Necessary Condition for Rectifiability Involving Wasserstein Distance W_2

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A Radon measure μ is n-rectifiable if it is absolutely continuous with respect to n-dimensional Hausdorff measure and μ-almost all of suppμ can be covered by Lipschitz images of $\mathbb{R}^n$. In this paper, we give a necessary condition for rectifiability in terms of the so-called α_2 numbers — coefficients quantifying flatness using Wasserstein distance W_2. In a recent article, we showed that the same condition is also sufficient for rectifiability, and so we get a new characterization of rectifiable measures.

1 Introduction

Let $1 \leq n \leq d$ be integers. We say that a Radon measure μ on $\mathbb{R}^d$ is n-rectifiable if there exist countably many Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mu(\mathbb{R}^d \setminus \bigcup_i f_i(\mathbb{R}^n)) = 0, \tag{1.1}$$

and moreover μ is absolutely continuous with respect to n-dimensional Hausdorff measure $\mathscr{H}^n$. A set $E \subset \mathbb{R}^d$ is n-rectifiable if the measure $\mathscr{H}^n\big|_E$ is n-rectifiable. We will often omit n and just write "rectifiable."

The study of rectifiable sets and measures lies at the very heart of geometric measure theory. We refer the reader to [19, Chapters 15–18] for some classical charac-

terizations of rectifiability involving densities, tangent measures, and projections. The aim of this paper is to prove a necessary condition for rectifiability involving the so-called α_2 coefficients.

1.1 α_p numbers

Coefficients α_p were introduced by Tolsa in [25]. In order to define them, we recall the definition of Wasserstein distance.

Let $1 \leq p < \infty$, and let μ, ν be two probability Borel measures on $\mathbb{R}^d$ satisfying $\int |x|^p \, d\mu < \infty$, $\int |x|^p \, d\nu < \infty$. The Wasserstein distance W_p between μ and ν is defined as

$$W_p(\mu, \nu) = \left(\inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, d\pi(x, y) \right)^{1/p},$$

where the infimum is taken over all transport plans between μ and ν, that is, Borel probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $\pi(A \times \mathbb{R}^d) = \mu(A)$ and $\pi(\mathbb{R}^d \times A) = \nu(A)$ for all measurable $A \subset \mathbb{R}^d$. The same definition makes sense if instead of probability measures we consider μ, ν, π of the same total mass.

Wasserstein distances are a way to measure the cost of transporting one measure to another, and they are of fundamental importance to the theory of optimal transport. For more information see for example [30, Chapter 7] or [31, Chapter 6].

The idea behind α_p numbers is to quantify how far a given measure is from being a flat measure, that is, from being of the form $c\mathscr{H}^n\big|_L$ for some constant $c > 0$ and some n-plane L. In order to measure it locally (say, in a ball B), we introduce the following auxiliary function.

Let $\varphi : \mathbb{R}^d \to [0, 1]$ be a radial Lipschitz function satisfying $\varphi \equiv 1$ in $B(0, 2)$, $\operatorname{supp}\varphi \subset B(0, 3)$, and for all $x \in B(0, 3)$

$$c^{-1}\operatorname{dist}(x, \partial B(0, 3))^2 \leq \varphi(x) \leq c\operatorname{dist}(x, \partial B(0, 3))^2,$$

$$|\nabla \varphi(x)| \leq c\operatorname{dist}(x, \partial B(0, 3)),$$

for some constant $c > 0$. For example, one could take $\varphi(x) = \phi(|x|)$ where $\phi : [0, \infty) \to [0, 1]$ is such that $\phi(r) = 1$ for $0 \leq r \leq 2$, $\phi(r) = 0$ for $r \geq 3$, and $\phi(r) = (3 - r)^2$ for $2 < r < 3$. Given a ball $B = B(x, r) \subset \mathbb{R}^d$ we set

$$\varphi_B(y) = \varphi\left(\frac{y - x}{r} \right). \tag{1.2}$$

φ_B can be seen as a regularized characteristic function of B.

For $1 \leq p < \infty$, a Radon measure μ on $\mathbb{R}^d$, a ball $B = B(x,r) \subset \mathbb{R}^d$ with $\mu(B) > 0$, and an n-plane L intersecting B, we define

$$\alpha_{\mu,p,L}(B) = \frac{1}{r\,\mu(B)^{1/p}}\,W_p(\varphi_B\mu, a_{B,L}\varphi_B\mathscr{H}^n\big|_L), \tag{1.3}$$

where $a_{B,L} = (\int \varphi_B\,\mathrm{d}\mu)/(\int \varphi_B\,\mathrm{d}\mathscr{H}^n\big|_L)$. We will usually omit the subscripts and just write a. We define also

$$\alpha_{\mu,p}(B) = \inf_L \alpha_{\mu,p,L}(B),$$

where the infimum is taken over all n-planes L intersecting B. For a ball $B = B(x,r)$ we will sometimes write $\alpha_{\mu,p}(x,r)$ instead of $\alpha_{\mu,p}(B)$, and we will do the same with all the other coefficients introduced below.

Coefficients α_p were first defined in [25] with the aim of characterizing *uniformly* rectifiable measures. The notion of uniform rectifiability, which can be seen as a more quantitative counterpart of rectifiability, was introduced by David and Semmes in [11, 12]. We say that a measure μ is uniformly n-rectifiable if

(i)   it is n-AD-regular, that is, there exists a constant C such that for all $x \in$ suppμ and $0 < r < \mathrm{diam}(\mathrm{supp}\mu)$ we have $C^{-1}r^n \leq \mu(B(x,r)) \leq Cr^n$,

(ii)   it has *big pieces of Lipschitz images*, that is, there exist constants $\theta, L > 0$ such that for any $x \in$ suppμ and $0 < r < \mathrm{diam}(\mathrm{supp}\mu)$ we may find an L-Lipschitz mapping g from the n-dimensional ball $B^n(0,r) \subset \mathbb{R}^n$ into $\mathbb{R}^d$ satisfying

$$\mu\big(B(x,r) \cap g(B^n(0,r))\big) \geq \theta r^n.$$

A trivial example of a uniformly rectifiable measure is the surface measure on a Lipschitz graph.

In [25] Tolsa showed the following characterization of uniformly rectifiable measures:

Theorem 1.1 ([25, Theorem 1.2]).   Let $1 \leq p \leq 2$. Suppose μ is an n-AD-regular measure on $\mathbb{R}^d$. Then, μ is uniformly rectifiable if and only if there exists $C > 0$ such that for any ball $B = B(z,R)$ centered at suppμ we have

$$\int_0^R \int_B \alpha_{\mu,p}(x,r)^2\,\mathrm{d}\mu(x)\frac{\mathrm{d}r}{r} \leq CR^n.$$

In this paper, we prove a necessary condition for rectifiability of measures, which is of similar spirit.

Theorem 1.2.   Let μ be an n-rectifiable measure on $\mathbb{R}^d$. Then for μ-a.e. $x \in \mathbb{R}^d$

$$\int_0^1 \alpha_{\mu,2}(x,r)^2 \, \frac{\mathrm{d}r}{r} < \infty. \tag{1.4}$$

In [9, Theorem 1.3], we show that (1.4) is also a sufficient condition for rectifiability (we use a slightly different version of α_2, but it does not matter, see Remark 1.5). Putting the two results together, we get the following characterization.

Corollary 1.3.   Let μ be a Radon measure on $\mathbb{R}^d$. Then μ is n-rectifiable if and only if for μ-a.e. $x \in \mathbb{R}^d$ we have

$$\int_0^1 \alpha_{\mu,2}(x,r)^2 \, \frac{\mathrm{d}r}{r} < \infty.$$

Remark 1.4.   The characterization above is sharp in the following sense. Suppose $1 \leq p \leq q < \infty$. Then it follows easily by Hölder's inequality, definition of α_p numbers, and the fact that $\mathrm{supp}\varphi_B \subset 3B$, that

$$\alpha_{\mu,p}(B) \leq \left(\frac{\mu(3B)}{\mu(B)}\right)^{1/p-1/q} \alpha_{\mu,q}(B).$$

Hence, for doubling measures, α_p numbers are increasing in p. It is well known that rectifiable measures are pointwise doubling, that is,

$$\limsup_{r\to 0^+} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \qquad \text{for } \mu-a.e. \ x \in \mathbb{R}^d \tag{1.5}$$

and so the finiteness of α_2 square function (1.4) implies finiteness of α_p square function for any $1 \leq p \leq 2$. However, in general, one cannot expect finiteness of α_p square function for $p > 2$, see Remark 1.7. In other words, Theorem 1.2 cannot be improved.

Remark 1.5.   For technical reasons, in [9], we define α_p numbers normalizing by $\mu(3B)$ (i.e., in (1.3) we replace $\mu(B)$ with $\mu(3B)$). Of course, the 3B-normalized coefficients are *smaller* than the B-normalized variant used here. Hence, if (1.4) is finite for B-normalized α_2 numbers, then it is finite for 3B-normalized α_2 numbers, and so [9,Theorem 1.3] may be applied to get Corollary 1.3.

It is worthwhile to compare this result with other recent characterizations of rectifiability, which all involve some sort of scale-invariant quantities measuring flatness.

1.2 β_p numbers

The first flatness-quantifying coefficients to be defined were Jones' β numbers, originating in [11, 12, 16]. For $1 \le p < \infty$ and a Radon measure μ on $\mathbb{R}^d$ set

$$\beta_{\mu,p}(x,r) = \inf_L \left(\frac{1}{r^n} \int_{B(x,r)} \left(\frac{\text{dist}(y,L)}{r} \right)^p \, d\mu(y) \right)^{1/p}, \tag{1.6}$$

where the infimum runs over all n-planes L intersecting $B(x,r)$. Let us also define upper and lower n-dimensional densities of a Radon measure μ at $x \in \mathbb{R}^d$ as

$$\Theta^{n,*}(x,\mu) = \limsup_{r\to0^+} \frac{\mu(B(x,r))}{r^n}, \qquad \Theta^n_*(x,\mu) = \liminf_{r\to0^+} \frac{\mu(B(x,r))}{r^n},$$

respectively. If both quantities are equal, we set $\Theta^n(x,\mu) = \Theta^{n,*}(x,\mu) = \Theta^n_*(x,\mu)$ and we call it n-dimensional density.

In [26] it was shown that for a rectifiable measure μ we have

$$\int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty \qquad \text{for } \mu - a.e. \ x \in \mathbb{R}^d. \tag{1.7}$$

On the other hand, Azzam and Tolsa proved in [3] that if a Radon measure μ satisfies (1.7) and

$$0 < \Theta^{n,*}(x,\mu) < \infty \qquad \text{for } \mu - a.e. \ x \in \mathbb{R}^d \tag{1.8}$$

then μ is n-rectifiable. More recently, Edelen, Naber, and Valtorta [13] managed to weaken the assumption (1.8) to

$$\Theta^{n,*}(x,\mu) > 0 \quad \text{and} \quad \Theta^n_*(x,\mu) < \infty \qquad \text{for } \mu - a.e. \ x \in \mathbb{R}^d. \tag{1.9}$$

An alternative proof showing that (1.7) and (1.9) are sufficient for rectifiability is given in [28].

Theorem 1.6 ([3, 13, 26]).   Let μ be a Radon measure on $\mathbb{R}^d$. Then, μ is n-rectifiable if and only if (1.7) and (1.9) hold for μ-a.e. $x \in \mathbb{R}^d$.

Contrary to Corollary 1.3, some sort of assumptions on densities of measure seem to be unavoidable because β_2 numbers are "weaker" than α_2 numbers (see Lemma 3.1). What we mean by that is the following: coefficients β_p measure how close

is suppμ to being contained in an n-plane, and so they do not see holes or high concentrations of measure. *Any* measure with support contained in an n-plane will have all β numbers equal to 0—even Dirac mass! Moreover, due to the normalizing factor r^n in (1.6), β numbers do not charge higher dimensional measures properly (note that the $(n+1)$-dimensional Lebesgue measure satisfies (1.7)). Coefficients α_p, on the other hand, penalize such phenomena.

The choice of $p = 2$ in the above considerations is not arbitrary. Condition (1.7) with $\beta_{\mu,2}(x,r)$ replaced by $\beta_{\mu,p}(x,r)$ is necessary for rectifiability only for $1 \leq p \leq 2$. On the other hand, (1.7) together with (1.8) imply rectifiability only for $p \geq 2$. See [28] for relevant counterexamples. Still, if instead of (1.8) we assume that $\Theta^n_*(\mu,x) > 0$ and $\Theta^{n,*}(\mu,x) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$, then the finiteness of β_p square function for certain $p < 2$ becomes sufficient for rectifiability, see [7, 22].

Remark 1.7. The example from [28] shows that one cannot expect finiteness of the α_p square function when $p > 2$. Indeed, it is easy to see that α_p numbers bound from above β_p numbers (see Lemma 3.1, the same proof works with arbitrary $1 \leq p < \infty$). Tolsa gave an example of a rectifiable measure such that for all $p > 2$ the square function involving β_p in infinite almost everywhere. Hence, the α_p square function of that measure is also infinite almost everywhere.

