adding a loop based at \tilde{x}_1 to γ and by modifying it by a suitable homotopy. We may further assume that γ is transverse to $Q^{-1}(x_i)$ for all i .

First, suppose that γ does not intersect with $Q^{-1}(x_i)$ for $i > 1$ and that γ intersects with $E_1 = Q^{-1}(x_1)$ only at the end points. In this case, $Q|_\gamma$ starts from x_1 in a certain, say positive, direction of the edge on which x_1 lies and returns to x_1 in the reverse, say negative, direction. Furthermore, $Q|_\gamma$ does not intersect x_i for $i > 1$. Thus, $Q|_\gamma$ is null homotopic in G relative to x_1 and such a homotopy can be constructed in such a way that the paths avoid $\{x_1, x_2, \dots, x_r\}$ during the homotopy except at the end points and except for the final constant path. Let $N(\gamma) \cong \gamma \times D^{m-1}$ be a small tubular neighborhood of γ in M such that $N(\gamma) \cap E_1 \cong \partial\gamma \times D^{m-1}$. We may assume that $N(\gamma)$ does not intersect $Q^{-1}(x_i)$ for $i > 1$. Let $N'(\gamma)$ be a smaller tubular neighborhood

corresponding to $\gamma \times D_{1/3}^{m-1}$, where D^{m-1} is the unit disk in \mathbb{R}^{m-1} and D_ρ^{m-1} is the disk with radius $\rho > 0$ with the same center. We can first modify Q by homotopy supported on $N(\gamma)$ so that $Q|_{\gamma \times \{p\}}$ coincides with $Q|_\gamma$ for all $p \in D_{2/3}^{m-1}$. As $Q|_\gamma$ is null-homotopic in G relative to x_1 as described above, we may further modify Q so that $Q|_{N'(\gamma)}$ is a constant map to x_1 .

At this stage, we have $Q^{-1}(x_1) = E_1 \cup N'(\gamma)$. Let E'_1 be the smooth $(m - 1)$ -dimensional submanifold of M obtained from $(E_1 \setminus (\partial\gamma \times \text{Int}D_{1/3}^{m-1})) \cup (\gamma \times \partial D_{1/3}^{m-1})$ by smoothing the corner. Then, we may further modify Q by homotopy supported on a neighborhood of $N'(\gamma)$ in such a way that Q is transverse to x_1 and that $Q^{-1}(x_1) = E'_1$. This can be achieved by sending the part $\gamma \times \text{Int}D_{1/3}^{m-1}$ to the negative side of x_1 . Then, the number of connected components of $Q^{-1}(x_1)$ decreases by 1. As $Q^{-1}(x_i)$ for $i > 1$ stay the same, the total number of connected components of $\cup_{i=1}^r Q^{-1}(x_i)$ decreases by 1.

Now, consider the case where $\text{Int} \gamma$ intersects with $Q^{-1}(x_i)$ for some $i \geq 1$. We fix a positive direction on each edge on which x_1, x_2, \dots, x_r lie. Let ω be the word on x_1, x_2, \dots, x_r constructed by associating x_i (or x_i^{-1}) every time $Q|_\gamma$ passes through x_i in the positive (resp. negative) direction. As $\pi_1(G, x_1)$ is a free group of rank r freely generated by the elements corresponding to $x_i, i = 1, 2, \dots, r$ and $Q|_\gamma$ represents the neutral element of $\pi_1(G, x_1)$, we see that $x_\ell x_\ell^{-1}$ or $x_\ell^{-1} x_\ell$ appears in the word ω for some ℓ with $1 \leq \ell \leq r$. Let γ_ℓ be the sub-arc of γ that corresponds to that sub-word, that is, $Q|_{\gamma_\ell}$ starts from x_ℓ in a certain direction and returns to x_ℓ in the reverse direction, where $Q|_{\text{Int} \gamma_\ell}$ does not intersect x_1, x_2, \dots, x_r .

If the end points of γ_ℓ lie on the same connected component of $Q^{-1}(x_\ell)$, then for γ , we can replace the sub-arc γ_ℓ by a path in $Q^{-1}(x_\ell)$ connecting the end points of γ_ℓ and slightly modify it so as to get a new smooth embedded curve γ' such that the corresponding word has strictly fewer letters.

On the other hand, if the end points lie on distinct components of $Q^{-1}(x_\ell)$, then we can modify Q by homotopy as described above so that we decrease the total number of connected components of $\cup_{i=1}^r Q^{-1}(x_i)$.