Let us mention that modified versions of β numbers are also used to study a competing notion of rectifiability for measures, the so-called *Federer rectifiability*. We say that a measure is n-rectifiable in the sense of Federer if it satisfies (1.1), and no absolute continuity with respect to $\mathscr{H}^n$ is required. Dropping the absolute continuity assumption makes such measures very difficult to characterize. A surprising example of a doubling, Federer 1-rectifiable measure supported on the whole plane was found by Garnett, Killip, and Schul [14]. Nevertheless, for $n = 1$, significant progress has been achieved in [2, 6–8, 17, 18]. See also a recent survey of Badger [5].

Theorem 1.2 yields an easy corollary involving *bilateral β numbers*. Set

$$b\beta_{\mu,2}(x,r)^2 = \inf_L \frac{1}{r^n} \left(\int_{B(x,r)} \left(\frac{\text{dist}(y,L)}{r} \right)^2 \, \mathrm{d}\mu(y) + \int_{B(x,r)} \left(\frac{\text{dist}(y,\text{supp}\mu)}{r} \right)^2 \, \mathrm{d}\mathscr{H}^n|_L(y) \right).$$

As shown in Lemma 3.1, if a ball $B(x,r)$ satisfies $\mu(B(x,r)) \approx r^n$ (see Subsection 2.1 for the precise meaning of $\approx$ symbol), then coefficients $\alpha_{\mu,2}(x,r)$ bound from above $b\beta_{\mu,2}(x,r)$. Since for n-rectifiable measure μ we have $0 < \Theta^n(\mu,x) < \infty$ μ-almost everywhere, we immediately get the following.

Corollary 1.8. Let μ be an n-rectifiable measure on $\mathbb{R}^d$. Then for μ-a.e. $x \in \mathbb{R}^d$ we have

$$\int_0^1 b\beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty.$$

1.3 α numbers

Another kind of coefficients quantifying flatness that has attracted a lot of interest is α numbers, first introduced in [24]. Their definition is very similar to that of α_p coefficients, and in fact they can be seen as a variant of α_1 numbers, see [25, Section 5].

Like before, we define a distance on the space of Radon measures. Given Radon measures μ, ν, and an open ball B we set

$$F_B(\mu, \nu) = \sup \left\{ \left| \int \phi \, d\mu - \int \phi \, d\nu \right| \; : \; \phi \in \mathrm{Lip}_1(B) \right\},$$

where

$$\mathrm{Lip}_1(B) = \{\phi \; : \; \mathrm{Lip}\,(\phi) \le 1, \; \mathrm{supp}\phi \subset B\}.$$

The coefficient α of a measure μ in a ball $B = B(x,r)$ is defined as

$$\alpha_\mu(B) = \inf_{c,L} \frac{1}{r\,\mu(B)} F_B(\mu, c\mathscr{H}^n\big|_L),$$

where the infimum is taken over all n-planes L and all $c \ge 0$ (we do not demand a priori that $\mu(B) = c\mathscr{H}^n\big|_L(B)$).

Tolsa showed in [26] that given a rectifiable measure μ we have

$$\int_0^1 \alpha_\mu(x,r)^2 \, \frac{dr}{r} < \infty \qquad \text{for } \mu - a.e. \; x \in \mathbb{R}^d. \tag{1.10}$$

One might ask if (1.10) is also a sufficient condition for rectifiability. Partial answers to that question were given in [1] and [21]. Very recently Azzam, Tolsa, and Toro [4] proved that a measure μ satisfying (1.10), which is also pointwise doubling, that is, such that (1.5) holds, is rectifiable. Since rectifiable measures satisfy (1.5), the following characterization holds.

Theorem 1.9 ([4, 26]). Let μ be a Radon measure on $\mathbb{R}^d$. Then μ is n-rectifiable if and only if (1.10) and (1.5) hold for μ-a.e. $x \in \mathbb{R}^d$.

In the same paper, the authors construct a purely unrectifiable measure satisfying (1.10), and so the pointwise doubling assumption (1.5) cannot be omitted. Let us remark that in the characterization from Corollary 1.3, we do not need to assume any doubling property.

We mention briefly yet another kind of square functions used to describe rectifiability. [29] and [27] are devoted to the so-called Δ numbers, defined as $\Delta_\mu(x,r) = |\frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n}|$. The results from [29] characterize rectifiable measures satisfying $0 < \Theta^n_*(\mu,x) \le \Theta^{n,*}(\mu,x) < \infty$ for μ-a.e $x \in \mathbb{R}^d$. In [27] it was shown that for $n = 1$, analogous results hold under the weaker assumption $0 < \Theta^{1,*}(x,\mu) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$.

1.4   Localizing Theorem 1.2 and organization of the paper

Theorem 1.2 follows easily from the following lemma.

Lemma 1.10.   Let μ be an n-rectifiable measure on $\mathbb{R}^d$, and let $\Gamma \subset \mathbb{R}^d$ be an n-dimensional 1-Lipschitz graph. Suppose $R \in \mathcal{D}_\Gamma$ with $\ell(R) = 1$ (see (2.2) for the definition of $\mathcal{D}_\Gamma$). Then, for any $0 < \varepsilon < 1$, there exists a set $R' \subset R$ such that $\mu(R') \ge (1-\varepsilon)\mu(R)$ and

$$\int_{R'} \int_0^1 \alpha_{\mu,2}(x,r)^2 \, \frac{\mathrm{d}r}{r} \, \mathrm{d}\mu(x) < \infty. \tag{1.11}$$

Proof of Theorem 1.2 using Lemma 1.10.   Let μ be n-rectifiable. It is well known that if one replaces Lipschitz images in (1.1) by Lipschitz graphs, or C^1 manifolds, the definition of rectifiability remains unchanged (see e.g., [19, Theorem 15.21]). Each C^1 manifold is contained in a countable union of (possibly rotated) Lipschitz graphs Γ with $\mathrm{Lip}(\Gamma) \le 1$. Hence, there exists a countable family of n-dimensional 1-Lipschitz graphs Γ_i such that

$$\mu\big(\mathbb{R}^d \setminus \bigcup_i \Gamma_i\big) = 0.$$

Each Γ_i is a countable union of dyadic Γ_i-cubes $R_i^j \in \mathcal{D}_{\Gamma_i}$ satisfying $\ell(R_i^j) = 1$. Clearly, $\mu(\mathbb{R}^d \setminus \bigcup_{i,j} R_i^j) = 0$.

Now, denote the set of x where (1.4) does *not* hold by $\mathcal{B}$, and suppose that $\mu(\mathcal{B}) > 0$. Then, there exists R_i^j such that $\mu(\mathcal{B} \cap R_i^j) > 0$. Let $\varepsilon > 0$ be such that $\mu(\mathcal{B} \cap R_i^j) > 2\varepsilon\mu(R_i^j)$. Applying Lemma 1.10 to R_i^j and ε as above we reach a contradiction. Thus, $\mu(\mathcal{B}) = 0$.   ∎

The rest of the article is dedicated to proving Lemma 1.10. Let us give a brief outline of the proof.

We introduce the necessary tools in Section 2. In Section 3, we show various estimates of α_2 coefficients, usually relying heavily on the results from [25]. In Section 4, we define a family of measures $\{\nu_Q\}_{Q \in \mathcal{D}_\Gamma}$, where $\nu_Q \ll \mathcal{H}^n|_\Gamma$, and each ν_Q approximates μ in some ball around Q. Roughly speaking, ν_Q is defined by projecting the measure of Whitney cubes onto the graph Γ— but only those Whitney cubes whose sidelength is not much bigger than $\ell(Q)$. Then, we construct a tree of good cubes satisfying

$$\sum_{Q \in \mathsf{Tree}} \alpha_{\nu_Q, 2}(\widetilde{B}_Q)^2 \ell(Q)^n < \infty,$$

where $\widetilde{B}_Q$ are balls with the same center as the corresponding cube Q. The stopping region of the tree of good cubes is small. In Section 5, we use the estimate above to show that actually

$$\sum_{Q \in \mathsf{Tree}} \alpha_{\mu, 2}(\widetilde{B}_Q)^2 \ell(Q)^n < \infty.$$

Using the inequality above, we prove (1.11) with $R' = R \setminus \bigcup_{Q \in \mathsf{Stop}(\mathsf{Tree})} Q$. This finishes the proof of Lemma 1.10.

2   Preliminaries

2.1   Notation

Throughout the paper we will write $A \lesssim B$ whenever $A \leq CB$ for some constant C, the so-called "implicit constant." All such implicit constants may depend on dimensions n, d, and we will not track this dependence. If the implicit constant depends also on some other parameter t, we will write $A \lesssim_t B$. The notation $A \approx B$ means $A \lesssim B \lesssim A$, and $A \approx_t B$ means $A \lesssim_t B \lesssim_t A$. Moreover, if symbols $\lesssim$ or $\approx$ appear in the assumptions of a lemma, then the implicit constant of the proven estimate will depend on the implicit constants from the assumptions (see Lemma 3.1 for example).

We denote by $B(z, r) \subset \mathbb{R}^d$ an open ball with center at $z \in \mathbb{R}^d$ and radius $r > 0$. Given a ball B, its center and radius are denoted by $z(B)$ and $r(B)$, respectively. If $\lambda > 0$, then λB is defined as a ball centered at $z(B)$ of radius $\lambda r(B)$.

Given two n-planes L_1, L_2, let L_1' and L_2' be the respective parallel n-planes passing through 0. Then,

$$\angle(L_1, L_2) = \mathrm{dist}_H(L_1' \cap B(0, 1), \ L_2' \cap B(0, 1)),$$

where dist_H stands for Hausdorff distance between two sets. Clearly, we always have $\angle(L_1, L_2) \in [0, 1]$, and $\angle(L_1, L_2) = 0$ if and only if L_1 and L_2 are parallel. Note that if L_1 and L_2 are lines in the plane, then $\angle(L_1, L_2)$ is the sine of the angle between L_1 and L_2.

Given an n-plane L, we will denote the orthogonal projection onto L by Π_L.

For a Borel measure ν on $\mathbb{R}^d$ and a Borel map $T : \mathbb{R}^d \to \mathbb{R}^d$, we denote by $T_* \nu$ the pushforward of ν, that is, a measure on $\mathbb{R}^d$ such that for all Borel $A \subset \mathbb{R}^d$

$$T_* \nu(A) = \nu(T^{-1}(A)).$$

In expressions of the form $W_p(\mu_1, a\mu_2)$, the letter a will always mean the unique constant for which the total mass of $a\mu_2$ is equal to that of μ_1. In other words,

$$a = \frac{\mu_1(\mathbb{R}^d)}{\mu_2(\mathbb{R}^d)}.$$

It may happen that a appears in the same line several times, and every time refers to a different quantity. We hope that this will not cause too much confusion.

Let us once and for all fix measure μ, an n-dimensional 1-Lipschitz graph Γ, and a constant $0 < \varepsilon < 1$ for which we are proving Lemma 1.10. We fix also a coordinate system such that $\Gamma = \{(x, A(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^d$, where $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is a 1-Lipschitz map.

We will denote by L_0 the subspace of $\mathbb{R}^d$ formed by the points whose last $d - n$ coordinates are zeros, so that Γ is a graph over L_0. We will write Π_0 and Π_Γ to denote projections onto L_0 and Γ, respectively, orthogonal to L_0. For the sake of convenience, instead of dealing with the usual surface measure on Γ we will work with

$$\sigma = (\Pi_\Gamma)_* \mathscr{H}^n\big|_{L_0},$$

which is comparable to $\mathscr{H}^n\big|_\Gamma$ (note that for $x \in \Gamma$ we have $\sigma(B(x, r)) \approx r^n$).

Given a ball $B \subset \mathbb{R}^d$ centered at Γ denote by L_B an n-plane minimizing $\alpha_{\sigma,2}(B)$ (note that for an open ball B, it could happen that $L_B \cap B = \varnothing$). Concerning the existence of minimizers, it follows easily from the fact that W_2 metrizes weak convergence of measures (see e.g., [31, Theorem 6.9]), from good compactness properties of weak convergence, and from the fact that the minimizing sequence is of the special form $\varphi_B a_{B,L_k} \mathscr{H}^n\big|_{L_k}$. There may be more than one minimizing plane; if that happens, we simply choose one of them.

For any Radon measure ν such that $\nu(B) > 0$ we set

$$\widehat{\alpha}_{\nu,2}(B) = \alpha_{\nu,2,L_B}(B).$$

Clearly, $\widehat{\alpha}_{\nu,2}(B) \geq \alpha_{\nu,2}(B)$. We will show that

$$\int_{R'} \int_0^1 \widehat{\alpha}_{\mu,2}(x,r)^2 \, \frac{\mathrm{d}r}{r} \, \mathrm{d}\mu(x) < \infty, \tag{2.1}$$

which implies (1.11).

2.2 Γ-cubes

We denote by $\mathcal{D}_{\mathbb{R}^n}, \mathcal{D}_{\mathbb{R}^d}$ the dyadic lattices on L_0 and $\mathbb{R}^d$, respectively. We assume the cubes to be half open-closed, that is, of the form

$$Q = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_i}{2^j}, \frac{k_i+1}{2^j}\right),$$

where $i = n$ for $\mathcal{D}_{\mathbb{R}^n}$, $i = d$ for $\mathcal{D}_{\mathbb{R}^d}$, and $k_1, \ldots, k_i, j$, are arbitrary integers. The sidelength of Q as above will be denoted by $\ell(Q) = 2^{-j}$.

The dyadic lattice on Γ is defined as

$$\mathcal{D}_\Gamma = \{\Pi_\Gamma(Q_0) : Q_0 \in \mathcal{D}_{\mathbb{R}^n}\}. \tag{2.2}$$

The elements of $\mathcal{D}_\Gamma$ will be called Γ-cubes, or just cubes. For every $Q \in \mathcal{D}_\Gamma$ and the corresponding $Q_0 \in \mathcal{D}_{\mathbb{R}^n}$ we define the sidelength of Q as $\ell(Q) = \ell(Q_0)$, and the center of Q as $z_Q = \Pi_\Gamma(z_{Q_0})$, where z_{Q_0} is the center of Q_0. We set

$$B_Q = B(z_Q, 3\mathrm{diam}(Q)),$$

$$\widetilde{B}_Q = \Lambda B_Q,$$

where $\Lambda = \Lambda(n) > 1$ is a constant fixed during the proof. We define also

$$\varphi_Q = \varphi_{B_Q},$$

$$L_Q = L_{B_Q},$$

$$V(Q) = \{x \in \mathbb{R}^d : \Pi_\Gamma(x) \in Q\}.$$

Recall that L_{B_Q} is the n-plane minimizing $\alpha_{\sigma,2}(B_Q)$ and that φ_{B_Q} was defined in (1.2). The "V" in $V(Q)$ stands for "vertical," since $V(Q)$ is a sort of vertical cube. Note also that $Q \subset B_Q \subset \widetilde{B}_Q$ and $r(B_Q) \approx \ell(Q)$.

Given $P \in \mathcal{D}_\Gamma$, we will write $\mathcal{D}_\Gamma(P)$ to denote the family of $Q \in \mathcal{D}_\Gamma$ such that $Q \subset P$.

Remark 2.1.   Let us fix $R \in \mathcal{D}_\Gamma$ with $\ell(R) = 1$ for which we are proving Lemma 1.10. Note that for $x \in R$ and $0 < r < 1$ computing $\alpha_{\mu,2}(x,r)$ involves only $\mu|_B$, where B is some ball containing R. Thus, when proving (2.1), we may and will assume that μ is a finite, compactly supported measure.

For every $e \in \{0,1\}^n$ consider the translated dyadic grid on L_0

$$\mathcal{D}_{\mathbb{R}^n}^e = \frac{1}{3}(e,0\ldots,0) + \mathcal{D}_{\mathbb{R}^n},$$

and the corresponding translated dyadic grid on Γ

$$\mathcal{D}_\Gamma^e = \{\Pi_\Gamma(Q) : Q \in \mathcal{D}_{\mathbb{R}^n}^e\}.$$

Let us also define the translated dyadic lattice on $\mathbb{R}^d$

$$\mathcal{D}_{\mathbb{R}^d}^e = \frac{1}{3}(e,0,\ldots,0) + \mathcal{D}_{\mathbb{R}^d}.$$

The union of all translated dyadic grids on Γ will be called an extended grid on Γ:

$$\widetilde{\mathcal{D}}_\Gamma = \bigcup_{e \in \{0,1\}^n} \mathcal{D}_\Gamma^e.$$

For each $Q \in \widetilde{\mathcal{D}}_\Gamma$ we define B_Q, φ_Q etc. in the same way as for $Q \in \mathcal{D}_\Gamma$.

The main reason for introducing the extended grid is to use a variant of the well-known one-third trick, which was already used in this context by Okikiolu [20].

Lemma 2.2.   There exists $k_0 = k_0(n, \Lambda) > 0$ such that for every $Q \in \mathcal{D}_\Gamma$ with $\ell(Q) \leq 2^{-k_0}$ there exists $P_Q \in \widetilde{\mathcal{D}}_\Gamma$ satisfying $\ell(P_Q) = 2^{k_0}\ell(Q)$ and $3\widetilde{B}_Q \subset V(P_Q)$.

Proof.   First, we remark that for every $j \geq 0$ and for every $x \in L_0$ there exists $e \in \{0,1\}^n$ and $P \in \mathcal{D}_{\mathbb{R}^n}^e$ with $\ell(P) = 2^{-j}$ and $x \in \frac{2}{3}P$. For a nice proof of this fact see [17, Section 3].

Now, consider the point $\Pi_0(z_Q)$. If we take $P \in \mathcal{D}_{\mathbb{R}^n}^e$ with $\ell(P) = 2^{k_0}\ell(Q)$ such that $\Pi_0(z_Q) \in \frac{2}{3}P$, we see that the n-dimensional ball $B^n(\Pi_0(z_Q), 9\Lambda \mathrm{diam}(Q))$ is contained in P as soon as $\frac{2^{k_0}}{3}\ell(Q) \geq 9\Lambda \mathrm{diam}(Q)$.

It follows that for $P_Q \in \mathcal{D}_\Gamma^e$ such that $\Pi_0(P_Q) = P$ we have $3\widetilde{B}_Q \subset V(P_Q)$.   ∎

It may happen that the cube $P_Q \in \widetilde{\mathcal{D}}_\Gamma$ from the lemma above is not unique, so let us just fix one for each $Q \in \mathcal{D}_\Gamma$. The direction $e \in \{0,1\}^n$ such that $P_Q \in \mathcal{D}_\Gamma^e$ will be denoted by $e(Q)$, and the integer k such that $\ell(P_Q) = 2^{k_0}\ell(Q) = 2^{-k}$ will be denoted by $k(Q)$.

We will use later on the fact that

$$9\mathrm{diam}(Q) \leq 2^{k_0}\ell(Q) = 2^{-k(Q)}. \tag{2.3}$$

2.3   Whitney cubes

A very useful tool for approximating the measure μ close to Γ is Whitney cubes. For each $e \in \{0,1\}^n$ we consider the decomposition of $\mathbb{R}^d \setminus \Gamma$ into a family $\mathcal{W}^e$ of Whitney dyadic cubes from $\mathcal{D}_{\mathbb{R}^d}^e$. That is, the elements of $\mathcal{W}^e \subset \mathcal{D}_{\mathbb{R}^d}^e$ are pairwise disjoint, their union equals $\mathbb{R}^d \setminus \Gamma$, and there exist dimensional constants $K > 20, D_0 \geq 1$ such that for every $Q \in \mathcal{W}^e$

   a) $10Q \subset \mathbb{R}^d \setminus \Gamma$,
   b) $KQ \cap \Gamma \neq \varnothing$,
   c)   there are at most D_0 cubes $Q' \in \mathcal{W}^e$ such that $10Q \cap 10Q' \neq \varnothing$. Furthermore, for such cubes Q' we have $\ell(Q') \approx \ell(Q)$.

For the proof see [23, Chapter VI, §1] or [15, Appendix J]. Moreover, it is not difficult to construct Whitney cubes in such a way that if $y \in \Gamma$, $Q \in \mathcal{W}^e$ and $B(y,r) \cap Q \neq \varnothing$, then

$$\mathrm{diam}(Q) \leq r,$$
$$Q \subset B(y, 3r), \tag{2.4}$$

see [26, Section 2.3] for details. We set

$$\mathcal{W}_k^e = \{Q \in \mathcal{W}^e : \ell(Q) \leq 2^{-k}\},$$

and also, for every $Q \in \mathcal{D}_\Gamma$ satisfying $\ell(Q) \leq 2^{-k_0}$,

$$\mathcal{W}_Q = \mathcal{W}^{e(Q)}_{k(Q)}.$$

Remark 2.3.   It follows immediately from the definition of $k(Q)$ that if $P \in \mathcal{W}_Q$, then

$$\ell(P) \leq 2^{-k(Q)} = 2^{k_0} \ell(Q).$$

2.4 Constants and parameters

For the reader's convenience, we collect here all the constants that appear in the proof. We indicate what depends on what, and when each constant gets fixed. As usually, the notation "$C_1 = C_1(C_2)$" means that C_1 is a constant whose precise value depends on some parameter C_2. An absolute constant is a constant that does not depend on any other parameter.

Recall that the measure μ, the Lipschitz graph Γ, and the constant $0 < \varepsilon < 1$ were fixed at the very beginning, in Subsection 2.1, and also that $\mathrm{Lip}(\Gamma) \leq 1$. Moreover, in Remark 2.1, we fixed $R \in \mathcal{D}_\Gamma$ with $\ell(R) = 1$, and without loss of generality we assumed that μ is finite and compactly supported.

- Λ is an absolute constant from the definition of $\widetilde{B}_Q = \Lambda B_Q$, it is fixed in (5.2) (actually, one can take $\Lambda = 9\sqrt{2}$);
- $k_0 = k_0(n, \Lambda)$ is an integer from Lemma 2.2;
- $\varepsilon_0 = \varepsilon_0(n)$ is the constant from Lemma 3.2;
- K and D_0 are dimensional constants from the definition of Whitney cubes;
- $\lambda = \lambda(k_0, K, n, d) > 3$ is fixed in Lemma 5.1, more precisely in equation (5.1) (one can choose e.g., $\lambda = C(n, d) K 2^{k_0}$);
- $M = M(\varepsilon, \lambda, \Lambda, n, d, \mu) > 100$ is chosen in Lemma 4.2.

3 Estimates of α_2 Coefficients

We begin by showing the relationship between $b\beta_2$ and α_2 coefficients.

Lemma 3.1.   Suppose that ν is a Radon measure, B is a ball satisfying $\nu(B) \approx r(B)^n$, and L is a plane minimizing $\alpha_{\nu,2}(B)$. Then

$$b\beta_{\nu,2}(B)^2 \lesssim r(B)^{-n-2} \int_B \mathrm{dist}(x, L)^2 \, \mathrm{d}\nu \lesssim \alpha_{\nu,2}(B).$$

Proof.   Let π be a minimizing transport plan between $\varphi_B \nu$ and $a_{B,L} \varphi_B \mathcal{H}^n\big|_L$ (where $a_{B,L}$ is as in the definition of $\alpha_{\nu,2}(B)$; note that $a_{B,L} \gtrsim 1$ since $\nu(B) \approx r(B)^n$). Then, by the definition of a transport plan, and the fact that $\varphi_B \equiv 1$ on B,

$$\alpha_{\nu,2}(B)^2 r(B)^2 \nu(B) = \int |x-y|^2 \, \mathrm{d}\pi(x,y)$$

$$\geq \frac{1}{2} \int_B \mathrm{dist}(x,L)^2 \, \mathrm{d}\nu + \frac{a_{B,L}}{2} \int_B \mathrm{dist}(y, \mathrm{supp}\nu)^2 \, \mathrm{d}\mathcal{H}^n\big|_L \gtrsim b\beta_{\nu,2}(B)^2 r(B)^{n+2}.$$

■

Recall that Γ is an n-dimensional 1-Lipschitz graph that was fixed in Subsection 2.1, $\sigma = (\Pi_\Gamma)_* \mathcal{H}^n\big|_{L_0}$, and that L_Q is the plane minimizing $\alpha_{\sigma,2}(B_Q)$. The next lemma states that Γ-cubes Q whose best approximating planes L_Q form big angle with L_0 have large α_2 numbers. In consequence, there are very few cubes of this kind (in fact, they form a Carleson family).

Lemma 3.2.   There exists $\varepsilon_0 = \varepsilon_0(n) > 0$ such that for every $Q \in \widetilde{\mathcal{D}}_\Gamma$ with $\measuredangle(L_Q, L_0) > 1 - \varepsilon_0$ we have

$$\alpha_{\sigma,2}(B_Q) \gtrsim 1.$$

Proof.   Suppose $Q \in \widetilde{\mathcal{D}}_\Gamma$. Take $x_k \in 0.5 B_Q \cap \Gamma$, $k = 1, \dots, n$, such that $|x_k - z_Q| = 0.5 r(B_Q)$, and the vectors $\{\Pi_0(x_k - z_Q)\}_k$ form an orthogonal basis of L_0. Set $B_0 = B(z_Q, \eta r(B_Q))$, $B_k = B(x_k, \eta r(B_Q))$, where $\eta = \eta(n) < 0.01$ is a small dimensional constant that will be chosen later. Clearly, for all $k = 0, \dots, n$ we have $B_k \subset B_Q$.

If L_Q does not intersect one of the balls, say B_k, then by Lemma 3.1

$$\alpha_{\sigma,2}(B_Q)^2 r(B_Q)^{n+2} \gtrsim \int_{B_Q} \mathrm{dist}(x, L_Q)^2 \, \mathrm{d}\sigma \geq \int_{\frac{1}{2} B_k} \mathrm{dist}(x, L_Q)^2 \, \mathrm{d}\sigma \gtrsim \eta^{n+2} r(B_Q)^{n+2}.$$

Now suppose that L_Q intersects all B_k. Then, since B_k are all centered at Γ, Γ is 1-Lipschitz, and x_k were chosen appropriately, it is easy to see that for $\eta = \eta(n)$ and $\varepsilon_0 = \varepsilon_0(n)$ small enough we have $\measuredangle(L_Q, L_0) \leq 1 - \varepsilon_0$. ■

The following two lemmas will let us compare α_2 coefficients at similar scales, so that we can pass from the integral form of α_2 square function (1.4) to its dyadic variant.