In this way, we get a continuous map Q homotopic to the original one such that Q is transverse to x_1, x_2, \dots, x_r and that $Q^{-1}(x_i)$ are all connected.

Let M' be the compact m -dimensional manifold with boundary obtained by cutting M along $\cup_{i=1}^r Q^{-1}(x_i)$. Note that M' is connected. Let G' be the graph obtained by cutting G at x_1, x_2, \dots, x_r . Note that G' is a tree and has $2r$ distinguished vertices corresponding to x_1, x_2, \dots, x_r . We denote the distinguished vertices corresponding to x_i by x_{i+} and $x_{i-}, i = 1, 2, \dots, r$. Then, we have a natural continuous map $Q' : M' \rightarrow$

G' induced by Q , where $\partial M' = \cup_{i=1}^r ((Q')^{-1}(x_{i+}) \cup (Q')^{-1}(x_{i-}))$. In the following, we will construct disjoint $(m - 1)$ -dimensional closed connected submanifolds of M' that correspond bijectively to the edges of G' in such a way that the closures of the connected components of the complement in M' correspond bijectively to the vertices of G' and that a condition similar to (5.1) is satisfied for each vertex of G' except for the distinguished vertices.

First, for each edge incident to a distinguished vertex $x_{i\pm}$, we associate to the edge a submanifold in $\text{Int } M'$ parallel and close to the boundary component $(Q')^{-1}(x_{i\pm})$ of M' . Note that if an edge is incident to two distinct distinguished vertices, then G must be a circle and has no vertex, which is a contradiction. So, this submanifold is well defined.

Let e be an edge, not incident to a distinguished vertex. We denote by x_e a point in the interior of e . Since G' is a tree, $G' \setminus \{x_e\}$ has exactly two connected components, say G'_1 and G'_2 . Let V_j denote the set of distinguished vertices belonging to G'_j , $j = 1, 2$. Then, we can construct an $(m - 1)$ -dimensional closed connected submanifold E_e of M' such that

- (1) $E_e \subset \text{Int } M'$;
- (2) E_e is disjoint from the submanifolds corresponding to the edges incident to distinguished vertices;
- (3) $M' \setminus E_e$ has exactly two connected components, say M'_1 and M'_2 ; and
- (4) the submanifolds corresponding to the edges incident to distinguished vertices in V_j are contained in M'_j , $j = 1, 2$, after renumbering M'_1 and M'_2 if necessary.

Such a submanifold E_e can be constructed, for example, as follows. Let F_k , $k = 1, 2, \dots, a$, be the $(m - 1)$ -dimensional submanifolds associated with the edges incident to the distinguished vertices in V_1 . Since M' is connected, and F_k are parallel to boundary components, we can find smoothly embedded arcs α_k , $k = 1, 2, \dots, a - 1$, in M' such that

- (1) α_k intersects F_k and F_{k+1} exactly at the end points and the intersections are transverse;
- (2) α_k does not intersect F_j , $j \neq k, k + 1$;
- (3) $\alpha_1, \alpha_2, \dots, \alpha_{a-1}$ are disjoint.

If $m = \dim M' \geq 3$, then such arcs as above can easily be found. When $m = 2$, consider an arbitrary properly embedded arc in a compact connected surface such that the end points lie in different components of the boundary. Then, such an arc is never separating. Therefore, a set of arcs as above can be found for this case as well. Now,

consider small tubular neighborhoods $h_k \cong [0, 1] \times D^{m-1}$ of α_k as 1-handles attached to F_1, F_2, \dots, F_a , and use them to perform surgery on $F_0 = \cup_{k=1}^a F_k$ so that we get

$$E_e = (F_0 \setminus ((\cup_{k=1}^{a-1} h_k) \cap F_0)) \cup (\cup_{k=1}^{a-1} dh_k),$$

where $dh_k \cong [0, 1] \times \partial D^{m-1}$. After the surgery, we smooth the corners so that E_e is a smoothly embedded $(m - 1)$ -dimensional submanifold of M' . By construction, it is connected and closed. We can further move E_e by isotopy so that it is disjoint from $\cup_{k=1}^a F_k$. Then, we can easily check that E_e has the desired properties.