Lemma 3.3 ([25, Lemma 5.3]).   Let ν be a finite measure supported inside the ball $B' \subset \mathbb{R}^d$. Let $B \subset \mathbb{R}^d$ be another ball such that $3B \subset B'$, with $r(B) \approx r(B')$ and $\nu(B) \approx \nu(B') \approx r(B)^n$. Let L be an n-plane that intersects B and let $f : L \to [0,1]$ be a function such that $f \equiv 1$ on $3B$, $f \equiv 0$ on $L \setminus B'$. Then

$$W_2(\varphi_B \nu, a\varphi_B \mathscr{H}^n\big|_L) \lesssim W_2(\nu, af\mathscr{H}^n\big|_L).$$

Recall that $\widehat{\alpha}_{\mu,2}(B) = \alpha_{\mu,2,L_B}(B)$.

Lemma 3.4.   Let ν be a Radon measure on $\mathbb{R}^d$, $B_1, B_2 \subset \mathbb{R}^d$ be balls centered at Γ with $3B_1 \subset B_2$, $r(B_1) \approx r(B_2)$, $\nu(B_1) \approx \nu(3B_2) \approx r(B_2)^n$. Then we have

$$\widehat{\alpha}_{\nu,2}(B_1) \lesssim \widehat{\alpha}_{\nu,2}(B_2) + \alpha_{\sigma,2}(B_2). \tag{3.1}$$

Proof.   We begin by noting that since $\nu(3B_1) \lesssim \nu(B_1)$, we have $\widehat{\alpha}_{\nu,2}(B_1) \lesssim 1$. As a result, it suffices to prove the lemma under the assumption $\alpha_{\sigma,2}(B_2) \leq \delta$ for some small constant $\delta > 0$, which will be fixed later on.

For brevity of notation set $\varphi_i = \varphi_{B_i}$, $L_i = L_{B_i}$ for $i = 1, 2$. We want to apply Lemma 3.3 with $B = B_1$, $B' = 3B_2$, $\nu = \varphi_2 \nu$, $L = L_2$, $f = \varphi_2\big|_L$. What needs to be checked is that $B_1 \cap L_2 \neq \varnothing$. If this intersection were empty, we would have by Lemma 3.1

$$\alpha_{\sigma,2}(B_2)^2 r(B_2)^{n+2} \gtrsim \int_{B_2} \operatorname{dist}(x, L_2)^2 \, \mathrm{d}\sigma \geq \int_{B_1} \operatorname{dist}(x, L_2)^2 \, \mathrm{d}\sigma$$
$$\geq \int_{\frac{1}{2}B_1} \frac{1}{2} r(B_1)^2 \, \mathrm{d}\sigma \approx r(B_1)^{n+2} \approx r(B_2)^{n+2}.$$

Thus, if $B_1 \cap L_2 = \varnothing$, then $\alpha_{\sigma,2}(B_2) \gtrsim 1$ and we arrive at a contradiction with $\alpha_{\sigma,2}(B_2) \leq \delta$ for δ small enough.

So the assumptions of Lemma 3.3 are met and we get

$$W_2(\varphi_1 \nu, a\varphi_1 \mathscr{H}^n\big|_{L_2}) \lesssim W_2(\varphi_2 \nu, a\varphi_2 \mathscr{H}^n\big|_{L_2}). \tag{3.2}$$

Similarly, taking $\nu = \varphi_2 \sigma$ and $B = B_1$, $B' = 3B_2$, $L = L_2$, $f = \varphi_2\big|_L$ it follows that

$$W_2(\varphi_1 \sigma, a\varphi_1 \mathscr{H}^n\big|_{L_2}) \lesssim W_2(\varphi_2 \sigma, a\varphi_2 \mathscr{H}^n\big|_{L_2}). \tag{3.3}$$

Using the triangle inequality, the scaling of W_2, the fact that L_1 minimizes $\alpha_{\sigma,2}(B_1)$, and the inequalities above, we arrive at

$$
\begin{aligned}
W_2(\varphi_1\nu, a\varphi_1\mathscr{H}^n\big|_{L_1}) &\leq W_2(\varphi_1\nu, a\varphi_1\mathscr{H}^n\big|_{L_2}) \\
&+ \left(\frac{\int \varphi_1 \, \mathrm{d}\nu}{\int \varphi_1 \, \mathrm{d}\sigma}\right)^{1/2} \left(W_2(\varphi_1\sigma, a\varphi_1\mathscr{H}^n\big|_{L_1}) + W_2(\varphi_1\sigma, a\varphi_1\mathscr{H}^n\big|_{L_2})\right) \\
&\overset{L_1 \text{ minimizer}}{\lesssim} W_2(\varphi_1\nu, a\varphi_1\mathscr{H}^n\big|_{L_2}) + \left(\frac{\nu(3B_1)}{r(B_1)^n}\right)^{1/2} W_2(\varphi_1\sigma, a\varphi_1\mathscr{H}^n\big|_{L_2}) \\
&\lesssim W_2(\varphi_1\nu, a\varphi_1\mathscr{H}^n\big|_{L_2}) + W_2(\varphi_1\sigma, a\varphi_1\mathscr{H}^n\big|_{L_2}) \\
&\overset{(3.2),(3.3)}{\lesssim} W_2(\varphi_2\nu, a\varphi_2\mathscr{H}^n\big|_{L_2}) + W_2(\varphi_2\sigma, a\varphi_2\mathscr{H}^n\big|_{L_2}). \quad (3.4)
\end{aligned}
$$

Dividing both sides by $r(B_1)^{1+n/2}$ yields

$$
\widehat{\alpha}_{\nu,2}(B_1) \lesssim \widehat{\alpha}_{\nu,2}(B_2) + \alpha_{\sigma,2}(B_2).
$$

∎

For technical reasons we define a modified version of α_2 coefficients. For any $Q \in \widetilde{\mathcal{D}}_\Gamma$ set

$$
\widetilde{\alpha}_{\nu,2}(Q) = \begin{cases} 1 & \text{if } \measuredangle(L_Q, L_0) > 1 - \varepsilon_0, \\ \ell(Q)^{-(1+\frac{n}{2})} W_2(\psi_Q\nu, a\psi_Q\mathscr{H}^n\big|_{L_Q}) & \text{otherwise,} \end{cases}
$$

where ε_0 is as in Lemma 3.2, and

$$
\psi_Q = \mathbb{1}_{V(Q)},
$$
$$
a = \frac{\int \psi_Q \, \mathrm{d}\nu}{\int \psi_Q \, \mathrm{d}\mathscr{H}^n\big|_{L_Q}}.
$$

Recall that $\sigma = (\Pi_\Gamma)_* \mathscr{H}^n\big|_{L_0} \approx \mathscr{H}^n\big|_\Gamma$.

Lemma 3.5. Let $\nu \ll \sigma$, $B \subset \mathbb{R}^d$ be a ball, $Q \in \widetilde{\mathcal{D}}_\Gamma$. Suppose they satisfy $3B \subset V(Q) \cap B_Q$, $r(B) \approx \ell(Q)$, $\nu(B) \approx \nu(Q) \approx \ell(Q)^n$. Then

$$
\widehat{\alpha}_{\nu,2}(B) \lesssim_{\varepsilon_0} \widetilde{\alpha}_{\nu,2}(Q) + \alpha_{\sigma,2}(B_Q).
$$

Proof. Since $\nu(B) > 0$ and supp$\nu \subset \Gamma$, we certainly have $\sigma(3B) \approx r(B)^n$. Moreover, our assumptions imply that $\nu(3B) \approx \nu(B)$, and so $\widehat{\alpha}_{\nu,2}(B) \lesssim 1$. Thus, we may argue in the same way as in the beginning of the proof of Lemma 3.4 to conclude that, without loss of generality, $L_Q \cap B \neq \varnothing$. Similarly, we may assume that $\angle(L_Q, L_0) \leq 1 - \varepsilon_0$, because otherwise it would follow from Lemma 3.2 that $\alpha_{\sigma,2}(B_Q)$ is big.

Now, since $\angle(L_Q, L_0) \leq 1 - \varepsilon_0$, we get that $V(Q) \cap L_Q \subset \kappa B_Q$ for some constant κ depending on ε_0; we may assume $\kappa > 10$.

We use Lemma 3.3 twice, first with $B = B$, $B' = \kappa B_Q$, $\nu = \psi_Q \nu$, $L = L_Q$, $f = \psi_Q|_L$, and then with $B = B$, $B' = \kappa B_Q$, $\nu = \varphi_Q \sigma$, $L = L_Q$, $f = \varphi_Q|_L$, to obtain

$$W_2(\varphi_B \nu, a\varphi_B \mathscr{H}^n\big|_{L_Q}) \lesssim_\kappa W_2(\psi_Q \nu, a\psi_Q \mathscr{H}^n\big|_{L_Q}),$$

$$W_2(\varphi_B \sigma, a\varphi_B \mathscr{H}^n\big|_{L_Q}) \lesssim_\kappa W_2(\varphi_Q \sigma, a\varphi_Q \mathscr{H}^n\big|_{L_Q}).$$

By the triangle inequality, the scaling of W_2, the fact that L_B minimizes $\alpha_{\sigma,2}(B)$, and the estimates above we get

$$
\begin{aligned}
W_2(\varphi_B \nu, a\varphi_B \mathscr{H}^n\big|_{L_B}) &\leq W_2(\varphi_B \nu, a\varphi_B \mathscr{H}^n\big|_{L_Q}) \\
&+ \left(\frac{\int \varphi_B \, \mathrm{d}\nu}{\int \varphi_B \, \mathrm{d}\sigma}\right)^{1/2} \left(W_2(\varphi_B \sigma, a\varphi_B \mathscr{H}^n\big|_{L_B}) + W_2(\varphi_B \sigma, a\varphi_B \mathscr{H}^n\big|_{L_Q})\right) \\
&\lesssim W_2(\varphi_B \nu, a\varphi_B \mathscr{H}^n\big|_{L_Q}) + \left(\frac{\nu(3B)}{r(B)^n}\right)^{1/2} W_2(\varphi_B \sigma, a\varphi_B \mathscr{H}^n\big|_{L_Q}) \\
&\lesssim_\kappa W_2(\psi_Q \nu, a\psi_Q \mathscr{H}^n\big|_{L_Q}) + W_2(\varphi_Q \sigma, a\varphi_Q \mathscr{H}^n\big|_{L_Q}).
\end{aligned}
$$

Dividing both sides by $r(B)^{1+n/2}$ yields the desired result. ∎

We will need an estimate that is a slight modification of [25, Lemma 6.2]. In order to formulate it, let us introduce the usual martingale difference operator. Recall that if $P \in \mathcal{D}_\Gamma^e$ for some $e \in \{0,1\}^n$, then $P' \in \mathcal{D}_\Gamma^e$ is a child of P if $P' \subset P$ and $\ell(P') = \frac{1}{2}\ell(P)$. Children of $P \in \mathcal{D}_{\mathbb{R}^n}^e$ are defined analogously.

Given $g \in L^1_{loc}(\sigma)$ and $P \in \mathcal{D}_\Gamma^e$ we set

$$
\Delta_P^\sigma g(x) = \begin{cases} \frac{\int_{P'} g \, \mathrm{d}\sigma}{\sigma(P')} - \frac{\int_P g \, \mathrm{d}\sigma}{\sigma(P)} & : x \in P', \ P' \text{ a child of } P, \\ 0 & : x \notin P. \end{cases}
$$

Given $h \in L^1_{loc}(\mathcal{H}^n|_{L_0})$ and $P \in \mathcal{D}^e_{\mathbb{R}^n}$ we define analogously $\Delta_P h(x)$:

$$\Delta_P h(x) = \begin{cases} \dfrac{\int_{P'} h \, d\mathcal{H}^n}{\ell(P')^n} - \dfrac{\int_P h \, d\mathcal{H}^n}{\ell(P)^n} & : x \in P', \ P' \text{ a child of } P, \\ 0 & : x \notin P. \end{cases}$$

Recall that for $g \in L^2(\sigma)$ we have

$$g = \sum_{P \in \mathcal{D}^e_\Gamma} \Delta^\sigma_P g,$$

in the sense of $L^2(\sigma)$, and

$$\|g\|^2_{L^2(\sigma)} = \sum_{P \in \mathcal{D}^e_\Gamma} \|\Delta^\sigma_P g\|^2_{L^2(\sigma)},$$

for details see for example [10, Part I] or [15, Section 5.4.2].

Let us introduce also some additional vocabulary. We will say that a family of cubes Tree $\subset \mathcal{D}^e_\Gamma$ is a tree with root R_0 if it satisfies:

(T1) $R_0 \in$ Tree, and for every $Q \in$ Tree we have $Q \subset R_0$,

(T2)   for every $Q \in$ Tree such that $Q \neq R_0$, the parent of Q also belongs to Tree.

By iterating (T2), we can actually see that if $Q \in$ Tree, then all the intermediate cubes $Q \subset P \subset R_0$ also belong to Tree.

The stopping region of Tree, denoted by Stop(Tree), is the family of all the cubes $P \in \mathcal{D}^e_\Gamma(R_0)$ satisfying:

(S) $P \notin$ Tree, but the parent of P belongs to Tree.

It is easy to see that the cubes from Stop(Tree) are pairwise disjoint, and that they are maximal descendants of R_0 not belonging to Tree. Moreover, for every $x \in R_0$, we have either $x \in P$ for some $P \in$ Stop(Tree) or $x \in Q_k$ for a sequence of cubes $\{Q_k\}_k \subset$ Tree satisfying $\ell(Q_k) \xrightarrow{k \to \infty} 0$.

The following lemma is a modified version of [25, Lemma 6.2].

Lemma 3.6.   Let ν be a Radon measure on Γ of the form $\nu = g\sigma$, with $g \in L^1(\sigma)$, $0 \leq g \leq C$ for some $C > 1$. Consider a cube $Q \in \widetilde{\mathcal{D}}_\Gamma$ and a tree Tree with root Q. Suppose that for all $P \in$ Tree we have $C^{-1}\ell(P)^n \leq \nu(P) \leq C\ell(P)^n$. Then, we have

$$\widetilde{\alpha}_{\nu,2}(Q)^2 \lesssim_{\varepsilon_0, C} \alpha_{\sigma,2}(B_Q)^2 + \sum_{P \in \text{Tree}} \|\Delta^\sigma_P g\|^2_{L^2(\sigma)} \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{S \in \text{Stop(Tree)}} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \nu(S), \tag{3.5}$$

and

$$\sum_{P \in \text{Tree}} \|\Delta_P^\sigma g\|_{L^2(\sigma)}^2 \leq C\|g\|_{L^1(\sigma)} = C\nu(\Gamma). \tag{3.6}$$

In the proof we will use [25, Remark 3.14]. It can be thought of as a flat counterpart of Lemma 3.6 — it is valid for more general measures ν (even more general than what we state below), but at the price of assuming $\Gamma = L_0 \simeq \mathbb{R}^n$.

Lemma 3.7 (simplified [25, Remark 3.14]). Suppose $Q \in \mathcal{D}_{\mathbb{R}^n}$ is a dyadic cube in $\mathbb{R}^n$ and Tree is a tree with root Q. Consider a measure $\nu = g\mathcal{H}^n|_Q$ such that $\nu(P) \approx \ell(P)^n$ for $P \in$ Tree. Then,

$$W_2(\nu, a\mathcal{H}^n|_Q) \lesssim \sum_{P \in \text{Tree}} \|\Delta_P g\|_{L^2(\mathcal{H}^n)}^2 \ell(P)\ell(Q) + \sum_{S \in \text{Stop(Tree)}} \ell(S)^2 \nu(S).$$

Remark 3.8. The definition of a tree of dyadic cubes in [25, p. 492] is slightly more restrictive than the one we adopted. Apart from conditions (T1) and (T2), they also satisfy

(T3)   if $Q \in$ Tree, then either all the children of Q belong to Tree, or none of them.

Equivalently, if $Q \in$ Tree, and Q is not the root, then all the brothers of Q also belong to Tree. To underline the difference between the two notions, sometimes the terms *coherent* and *semicoherent* family of cubes are used. The former refers to trees satisfying (T1–T3), the latter to those satisfying (T1–T2).

Nevertheless, [25, Remark 3.14] cited above is true for both coherent and semicoherent families of cubes. That is, property (T3) is never used in the proof of either [25, Remark 3.14] or the preceding "key lemma" [25, Lemma 3.13].

We are finally ready to prove Lemma 3.6.

Proof of Lemma 3.6. Let $L = L_Q$. If $\angle(L, L_0) > 1 - \varepsilon_0$, then by Lemma 3.2 and the definition of $\widetilde{\alpha}_{\nu,2}(Q)$

$$\widetilde{\alpha}_{\nu,2}(Q)^2 = 1 \lesssim \alpha_{\sigma,2}(B_Q)^2,$$

and we are done. Now assume that $\angle(L, L_0) \leq 1 - \varepsilon_0$.

Let $\tilde{\Pi}_L$ be the projection from $\mathbb{R}^d$ onto L, orthogonal to L_0. We also consider the flat measure $\sigma_L = (\tilde{\Pi}_L)_*\sigma = (\tilde{\Pi}_L)_*\mathcal{H}^n|_{L_0} = c_L\mathcal{H}^n|_L$ (recall that Π_Γ is a projection orthogonal to L_0, so that $\tilde{\Pi}_L \circ \Pi_\Gamma = \tilde{\Pi}_L$). Define $g_0 : L_0 \to \mathbb{R}$ as $g_0 = g \circ \Pi_\Gamma$.

By triangle inequality

$$W_2(\psi_Q \nu, a\psi_Q \mathscr{H}^n\big|_L) = W_2(\psi_Q \nu, a\psi_Q \sigma_L) \leq W_2(\psi_Q \nu, \psi_Q(\tilde{\Pi}_L)_* \nu) + W_2(\psi_Q(\tilde{\Pi}_L)_* \nu, a\psi_Q \sigma_L).$$
(3.7)

The first term from the right hand side is estimated by $\alpha_{\sigma,2}(B_Q)$:

$$W_2(\psi_Q \nu, \psi_Q(\tilde{\Pi}_L)_* \nu)^2 \leq \int_Q |x - \tilde{\Pi}_L(x)|^2 \, \mathrm{d}\nu(x) \approx_{\varepsilon_0} \int_Q \mathrm{dist}(x, L)^2 \, \mathrm{d}\nu(x)$$
$$\lesssim_C \int_Q \mathrm{dist}(x, L)^2 \, \mathrm{d}\sigma(x) \lesssim \alpha_{\sigma,2}(B_Q)^2 \ell(Q)^{n+2}.$$

We estimate the second term from the right hand side of (3.7) using the fact that $\Pi_0\big|_{L \cap V(Q)} : L \cap V(Q) \to L_0 \cap V(Q)$ is bilipschitz, with a constant depending on ε_0 (because $\angle(L, L_0) \leq 1 - \varepsilon_0$):

$$W_2(\psi_Q(\tilde{\Pi}_L)_* \nu, a\psi_Q \sigma_L) \approx_{\varepsilon_0} W_2(\psi_Q(\Pi_0)_*((\tilde{\Pi}_L)_* \nu), a\psi_Q(\Pi_0)_* \sigma_L) = W_2(\psi_Q g_0 \mathscr{H}^n\big|_{L_0}, a\psi_Q \mathscr{H}^n\big|_{L_0}).$$

By Lemma 3.7 we have

$$W_2(\psi_Q g_0 \mathscr{H}^n\big|_{L_0}, a\psi_Q \mathscr{H}^n\big|_{L_0})^2 \lesssim \sum_{P' \in \mathsf{Tree}_{\mathbb{R}^n}} \|\Delta_{P'} g_0\|^2_{L^2(L_0)} \ell(P')\ell(Q) + \sum_{S \in \mathsf{Stop}(\mathsf{Tree})} \ell(S)^2 \nu(S),$$

where $\mathsf{Tree}_{\mathbb{R}^n} \subset \mathcal{D}_{\mathbb{R}^n}$ is the tree formed by cubes $P' = \Pi_0(P)$, $P \in \mathsf{Tree}$, and $L^2(L_0) = L^2(\mathscr{H}^n\big|_{L_0})$.

Using (3.7) and the estimates above we get

$$W_2(\psi_Q \nu, a\psi_Q \mathscr{H}^n\big|_L)^2 \lesssim_{\varepsilon_0} \alpha_{\sigma,2}(B_Q)^2 \ell(Q)^{n+2} + \sum_{P' \in \mathsf{Tree}_{\mathbb{R}^n}} \|\Delta_{P'} g_0\|^2_{L^2(L_0)} \ell(P')\ell(Q)$$
$$+ \sum_{S \in \mathsf{Stop}(\mathsf{Tree})} \ell(S)^2 \nu(S).$$

We conclude the proof of (3.5) by noting that for each $P \in \mathsf{Tree}$

$$\|\Delta_P^\sigma g\|_{L^2(\sigma)} = \|\Delta_{\Pi_0(P)} g_0\|_{L^2(L_0)}.$$

The estimate (3.6) follows trivially from the fact that if $e \in \{0,1\}^n$ is such that $Q \in \mathcal{D}_\Gamma^e$, then

$$\sum_{P \in \mathsf{Tree}} \|\Delta_P^\sigma g\|_{L^2(\sigma)}^2 \leq \sum_{P \in \mathcal{D}_\Gamma^e} \|\Delta_P^\sigma g\|_{L^2(\sigma)}^2 = \|g\|_{L^2(\sigma)}^2 \leq C\|g\|_{L^1(\sigma)}. \qquad \blacksquare$$

We would like to use Lemma 3.6 also on measures with unbounded density. An approximation argument allows us to get rid of the boundedness assumption, at least if we assume additionally that $\nu(B_P) \leq C\ell(P)^n$ for $P \in \mathsf{Tree}$.

Lemma 3.9. Let $\nu = g\sigma$ with $g \in L^1(\sigma)$, $g \geq 0$. Consider a cube $Q \in \widetilde{\mathcal{D}}_\Gamma$ and a tree Tree with root Q. Suppose there exists $C > 1$ such that for all $P \in \mathsf{Tree}$ we have $C^{-1}\ell(P)^n \leq \nu(P) \leq \nu(B_P) \leq C\ell(P)^n$. Then, we have

$$\widetilde{\alpha}_{\nu,2}(Q)^2 \lesssim_{\varepsilon_0, C} \alpha_{\sigma,2}(B_Q)^2 + \sum_{P \in \mathsf{Tree}} \|\Delta_P^\sigma g\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{S \in \mathsf{Stop(Tree)}} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \nu(S), \qquad (3.8)$$

and

$$\sum_{P \in \mathsf{Tree}} \|\Delta_P^\sigma g\|_{L^2(\sigma)}^2 \leq C\|g\|_{L^1(\sigma)} = C\nu(\Gamma). \qquad (3.9)$$

We divide the proof into smaller pieces. Let $\mathsf{Stop} = \mathsf{Stop(Tree)}$. First, we define the set of good points as

$$G = Q \setminus \bigcup_{P \in \mathsf{Stop}} P.$$

Note that the points from $x \in G$ are not contained in any stopping cube, and so there are arbitrarily small cubes $P \in \mathsf{Tree}$ containing x. We introduce the following approximating measure:

$$\tilde{\nu} = \nu|_G + \sum_{S \in \mathsf{Stop}} \frac{\nu(S)}{\sigma(S)} \sigma|_S.$$

It is clear that for $Q \in \mathsf{Tree} \cup \mathsf{Stop}$ we have $\tilde{\nu}(Q) = \nu(Q)$. Moreover, for $Q \in \mathsf{Tree}$

$$C^{-1}\ell(Q)^n \leq \tilde{\nu}(Q) = \nu(Q) \leq C\ell(Q)^n. \qquad (3.10)$$

On the other hand, each $S \in \mathsf{Stop}$ is a child of some $Q \in \mathsf{Tree}$, so that

$$\tilde{\nu}(S) = \nu(S) \leq \nu(Q) \leq C\ell(Q)^n = 2^n C\ell(S)^n. \qquad (3.11)$$

Lemma 3.10.   We have

$$\left\|\frac{d\tilde{\nu}}{d\sigma}\right\|_{L^\infty(\sigma)} \lesssim C.$$

Proof.   It is trivial that for $x \in S \in \mathsf{Stop}$ the density is constant and

$$\frac{d\tilde{\nu}}{d\sigma}(x) = \frac{\nu(S)}{\sigma(S)} = \frac{\nu(S)}{\ell(S)^n} \overset{(3.11)}{\leq} 2^n C.$$

On the other hand, by the definition of $\tilde{\nu}$, for σ-a.e. $x \in G$ we have $\frac{d\tilde{\nu}}{d\sigma}(x) = \frac{d\nu}{d\sigma}(x) = g(x)$. Moreover, for σ-a.e. $x \in G$ we have a sequence of cubes $Q_j \in \mathsf{Tree}$ such that $\ell(Q_j) = 2^{-j}$ and $x \in Q_j$. Note that there exists some integer $j_0 > 0$ (depending on dimension) such that

$$Q_{j+j_0} \subset B(x, 2^{-j}) \subset B_{Q_j}.$$

It follows that

$$\frac{d\tilde{\nu}}{d\sigma}(x) = \frac{d\nu}{d\sigma}(x) = \lim_{j\to\infty} \frac{\nu(B(x, 2^{-j}))}{\sigma(B(x, 2^{-j}))} \leq \lim_{j\to\infty} \frac{\nu(B_{Q_j})}{\sigma(Q_{j+j_0})} \leq \lim_{j\to\infty} \frac{C\ell(Q_j)^n}{\ell(Q_{j+j_0})^n} = C\, 2^{nj_0}.$$

Thus,

$$\left\|\frac{d\tilde{\nu}}{d\sigma}\right\|_{L^\infty(\sigma)} \lesssim C.$$

∎

Let $\tilde{g} \in L^1(\sigma) \cap L^\infty(\sigma)$ be such that $\tilde{\nu} = \tilde{g}\sigma$. Applying Lemma 3.6 to $\tilde{\nu}$ yields

$$\widetilde{\alpha}_{\tilde{\nu},2}(Q)^2 \lesssim_{\varepsilon_0,C} \alpha_{\sigma,2}(B_Q)^2 + \sum_{P\in\mathsf{Tree}} \|\Delta_P^\sigma \tilde{g}\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{S\in\mathsf{Stop}} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \tilde{\nu}(S), \tag{3.12}$$

and

$$\sum_{P\in\mathsf{Tree}} \|\Delta_P^\sigma \tilde{g}\|_{L^2(\sigma)}^2 \leq C\|\tilde{g}\|_{L^1(\sigma)} = C\tilde{\nu}(\Gamma) = C\nu(\Gamma). \tag{3.13}$$

Observe that for $P \in \mathsf{Tree}$ we have

$$\Delta_P^\sigma \tilde{g} = \Delta_P^\sigma g. \tag{3.14}$$

Indeed, for $x \notin P$, both quantities are equal to zero. For $x \in P' \subset P$, where P' is a child of P, we have $P' \in \mathsf{Tree} \cup \mathsf{Stop}$, and so

$$\Delta_P^\sigma \tilde{g}(x) = \frac{\int_{P'} \tilde{g} \, d\sigma}{\sigma(P')} - \frac{\int_P \tilde{g} \, d\sigma}{\sigma(P)} = \frac{\tilde{\nu}(P')}{\sigma(P')} - \frac{\tilde{\nu}(P)}{\sigma(P)} = \frac{\nu(P')}{\sigma(P')} - \frac{\nu(P)}{\sigma(P)} = \Delta_P^\sigma g.$$

Hence, (3.9) follows immediately from (3.13).