For the moment, we ignore Q' and cut M' along E_e to get two compact connected manifolds. We also cut G' at x_e into two trees, where each tree has an additional distinguished vertex. Then, we continue the same procedures to get an $(m - 1)$ -dimensional connected closed submanifold corresponding to an edge not incident to a distinguished vertex. As the number of edges is finite, this process will terminate in a finite number of steps. Finally, we get a family of closed connected $(m - 1)$ -dimensional submanifolds of M' that correspond bijectively to the edges of G' . By construction, each component of the complement corresponds to a unique vertex of G' in such a way that the closure contains an $(m - 1)$ -dimensional submanifold E_e corresponding to an edge e if and only if the vertex is incident to the edge e .

Recall that M can be reconstructed from M' by identifying pairs of boundary components. In this sense, we identify the submanifolds in $\text{Int} M'$ with those in M . Likewise, G is also reconstructed from G' . Now, we associate the above constructed $(m - 1)$ -dimensional submanifold E_e to each edge e of G , where for the edge e containing x_i , we put $E_e = Q^{-1}(x_i)$, $i = 1, 2, \dots, r$. Furthermore, we associate to each vertex of G the closure of the component as described in the previous paragraph. All these ingredients show that we have constructed an m -decorated graph as in Section 5. By construction, condition (5.1) is automatically satisfied.

Then, by using the techniques used in the proof of Theorem 5.3, we can construct a smooth function $f : M'' \rightarrow \mathbb{R}$ with finitely many critical values that realizes the m -decorated graph as the Reeb space. By using the original identification maps for diffeomorphisms for gluing the pieces when constructing M'' , we can arrange so that M'' is naturally identified with M . Then, the Reeb space W_f can also be naturally identified with G .

Finally, we should note that the quotient map $q_f : M \rightarrow W_f = G$ is homotopic to Q . This follows from the fact that $G \setminus \{x_1, x_2, \dots, x_r\}$ is contractible. This completes the proof. ■

For a finitely generated group H , set

$$\text{corank}(H) = \max\{r \mid \text{There exists an epimorphism } H \rightarrow F_r\},$$

where F_r is the free group of rank $r \geq 0$. This is called the *co-rank* of the group H (see, e.g., [2, 3]).

As an immediate corollary to Theorem 6.1, we get the following.

Corollary 6.4. Let M be a smooth closed connected manifold of dimension $m \geq 2$ and G a connected graph without loops. Then, G arises as the Reeb space of a certain smooth function on M with finitely many critical values if and only if $\beta_1(G) \leq \text{corank}(\pi_1(M))$, where β_1 denotes the 1st Betti number.

Remark 6.5. There has been a lot of research on the realization of Reeb graphs, for example, by Sharko [29], Martínez-Alfaro *et al.* [18–20], Masumoto and Saeki [21], Gelbukh [2–5], Kaluba *et al.* [12], Michalak [22, 23], Michalak and Marzantowicz [17], Kitazawa [13–15], etc. Our theorems generalize some of them.

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References

- [1] Calcut, J. S., R. E. Gompf, and J. D. McCarthy. "On fundamental groups of quotient spaces." *Topology Appl.* 159 (2012): 322–30.
- [2] Gelbukh, I. "Co-rank and Betti number of a group." *Czechoslovak Math. J.* 65, no. 140 (2015): 565–7.
- [3] Gelbukh, I. "The co-rank of the fundamental group: the direct product, the first Betti number, and the topology of foliations." *Math. Slovaca* 67 (2017): 645–56.
- [4] Gelbukh, I. "Loops in Reeb graphs of n -manifolds." *Discrete Comput. Geom.* 59 (2018): 843–63.
- [5] Gelbukh, I. "Approximation of metric spaces by Reeb graphs: cycle rank of a Reeb graph,