Since for $S \in \mathsf{Stop}$ we have $\tilde{\nu}(S) = \nu(S)$, we can use (3.14) to transform (3.12) into

$$\widetilde{\alpha}_{\tilde{\nu},2}(Q)^2 \lesssim_{\varepsilon_0, C} \alpha_{\sigma,2}(B_Q)^2 + \sum_{P \in \mathsf{Tree}} \|\Delta_P^\sigma g\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{S \in \mathsf{Stop}} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \nu(S). \tag{3.15}$$

In order to reach (3.8) and finish the proof of Lemma 3.9, we only need to show how to pass from the estimate on $\widetilde{\alpha}_{\tilde{\nu},2}(Q)$ (3.15) to one on $\widetilde{\alpha}_{\nu,2}(Q)$.

Proof of Lemma 3.9.   Recall that if $\angle(L_Q, L_0) > 1 - \varepsilon_0$, then $\widetilde{\alpha}_{\nu,2}(Q) = 1$, but at the same time $\alpha_{\sigma,2}(B_Q) \gtrsim 1$ by Lemma 3.2, so this case is trivial. Suppose $\angle(L_Q, L_0) \leq 1 - \varepsilon_0$. We define a transport plan between $\psi_Q \tilde{\nu}$ and $\psi_Q \nu$:

$$d\pi(x, y) = \mathbb{1}_{Q \cap G}(x) d\nu(x) d\delta_x(y) + \sum_{S \in \mathsf{Stop}} \frac{\mathbb{1}_S(x) \mathbb{1}_S(y)}{\sigma(S)} d\nu(x) d\sigma(y),$$

and we estimate

$$W_2(\psi_Q \tilde{\nu}, \psi_Q \nu)^2 \leq \int |x - y|^2 \, d\pi(x, y) \lesssim \sum_{S \in \mathsf{Stop}} \ell(S)^2 \nu(S).$$

From the triangle inequality, the bound above, and (3.15), we get that

$$\widetilde{\alpha}_{\nu,2}(Q)^2 \approx \ell(Q)^{-(n+2)} W_2(\psi_Q \nu, a\psi_Q \mathscr{H}^n|_{L_Q})^2$$

$$\lesssim \ell(Q)^{-(n+2)} \left(W_2(\psi_Q \tilde{\nu}, \psi_Q \nu)^2 + W_2(\psi_Q \tilde{\nu}, a\psi_Q \mathscr{H}^n|_{L_Q})^2 \right)$$

$$\lesssim_{\varepsilon_0, C} \alpha_{\sigma,2}(B_Q)^2 + \sum_{P \in \mathsf{Tree}} \|\Delta_P^\sigma g\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{S \in \mathsf{Stop}} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \nu(S).$$

∎

4 Approximating Measures

We will construct a family of measures on Γ that will approximate μ. For every Whitney cube $P \in \mathcal{W}^e$ we define $g_P : \Gamma \to \mathbb{R}$ as

$$g_P(x) = \frac{\mu(P)}{\ell(P)^n} \mathbb{1}_{\Pi_\Gamma(P)}(x).$$

Note that $\int g_P \, d\sigma = \mu(P)$.

Given $e \in \{0, 1\}^n$, $k \in \mathbb{Z}$, we define the following measures supported on Γ:

$$\nu^e = \mu|_\Gamma + \left(\sum_{P \in \mathcal{W}^e} g_P \right) \sigma,$$

$$\nu_k^e = \mu|_\Gamma + \left(\sum_{P \in \mathcal{W}_k^e} g_P \right) \sigma.$$

Moreover, for every $Q \in \mathcal{D}_\Gamma$ with $\ell(Q) \leq 2^{-k_0}$ we set

$$\nu_Q = \nu_{k(Q)}^{e(Q)} = \mu|_\Gamma + \left(\sum_{P \in \mathcal{W}_Q} g_P \right) \sigma.$$

Note that, since we assume μ is finite and compactly supported (see Remark 2.1), all the measures ν^e, ν_k^e, are also finite and compactly supported.

We defined ν_Q in such a way that, for "good" $Q \in \mathcal{D}_\Gamma$, the measures $\mu|_{B_Q}$ and $\nu_Q|_{B_Q}$ are close in the W_2 distance. This will be shown in Section 5. The rest of this section is dedicated to the construction of a tree of "good cubes."

Recall that $R \in \mathcal{D}_\Gamma$ is a Γ-cube fixed in Remark 2.1, and $0 < \varepsilon \ll 1$ is a small constant fixed in Subsection 2.1.

Lemma 4.1.   Let $\lambda > 3$. Then, there exist a big constant $M = M(\varepsilon, \lambda, \Lambda, n, d, \mu) \gg 1$ and a tree of good cubes $\mathsf{Tree} = \mathsf{Tree}(\lambda, \varepsilon, M) \subset \mathcal{D}_\Gamma(R)$ with root R, such that for every $Q \in \mathsf{Tree}$ we have

$$\mu(\lambda \widetilde{B}_Q) \leq M\ell(Q)^n,$$

$$\mu(Q) \geq M^{-1}\ell(Q)^n,$$

the stopping region $\mathsf{Stop} = \mathsf{Stop}(\mathsf{Tree})$ is small:

$$\mu\left(\bigcup_{Q \in \mathsf{Stop}} Q\right) < \varepsilon,$$

and $\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2$ satisfy the packing condition:

$$\sum_{Q \in \mathsf{Tree}} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2 \ell(Q)^n < \infty. \tag{4.1}$$

We split the proof into several small lemmas. First, we define auxiliary families of good cubes in $\mathcal{D}_\Gamma^e$ using a standard stopping time argument.

For each $e \in \{0,1\}^n$ there exists a finite collection of cubes $\{R_i^e\} \subset \mathcal{D}_\Gamma^e$ such that $\ell(R_i^e) = 1$, $R_i^e \cap R \neq \varnothing$. Set $R^e = \bigcup_i R_i^e$. Let $M \gg 1$ be constant to be fixed later on, and set

$$\mathsf{HD}_{\nu,0}^e = \{Q \in \mathcal{D}_\Gamma^e :\; Q \subset R^e,\; \nu^e(\lambda \widetilde{B}_Q) > M\ell(Q)^n\},$$

$$\mathsf{HD}_{\mu,0}^e = \{Q \in \mathcal{D}_\Gamma^e :\; Q \subset R^e,\; \mu(\lambda \widetilde{B}_Q) > M\ell(Q)^n\},$$

$$\mathsf{LD}_0^e = \{Q \in \mathcal{D}_\Gamma^e :\; Q \subset R^e,\; \mu(Q) < M^{-1}\ell(Q)^n\}.$$

HD and LD stand for "high density" and "low density." Let $\mathsf{Stop}^e \subset \mathcal{D}_\Gamma^e$ be the family of maximal with respect to inclusion cubes from $\mathsf{HD}_{\nu,0}^e \cup \mathsf{HD}_{\mu,0}^e \cup \mathsf{LD}_0^e$, and set $\mathsf{HD}_\nu^e = \mathsf{HD}_{\nu,0}^e \cap \mathsf{Stop}^e$, $\mathsf{HD}_\mu^e = \mathsf{HD}_{\mu,0}^e \cap \mathsf{Stop}^e$, $\mathsf{LD}^e = \mathsf{LD}_0^e \cap \mathsf{Stop}^e$. Note that cubes from Stop^e are pairwise disjoint. We define Tree^e as the family of those cubes from $\bigcup_i \mathcal{D}_\Gamma^e(R_i^e)$, which are not contained in any cube from Stop^e. Actually, this might not be a tree, but it is a finite collection of trees with roots R_i^e.

Lemma 4.2.   For $M = M(\varepsilon, \lambda, \Lambda, n, d, \mu)$ big enough, we have for all $e \in \{0,1\}^n$

$$\mu\left(\bigcup_{Q \in \mathsf{Stop}^e} Q\right) < \frac{\varepsilon}{2^n}. \tag{4.2}$$

Proof.   Let $e \in \{0,1\}^n$. It is easy to see that the measure of LD^e is small: for every $Q \in \mathsf{LD}^e$ we have $\mu(Q) \leq M^{-1}\sigma(Q)$, so

$$\mu\left(\bigcup_{Q \in \mathsf{LD}^e} Q\right) \leq M^{-1}\sigma(R^e) \approx M^{-1}. \tag{4.3}$$

To estimate the measure of HD_μ^e, define for some big $N \gg 1$

$$H_N = \{x \in \mathbb{R}^d : \mu(B(x,r)) > Nr^n \text{ for some } r \in (0,1)\}.$$

Since μ is n-rectifiable, the density $\Theta^n(x,\mu)$ exists and is positive and finite μ-a.e. Moreover, recall that $\mu(\mathbb{R}^d)$ is finite. This implies that for $N = N(\mu, \varepsilon, n)$ big enough

$$\mu(H_N) \leq \frac{\varepsilon}{2^{n+2}}.$$

We will show that, if M is chosen big enough, then for all $Q \in \mathsf{HD}_\mu^e$ we have $Q \subset H_N$. Indeed, let $x \in Q \in \mathsf{HD}_\mu^e$. Then $B(x, 2\lambda r(\widetilde{B}_Q)) \supset \lambda \widetilde{B}_Q$, and so

$$\mu(B(x, 2\lambda r(\widetilde{B}_Q))) \geq \mu(\lambda \widetilde{B}_Q) > M\ell(Q)^n > N(6\lambda \Lambda \operatorname{diam}(Q))^n = N(2\lambda r(\widetilde{B}_Q))^n,$$

for M big enough with respect to N, λ, Λ, n. Moreover, note that for $Q \in \mathsf{HD}_\mu^e$ we have

$$\frac{\mu(\mathbb{R}^d)}{M} > \ell(Q)^n \approx_\Lambda r(\widetilde{B}_Q)^n,$$

and so taking M big enough (depending on $\mu(\mathbb{R}^d), \lambda, \Lambda, n$) we can ensure that all $Q \in \mathsf{HD}_\mu^e$ satisfy $2\lambda r(\widetilde{B}_Q) < 1$. Thus, $x \in H_N$, and we conclude that

$$\mu\left(\bigcup_{Q \in \mathsf{HD}_\mu^e} Q\right) \leq \mu(H_N) \leq \frac{\varepsilon}{2^{n+2}}. \tag{4.4}$$

Since ν^e is a finite n-rectifiable measure, we can argue in the same way as above to get

$$\nu^e\left(\bigcup_{Q \in \mathsf{HD}_\nu^e} Q\right) \leq \frac{\varepsilon}{2^{n+2}}.$$

Smallness of $\mu(\bigcup_{Q \in \mathsf{HD}_\nu^e} Q)$ follows from the fact that $\mu|_\Gamma \leq \nu^e$. Putting this together with (4.3) and (4.4) we get

$$\mu\left(\bigcup_{Q \in \mathsf{Stop}^e} Q\right) < \frac{\varepsilon}{2^n}.$$

We take M so big that the above holds for all $e \in \{0,1\}^n$, and the proof is finished. ∎

For each $e \in \{0,1\}^n$, $k = 0,1,2,\ldots$, let g_k^e be the density of ν_k^e with respect to σ. Note that, due to the definition of Tree^e, for any $Q \in \mathsf{Tree}^e$ we have

$$M^{-1}\,\ell(Q)^n \leq \nu_k^e(Q) \leq \nu_k^e(B_Q) \leq M\,\ell(Q)^n.$$

Hence, given a cube $Q \in \mathsf{Tree}^e$ with $\ell(Q) = 2^{-k}$, we can estimate $\widetilde{\alpha}_{\nu_k^e,2}(Q)^2$ using Lemma 3.9 (applied to ν_k^e and $\mathsf{Tree} = \{P \in \mathsf{Tree}^e\ P \subset Q\}$) to get

$$\widetilde{\alpha}_{\nu_k^e,2}(Q)^2 \lesssim_{\varepsilon_0,M} \alpha_{\sigma,2}(B_Q)^2 + \sum_{\substack{P \in \mathsf{Tree}^e \\ P \subset Q}} \|\Delta_P^\sigma g_k^e\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{\substack{S \in \mathsf{Stop}^e \\ S \subset Q}} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \nu_k^e(S). \qquad (4.5)$$

The following lemma states that the right hand side of this estimate can be made independent of k.

Lemma 4.3.   For all $Q \in \mathsf{Tree}^e$ with $\ell(Q) = 2^{-k}$, $k \geq 0$, we have

$$\widetilde{\alpha}_{\nu_k^e,2}(Q)^2 \lesssim_{\varepsilon_0,M} \alpha_{\sigma,2}(B_Q)^2 + \sum_{\substack{P \in \mathsf{Tree}^e \\ P \subset Q}} \|\Delta_P^\sigma g_0^e\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{\substack{S \in \mathsf{Stop}^e \\ S \subset Q}} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \nu^e(S). \qquad (4.6)$$

Moreover,

$$\sum_{P \in \mathsf{Tree}^e} \|\Delta_P^\sigma g_0^e\|_{L^2(\sigma)}^2 \lesssim M \|g_0^e\|_{L^1(\sigma)} = M\nu_0^e(\Gamma) \leq M\mu(\mathbb{R}^d). \qquad (4.7)$$

Proof.   We claim that for $P \in \mathsf{Tree}^e$ with $\ell(P) \leq 2^{-k}$ (in particular, for $P \in \mathsf{Tree}^e$ such that $P \subset Q$) we have

$$\Delta_P^\sigma g_k^e = \Delta_P^\sigma g_0^e. \qquad (4.8)$$

Indeed, for $x \notin P$ both sides of (4.8) are zero. For $x \in P' \subset P$, where $P' \in \mathsf{Tree}^e \cup \mathsf{Stop}^e$ is a child of P, we have

$$\Delta_P^\sigma g_0^e(x) - \Delta_P^\sigma g_k^e(x) = \frac{\nu_0^e(P') - \nu_k^e(P')}{\ell(P')^n} - \frac{\nu_0^e(P) - \nu_k^e(P)}{\ell(P)^n}$$

$$= \ell(P')^{-n} \left(\sum_{S \in \mathcal{W}_0^e \setminus \mathcal{W}_k^e} \frac{\mu(S)}{\ell(S)^n} \sigma(P' \cap \Pi_\Gamma(S)) \right) - \ell(P)^{-n} \left(\sum_{S \in \mathcal{W}_0^e \setminus \mathcal{W}_k^e} \frac{\mu(S)}{\ell(S)^n} \sigma(P \cap \Pi_\Gamma(S)) \right).$$

The Whitney cubes S in the sums above satisfy $\ell(S) > 2^{-k} \geq \ell(P)$, and moreover we have $\Pi_\Gamma(S) \in \mathcal{D}_\Gamma^e$. Hence, we either have $P \cap \Pi_\Gamma(S) = P$ or $P \cap \Pi_\Gamma(S) = \varnothing$. The same is true for

P'. Moreover, we have $P \cap \Pi_\Gamma(S) \neq \varnothing$ if and only if $P' \cap \Pi_\Gamma(S) \neq \varnothing$. It follows that the right hand side above is equal to

$$\sum_{\substack{S \in \mathcal{W}_0^e \setminus \mathcal{W}_k^e \\ P' \cap \Pi_\Gamma(S) \neq \varnothing}} \frac{\mu(S)}{\ell(S)^n} - \sum_{\substack{S \in \mathcal{W}_0^e \setminus \mathcal{W}_k^e \\ P \cap \Pi_\Gamma(S) \neq \varnothing}} \frac{\mu(S)}{\ell(S)^n} = 0.$$

Thus $\Delta_P^\sigma g_k^e = \Delta_P^\sigma g_0^e$. Using this equality, and also the fact that $\nu_k^e \leq \nu^e$, we transform (4.5) into

$$\widetilde{\alpha}_{\nu_k^e,2}(Q)^2 \lesssim_{\varepsilon_0,M} \alpha_{\sigma,2}(B_Q)^2 + \sum_{\substack{P \in \mathsf{Tree}^e \\ P \subset Q}} \|\Delta_P^\sigma g_0^e\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{\substack{P \in \mathsf{Stop}^e \\ P \subset Q}} \frac{\ell(P)^2}{\ell(Q)^{n+2}} \nu^e(P). \qquad (4.9)$$

Concerning (4.7), it is an immediate consequence of (3.9) when we apply Lemma 3.9 to ν_0^e and the trees $\{Q \in \mathsf{Tree}^e \ Q \subset R_i^e\}$ (recall that the union of such trees gives the entire Tree^e). ∎

We finally define Tree as the collection of cubes $Q \in \mathcal{D}_\Gamma$ such that for every $e \in \{0,1\}^n$ there exists $P \in \mathsf{Tree}^e$ satisfying $\ell(P) = \ell(Q)$ and $P \cap Q \neq \varnothing$. It is easy to check that Tree is indeed a tree, and that the stopping cubes $\mathsf{Stop} = \mathsf{Stop}(\mathsf{Tree})$ satisfy $\bigcup_{Q \in \mathsf{Stop}} Q \subset \bigcup_e \bigcup_{Q \in \mathsf{Stop}^e} Q$. Thus,

$$\mu\left(\bigcup_{Q \in \mathsf{Stop}} Q\right) \leq \sum_{e \in \{0,1\}^n} \mu\left(\bigcup_{Q \in \mathsf{Stop}^e} Q\right) \overset{(4.2)}{\leq} \varepsilon.$$

Moreover, $\mathsf{Tree} \subset \mathsf{Tree}_{(0,\dots,0)}$, so for all $Q \in \mathsf{Tree}$

$$\mu(\lambda \widetilde{B}_Q) \leq M\ell(Q)^n,$$

$$\mu(Q) \geq M^{-1}\ell(Q)^n.$$

The only thing that remains to be shown is the packing condition (4.1).

Lemma 4.4.   We have

$$\sum_{Q \in \mathsf{Tree}} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2 \ell(Q)^n < \infty.$$

Proof.   Recall that in Lemma 2.2 we defined a constant $k_0 > 0$ such that for any $Q \in \mathcal{D}_\Gamma$, $\ell(Q) \leq 2^{-k_0}$, there exists a cube $P_Q \in \widetilde{\mathcal{D}}_\Gamma$ satisfying $3\widetilde{B}_Q \subset V(P_Q)$, $\ell(P_Q) = 2^{k_0}\ell(Q)$.

Since there are only finitely many $Q \in \mathsf{Tree}$ with $\ell(Q) > 2^{-k_0}$, we may ignore them in the estimates that follow.

Suppose $Q \in \mathsf{Tree}$ and $\ell(Q) \leq 2^{-k_0}$, let P_Q be as above. Recall that $\nu_Q = \nu_{k(Q)}^{e(Q)}$, where $e = e(Q)$, $k = k(Q)$ are such that $P_Q \in \mathcal{D}_\Gamma^e$ and $\ell(P_Q) = 2^{-k}$.

We defined Tree in such a way that necessarily $P_Q \in \mathsf{Tree}^e$. It follows from Lemma 3.5 applied with $\nu = \nu_Q$, $B = \widetilde{B}_Q$, $Q = P_Q$, that

$$\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q) \lesssim_{\varepsilon_0,M,k_0} \widetilde{\alpha}_{\nu_Q,2}(P_Q) + \alpha_{\sigma,2}(B_{P_Q}).$$

We use (4.6) and the inequality above to obtain

$$\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2 \lesssim_{\varepsilon_0,M,k_0} \alpha_{\sigma,2}(B_{P_Q})^2 + \sum_{\substack{P\in\mathsf{Tree}^e\\P\subset P_Q}} \|\Delta_P^\sigma g_0^e\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(P_Q)^{n+1}} + \sum_{\substack{S\in\mathsf{Stop}^e\\S\subset P_Q}} \frac{\ell(S)^2}{\ell(P_Q)^{n+2}} \nu^e(S).$$

Taking into account that each $P_Q \in \mathsf{Tree}^e$ may correspond to only a bounded number of $Q \in \mathsf{Tree}$, and that $\ell(Q) \approx_{k_0} \ell(P_Q)$, we get

$$\sum_{Q\in\mathsf{Tree}:P_Q\in\mathsf{Tree}^e} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2 \ell(Q)^n \lesssim_{\varepsilon_0,M,k_0} \sum_{Q'\in\mathsf{Tree}^e} \alpha_{\sigma,2}(B_{Q'})^2 \ell(Q')^n$$

$$+ \sum_{Q'\in\mathsf{Tree}^e} \sum_{\substack{P\in\mathsf{Tree}^e\\P\subset Q'}} \|\Delta_P^\sigma g_0^e\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q')} + \sum_{Q'\in\mathsf{Tree}^e} \sum_{\substack{S\in\mathsf{Stop}^e\\S\subset Q'}} \frac{\ell(S)^2}{\ell(Q')^2} \nu^e(S).$$

The first sum from the right hand side is finite because σ is uniformly rectifiable, see Theorem 1.1. We estimate the second sum by changing the order of summation:

$$\sum_{Q'\in\mathsf{Tree}^e} \sum_{\substack{P\in\mathsf{Tree}^e\\P\subset Q'}} \|\Delta_P^\sigma g_0^e\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q')} = \sum_{P\in\mathsf{Tree}^e} \|\Delta_P^\sigma g_0^e\|_{L^2(\sigma)}^2 \sum_{\substack{Q'\in\mathsf{Tree}^e\\Q'\supset P}} \frac{\ell(P)}{\ell(Q')}$$

$$\lesssim \sum_{P\in\mathsf{Tree}^e} \|\Delta_P^\sigma g_0^e\|_{L^2(\sigma)}^2 \overset{(4.7)}{\lesssim} M\mu(\mathbb{R}^d) < \infty.$$

The third sum is treated similarly:

$$\sum_{Q'\in\mathsf{Tree}^e} \sum_{\substack{S\in\mathsf{Stop}^e\\S\subset Q'}} \frac{\ell(S)^2}{\ell(Q')^2} \nu^e(S) = \sum_{S\in\mathsf{Stop}^e} \nu^e(S) \sum_{\substack{Q'\in\mathsf{Tree}^e\\Q'\supset S}} \frac{\ell(S)^2}{\ell(Q')^2} \lesssim \sum_{S\in\mathsf{Stop}^e} \nu^e(S) < \infty.$$

Thus,

$$\sum_{Q \in \text{Tree}} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2 \ell(Q)^n = \sum_{e \in \{0,1\}^n} \sum_{Q \in \text{Tree}:P_Q \in \text{Tree}^e} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2 \ell(Q)^n < \infty.$$

∎

5   From Approximating Measures to μ

To prove Lemma 1.10 we need to pass from the estimates on $\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)$ shown in Lemma 4.1 to estimates on $\widehat{\alpha}_{\mu,2}(B_Q)$.

Recall that $K > 20$ is the constant such that for all Whitney cubes $Q \in \mathcal{W}^e$ we have $KQ \cap \Gamma \neq \varnothing$, and $k_0 = k_0(n, \Lambda)$ is an integer from Lemma 2.2.

Lemma 5.1.   There exists $\lambda = \lambda(k_0, K, n, d) > 3$ such that if $M = M(\varepsilon, \lambda, \Lambda, n, d, \mu)$ and $\text{Tree} = \text{Tree}(\lambda, M, \varepsilon)$ are as in Lemma 4.1, then for all $Q \in \text{Tree}$ with $\ell(Q) \leq 2^{-k_0}$

$$\widehat{\alpha}_{\mu,2}(B_Q)^2 \lesssim_{M,\lambda,\Lambda} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2 + \alpha_{\sigma,2}(\widetilde{B}_Q)^2 + \frac{1}{\ell(Q)^{n+2}} \sum_{\substack{P \in \mathcal{W}_Q \\ P \subset \lambda \widetilde{B}_Q}} \mu(P)\ell(P)^2.$$

Proof.   Let $Q \in \text{Tree}$ with $\ell(Q) \leq 2^{-k_0}$. We will define an auxiliary measure μ_Q. Set

$$I_Q = \{P \in \mathcal{W}_Q : \Pi_\Gamma(P) \cap 3\widetilde{B}_Q \neq \varnothing\}.$$

It is easy to check that

$$\bigcup_{P \in I_Q} P \subset \lambda \widetilde{B}_Q, \tag{5.1}$$

for $\lambda = \lambda(k_0, K, n, d)$ big enough (e.g., $\lambda = C(n, d)K2^{k_0}$ works). It is crucial that all cubes in I_Q have sidelength bounded by $2^{k_0}\ell(Q)$, otherwise no such λ would exist.