- the co-rank of the fundamental group, and large components of level sets on Riemannian manifolds." *Filomat* 33 (2019): 2031–49.
- [6] Hiratuka, J. T. and O. Saeki. "Triangulating Stein factorizations of generic maps and Euler characteristic formulas." In *Singularity Theory, Geometry and Topology*. 61–89. RIMS Kôkyûroku Bessatsu B38, Res. Inst. Math. Sci. (RIMS), Kyoto, 2013.
- [7] Izar, S. A. "Funções de Morse: um teorema de classificações em dimensão 2." Master thesis, University of São Paulo, 1978.
- [8] Izar, S. A. "Funções de Morse: uma teoria combinatória em dimensão três." Doctor thesis, University of São Paulo, 1983.
- [9] Izar, S. A. "Funções de Morse e Topologia das Superfícies I: O Grafo de Reeb de $f : M \rightarrow \mathbb{R}$." *Métrica* no. 31. In *Estudo e Pesquisas em Matemática*. Brazil: IBILCE, 1988. <https://www.ibilce.unesp.br/Home/Departamentos/Matematica/metrica-31.pdf>.
- [10] Izar, S. A. "Funções de Morse e Topologia das Superfícies II: Classificação das Funções de Morse Estáveis Sobre Superfícies." *Métrica* no. 35. In *Estudo e Pesquisas em Matemática*. Brazil: IBILCE, 1989. <https://www.ibilce.unesp.br/Home/Departamentos/Matematica/metrica-35.pdf>.
- [11] Izar, S. A. "Funções de Morse e Topologia das Superfícies III: Campos Pseudo-gradientes de uma Função de Morse Sobre uma Superfície." *Métrica* no. 44. In *Estudo e Pesquisas em Matemática*. Brazil: IBILCE, 1992. <https://www.ibilce.unesp.br/Home/Departamentos/Matematica/metrica-44.pdf>.
- [12] Kaluba, M., W. Marzantowicz, and N. Silva. "On representation of the Reeb graph as a sub-complex of manifold." *Topol. Methods Nonlinear Anal.* 45 (2015): 287–307.
- [13] Kitazawa, N. "On Reeb graphs induced from smooth functions on 3-dimensional closed orientable manifolds with finitely many singular values." Preprint arXiv:1902.08841 [math.GT].
- [14] Kitazawa, N. "On Reeb graphs induced from smooth functions on closed or open manifolds." Preprint arXiv:1908.04340 [math.GT].
- [15] Kitazawa, N. "Maps on manifolds onto graphs locally regarded as a quotient map onto a Reeb space and construction problem." Preprint arXiv:1909.10315 [math.GT].
- [16] Levine, H. *Classifying Immersions into \mathbb{R}^4 over Stable Maps of 3-Manifolds into \mathbb{R}^2* , vol. 1157. Lecture Notes in Mathematics. Berlin: Springer, 1985.
- [17] Marzantowicz, W. and Ł. P. Michalak. "Relations between Reeb graphs, systems of hypersurfaces and epimorphisms onto free groups." Preprint arXiv:2002.02388 [math.GT].
- [18] Martínez-Alfaro, J., I. S. Meza-Sarmiento, and R. D. S. Oliveira. "Singular levels and topological invariants of Morse Bott integrable systems on surfaces." *J. Differential Equations* 260 (2016): 688–707.
- [19] Martínez-Alfaro, J., I. S. Meza-Sarmiento, and R. D. S. Oliveira. "Topological classification of simple Morse Bott functions on surfaces." In *Real and Complex Singularities*, vol. 675. Contemporary Mathematics. 165–79. Providence, RI: American Mathematical Society, 2016.
- [20] Martínez-Alfaro, J., I. S. Meza-Sarmiento, and R. D. S. Oliveira. "Singular levels and topological invariants of Morse–Bott foliations on non-orientable surfaces." *Topol. Methods Nonlinear Anal.* 51 (2018): 183–213.

- [21] Masumoto, Y. and O. Saeki. "A smooth function on a manifold with given Reeb graph." *Kyushu J. Math.* 65 (2011): 75–84.
- [22] Michalak, Ł. P. "Realization of a graph as the Reeb graph of a Morse function on a manifold." *Topol. Methods Nonlinear Anal.* 52 (2018): 749–62.
- [23] Michalak, Ł. P. "Combinatorial modifications of Reeb graphs and the realization problem." Preprint arXiv:1811.08031 [math.GT].
- [24] Milnor, J. W. "Sommes de variétés différentiables et structures différentiables des sphères." *Bull. Soc. Math. France* 87 (1959): 439–44.
- [25] Milnor, J. *Differential Topology*, Lectures on Modern Mathematics. vol. II. 165–83. New York: Wiley, 1964.
- [26] Reeb, G. "Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique." *C. R. Hebd. Séances Acad. Sci.* 222 (1946): 847–9.
- [27] Saeki, O. "Reeb graphs of smooth functions on manifolds." *RIMS Kôkyûroku*, Singularity Theory of Differentiable Maps and their Applications, 2156 (2020).
- [28] Sharko, V. V. "About Kronrod–Reeb graph of a function on a manifold." *Methods Funct. Anal. Topology* 12 (2006): 389–96.
- [29] Shiota, M. "Thom's conjecture on triangulations of maps." *Topology* 39 (2000): 383–99.
- [30] Whitney, H. "Analytic extensions of differentiable functions defined in closed sets." *Trans. Amer. Math. Soc.* 36 (1934): 63–89.