Recall that the functions $g_P(x) = \frac{\mu(P)}{\ell(P)^n} \mathbb{1}_{\Pi_\Gamma(P)}(x)$, $P \in \mathcal{W}_Q$, were used to define ν_Q at the beginning of Section 4. Let

$$a_P = \frac{\int \varphi_{\widetilde{B}_Q} g_P \, d\sigma}{\mu(P)}.$$

Note that for $P \in \mathcal{W}_Q \setminus I_Q$ we have $a_P = 0$. The measure μ_Q is defined as

$$\mu_Q = \varphi_{\widetilde{B}_Q}\mu|_\Gamma + \sum_{P \in I_Q} a_P \mu|_P.$$

First, let us show that if Λ (the constant from the definition of $\widetilde{B}_Q = \Lambda B_Q$) is big enough, then $\mu|_{3B_Q} = \mu_Q\big|_{3B_Q}$. We need to check the following: if $P \in \mathcal{W}^{e(Q)}$ is such that $P \cap 3B_Q \neq \varnothing$, then $P \in I_Q$ and $a_P = 1$.

Note that for all such P we have

$$\ell(P) \leq \text{diam}(P) \overset{(2.4)}{\leq} r(3B_Q) = 9\text{diam}(Q) \overset{(2.3)}{\leq} 2^{-k(Q)},$$

and so $P \in \mathcal{W}_Q$. Furthermore, the fact that $P \cap 3B_Q \neq \varnothing$ and (2.4) imply that $P \subset 9B_Q$. Since Π_Γ is $\sqrt{2}$-Lipschitz continuous, and B_Q is centered at Γ, we get that for Λ big enough (e.g., $\Lambda = 9\sqrt{2}$)

$$\Pi_\Gamma(P) \subset \Lambda B_Q = \widetilde{B}_Q. \tag{5.2}$$

We conclude that $P \in I_Q$ and $a_P = 1$, and so,

$$\mu|_{3B_Q} = \mu_Q\big|_{3B_Q}. \tag{5.3}$$

Set $L = L_{\widetilde{B}_Q}$. We will apply Lemma 3.3 with $\nu = \mu_Q$, $B_1 = B_Q$, $B_2 = \lambda\widetilde{B}_Q$, $L = L$, and $f = \varphi_{\widetilde{B}_Q}$. Notice that $\text{supp}\,\mu_Q \subset \lambda\widetilde{B}_Q$ by (5.1). Moreover, using the same trick as in the beginning of the proof of Lemma 3.4, we may assume that $L \cap B_Q \neq \varnothing$. Since $\mu_Q(B_Q) \approx_M \mu_Q(\lambda\widetilde{B}_Q) \approx_M \ell(Q)^n$ by Lemma 4.1, and $r(\lambda\widetilde{B}_Q) = \lambda\Lambda r(B_Q)$, the assumptions of Lemma 3.3 are met, and we get that

$$W_2(\varphi_Q\mu_Q, a\varphi_Q\mathscr{H}^n\big|_L) \lesssim_{M,\lambda,\Lambda} W_2(\mu_Q, a\varphi_{\widetilde{B}_Q}\mathscr{H}^n\big|_L). \tag{5.4}$$

Applying the triangle inequality yields

$$W_2(\mu_Q, a\varphi_{\widetilde{B}_Q}\mathscr{H}^n\big|_L)^2 \lesssim W_2(\mu_Q, \varphi_{\widetilde{B}_Q}\nu_Q)^2 + W_2(\varphi_{\widetilde{B}_Q}\nu_Q, a\varphi_{\widetilde{B}_Q}\mathscr{H}^n\big|_L)^2$$
$$\approx_M W_2(\mu_Q, \varphi_{\widetilde{B}_Q}\nu_Q)^2 + \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2\ell(Q)^{n+2}. \tag{5.5}$$

To estimate $W_2(\mu_Q, \varphi_{\widetilde{B}_Q}\nu_Q)$ we define the following transport plan:

$$d\pi(x,y) = \varphi_{\widetilde{B}_Q}(x)d\mu|_\Gamma(x)d\delta_x(y) + \sum_{P \in I_Q}\frac{1}{\mu_Q(P)}d\mu_Q\big|_P(x)\varphi_{\widetilde{B}_Q}(y)g_P(y)d\sigma(y).$$

Then,

$$W_2(\mu_Q, \varphi_{\widetilde{B}_Q} \nu_Q)^2 \leq \int |x-y|^2 \, \mathrm{d}\pi(x,y) \lesssim \sum_{P \in I_Q} \ell(P)^2 \int \varphi_{\widetilde{B}_Q}(y) g_P(y) \mathrm{d}\sigma(y).$$

$$\leq \sum_{P \in I_Q} \mu(P)\ell(P)^2 \overset{(5.1)}{\leq} \sum_{\substack{P \in \mathcal{W}_Q \\ P \subset \lambda \widetilde{B}_Q}} \mu(P)\ell(P)^2.$$

Putting together (5.3), (5.4), (5.5), and the estimate above, we get

$$W_2(\varphi_Q \mu, a\varphi_Q \mathscr{H}^n|_L) \lesssim_{M,\lambda,\Lambda} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2 \ell(Q)^{n+2} + \sum_{\substack{P \in \mathcal{W}_Q \\ P \subset \lambda \widetilde{B}_Q}} \mu(P)\ell(P)^2.$$

Finally, we use the triangle inequality, the estimate $\mu(3B_Q) \approx_M \sigma(B_Q) \approx r(B_Q)^n$, and the fact that L_Q minimizes $\alpha_{\sigma,2}(B_Q)$, to get

$$\widehat{\alpha}_{\mu,2}(B_Q)^2 \ell(Q)^{n+2} \approx_M W_2(\varphi_Q \mu, a\varphi_Q \mathscr{H}^n|_{L_Q}) \leq W_2(\varphi_Q \mu, a\varphi_Q \mathscr{H}^n|_L)$$

$$+ \left(\frac{\int \varphi_Q \, \mathrm{d}\mu}{\int \varphi_Q \, \mathrm{d}\sigma} \right)^{1/2} \left(W_2(\varphi_Q \sigma, a\varphi_Q \mathscr{H}^n|_{L_Q}) + W_2(\varphi_Q \sigma, a\varphi_Q \mathscr{H}^n|_L) \right)$$

$$\lesssim_M W_2(\varphi_Q \mu, a\varphi_Q \mathscr{H}^n|_L) + W_2(\varphi_Q \sigma, a\varphi_Q \mathscr{H}^n|_L)$$

$$\lesssim W_2(\varphi_Q \mu, a\varphi_Q \mathscr{H}^n|_L) + \alpha_{\sigma,2}(\widetilde{B}_Q)^2 \ell(Q)^{n+2},$$

and so the proof is complete. ∎

We are ready to finish the proof of Lemma 1.10.

Proof of Lemma 1.10.   Recall that R is a Γ-cube with $\ell(R) = 1$, and $\varepsilon > 0$ is an arbitrary small constant, and that they were both fixed in Subsection 2.1. Let λ, M, Tree, and Stop be as in Lemma 5.1 and Lemma 4.1. Set

$$R' = R \setminus \bigcup_{P \in \text{Stop}} P.$$

By Lemma 4.1, we have $\mu(R') \geq (1-\varepsilon)\mu(R)$. Our aim is to show that

$$\int_{R'} \int_0^1 \alpha_{\mu,2}(x,r)^2 \, \frac{\mathrm{d}r}{r} \, \mathrm{d}\mu(x) < \infty.$$

For any $x \in R'$ we have arbitrarily small cubes from Tree containing x. Hence, for any $k \geq k_0 + 3$, $r \in (2^{-k}, 2^{-k+1}]$, we have $3B(x, r) \subset B_Q$ for the cube $Q \in$ Tree containing x and satisfying $\ell(Q) = 2^{-k+3}$. Thus, by Lemma 3.4,

$$\widehat{\alpha}_{\mu,2}(B(x,r))^2 \lesssim_M \widehat{\alpha}_{\mu,2}(B_Q)^2 + \alpha_{\sigma,2}(B_Q)^2.$$

Integrating both sides with respect to r yields

$$\int_{2^{-k}}^{2^{-k+1}} \widehat{\alpha}_{\mu,2}(B(x,r))^2 \frac{dr}{r} \lesssim_M \int_{2^{-k}}^{2^{-k+1}} (\widehat{\alpha}_{\mu,2}(B_Q)^2 + \alpha_{\sigma,2}(B_Q)^2) \frac{dr}{r} \approx \widehat{\alpha}_{\mu,2}(B_Q)^2 + \alpha_{\sigma,2}(B_Q)^2.$$

The inequality above holds for all $x \in Q \cap R'$, so

$$\int_{Q \cap R'} \int_{2^{-k}}^{2^{-k+1}} \widehat{\alpha}_{\mu,2}(B(x,r))^2 \frac{dr}{r} \, d\mu(x) \lesssim_M (\widehat{\alpha}_{\mu,2}(B_Q)^2 + \alpha_{\sigma,2}(B_Q)^2)\mu(Q)$$

$$\approx_M (\widehat{\alpha}_{\mu,2}(B_Q)^2 + \alpha_{\sigma,2}(B_Q)^2)\ell(Q)^n.$$

Summing over all $Q \in$ Tree with $\ell(Q) = 2^{-k+3}$, and then over all $k \geq k_0 + 3$, we get

$$\int_{R'} \int_0^{2^{-k_0-2}} \widehat{\alpha}_{\mu,2}(B(x,r))^2 \frac{dr}{r} \, d\mu(x) \lesssim_M \sum_{\substack{Q \in \text{Tree} \\ \ell(Q) \leq 2^{-k_0}}} \widehat{\alpha}_{\mu,2}(B_Q)^2 \ell(Q)^n + \sum_{\substack{Q \in \text{Tree} \\ \ell(Q) \leq 2^{-k_0}}} \alpha_{\sigma,2}(B_Q)^2 \ell(Q)^n.$$

(5.6)

On the other hand, for any $r > 0$ we have

$$\widehat{\alpha}_{\mu,2}(B(x,r))^2 \lesssim \frac{\mu(\mathbb{R}^d)}{r^n},$$

so

$$\int_{R'} \int_{2^{-k_0-2}}^1 \widehat{\alpha}_{\mu,2}(B(x,r))^2 \frac{dr}{r} \, d\mu(x) < \infty.$$

Thus, in order to prove Lemma 1.10, it suffices to show that the sums on the right hand side of (5.6) are finite.

The finiteness of

$$\sum_{Q \in \mathcal{D}_\Gamma, \, Q \subset R} \alpha_{\sigma,2}(B_Q)^2 \ell(Q)^n$$

follows by Theorem 1.1. To estimate the other sum we apply Lemma 5.1:

$$\sum_{\substack{Q\in\mathsf{Tree}\\ \ell(Q)\leq 2^{-k_0}}} \widehat{\alpha}_{\mu,2}(B_Q)^2\ell(Q)^n \lesssim \sum_{\substack{Q\in\mathsf{Tree}\\ \ell(Q)\leq 2^{-k_0}}} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2\ell(Q)^n + \sum_{\substack{Q\in\mathsf{Tree}\\ \ell(Q)\leq 2^{-k_0}}} \alpha_{\sigma,2}(\widetilde{B}_Q)^2\ell(Q)^n$$

$$+ \sum_{\substack{Q\in\mathsf{Tree}\\ \ell(Q)\leq 2^{-k_0}}} \sum_{\substack{P\in\mathcal{W}_Q\\ P\subset\lambda\tilde{B}_Q}} \mu(P)\frac{\ell(P)^2}{\ell(Q)^2}.$$

The first sum is finite by Lemma 4.1, the second by Theorem 1.1. Concerning the last sum, we may estimate it in the following way:

$$\sum_{\substack{Q\in\mathsf{Tree}\\ \ell(Q)\leq 2^{-k_0}}} \sum_{\substack{P\in\mathcal{W}_Q\\ P\subset\lambda\tilde{B}_Q}} \mu(P)\frac{\ell(P)^2}{\ell(Q)^2} \lesssim \sum_{e\in\{0,1\}^n} \sum_{\substack{P\in\mathcal{W}^e\\ P\subset\lambda\tilde{B}_R}} \mu(P) \sum_{\substack{Q\in\mathsf{Tree}\\ \lambda\tilde{B}_Q\supset P}} \frac{\ell(P)^2}{\ell(Q)^2}$$

$$\lesssim \sum_{e\in\{0,1\}^n} \sum_{\substack{P\in\mathcal{W}^e\\ P\subset\lambda\tilde{B}_R}} \mu(P) \leq \sum_{e\in\{0,1\}^n} \mu(\lambda\widetilde{B}_R) = 2^n\mu(\lambda\widetilde{B}_R) < \infty.$$

Thus,

$$\sum_{Q\in\mathsf{Tree}} \widehat{\alpha}_{\mu,2}(B_Q)^2\ell(Q)^n < \infty.$$

∎

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