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A reformulation of the generalized q-Painlevé VI system with $W(A^{(1)}_{2n+1})$ symmetry

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In the previous work, we introduced the higher order q-Painlevé system q-$P_{(n+1,n+1)}$ as a generalization of the Jimbo–Sakai's q-Painlevé VI equation. It is derived from a q-analogue of the Drinfeld–Sokolov hierarchy of type $A^{(1)}_{2n+1}$ and admits a particular solution in terms of the Heine's q-hypergeometric function ${}_{n+1}\phi_n$. However, the obtained system is insufficient as a generalization of q-$P_{\rm VI}$ due to some reasons. In this article, we rewrite the system q-$P_{(n+1,n+1)}$ to a more suitable one.

Keywords: discrete Painlevé equations; affine Weyl groups; basic hypergeometric functions.

1. Introduction

Several generalizations of the Painlevé VI equation ($P_{\rm VI}$) have been proposed ([1, 3, 5, 11, 12, 14, 15, 19]). We focus on the higher order Painlevé system $P_{(n+1,n+1)}$ given in [1, 14], or equivalently the Schlesinger system $\mathcal{H}_{n+1,1}$ given in [19], among them. It can be regarded as a generalization from a viewpoint of a particular solution in terms of the hypergeometric function ${}_{n+1}F_n$ ([13, 20]). The aim of this article is to introduce its q-analogue. This q-difference equation becomes a generalization of the q-Painlevé VI equation (q-$P_{\rm VI}$) given in [4].

The investigation of generalizations of q-$P_{\rm VI}$ has been developed in recent years ([8–10, 16, 18]). In the previous work [16], we proposed the higher order q-Painlevé system q-$P_{(n+1,n+1)}$, whose explicit formula will be given in Section 2, as a q-analogue of $P_{(n+1,n+1)}$. It is derived from the q-Drinfeld–Sokolov hierarchy of type $A^{(1)}_{2n+1}$, contains q-$P_{\rm VI}$ in the case of $n = 1$ and admits a particular solution in terms of the q-hypergeometric function ${}_{n+1}\phi_n$. However, this system is insufficient as a generalization of q-$P_{\rm VI}$ due to the following two reasons.

(1) The system q-$P_{(n+1,n+1)}$ is probably reducible and reduces to a system of $2n$-th order.

(2) We do not express the backward q-shifts $(\underline{x_i}, \underline{y_i})$ as functions in (x_i, y_i).

The aim of this article is to solve those two problems. We reduce the system q-$P_{(n+1,n+1)}$ to a more suitable one as a generalization of q-$P_{\rm VI}$.

This article is organized as follows. In Section 2, we recall the definition of q-$P_{(n+1,n+1)}$ and its properties, namely a Lax pair, an affine Weyl group symmetry, a relationship with q-$P_{\rm VI}$ and a particular solution in terms of ${}_{n+1}\phi_n$. In Section 3, we formulate a system of q-difference equations of $2n$-th order which is equivalent to q-P_{VI} in the case of $n = 1$. It is the main result of this article. In Section 4, we describe an action of the affine Weyl group on the $2n$-th order system given in the previous section.

2. Review: higher order q-Painlevé system q-$P_{(n+1,n+1)}$

In the following, we use notations

$$\overline{x(t)} = x(qt), \quad \underline{x(t)} = x(q^{-1}t),$$

where $t, q \in \mathbb{C}$ and $|q| < 1$.

The system q-$P_{(n+1,n+1)}$ given in [16] is expressed as a system of q-difference equations

$$\begin{cases} x_{i-1} - x_i = \dfrac{b_{i-1}\underline{x_{i-1}}}{1 + \underline{x_{i-1}}y_{i-1}} - \dfrac{a_i \underline{x_i}}{1 + \underline{x_i}y_{i-1}} \\ \underline{y_{i-1}} - \underline{y_i} = \dfrac{a_i y_{i-1}}{1 + \underline{x_i}y_{i-1}} - \dfrac{b_i y_i}{1 + \underline{x_i}y_i} \end{cases} \quad (i = 1, \ldots, n+1),$$

with a constraint

$$\prod_{i=1}^{n+1} a_i \frac{1 + \underline{x_i}y_i}{1 + \underline{x_i}y_{i-1}} = q^{-n/2}, \tag{2.1}$$

where

$$b_0 = qb_{n+1}, \quad x_0 = tx_{n+1}, \quad y_0 = \frac{q}{t}y_{n+1}.$$

REMARK 2.1 In the previous work [18], a higher order generalizations of q-P_{VI} were presented by Tsuda. Since his q-Painlevé system can be regarded as a q-analogue of the Schlesinger system $\mathcal{H}_{n+1,1}$, we conjecture that his system coinsides with q-$P_{(n+1,n+1)}$. However, a relationship between both q-Painlevé systems has not been clarified yet.

We derived the system q-$P_{(n+1,n+1)}$ by a similarity reduction from the q-Drinfeld–Sokolov hierarchy of type $A^{(1)}_{2n+1}$. Hence the following theorem is obtained naturally via the construction of the system.

THEOREM 2.2 ([16]) The system q-$P_{(n+1,n+1)}$ is given as the compatibility condition of a system of linear q-difference equations

$$\Psi(q^{-1}z, t) = M(z, t)\Psi(z, t), \quad \Psi(z, q^{-1}t) = B(z, t)\Psi(z, t). \tag{2.2}$$

with $(2n+2)\times(2n+2)$ matrices

$$M(z,t)=\begin{bmatrix} a_1 & \varphi_1 & -1 & & & & & & \\ & b_1 & \varphi_2 & -1 & & & & & \\ & & a_2 & \varphi_3 & -1 & & & & \\ & & & b_2 & \varphi_4 & & & & \\ & & & & & \ddots & & & \\ & & & & & & \varphi_{2n-1} & -1 & \\ & & & & & & b_n & \varphi_{2n} & -1 \\ -tz & & & & & & & a_{n+1} & \varphi_{2n+1} \\ \varphi_0 z & -z & & & & & & & b_{n+1} \end{bmatrix},$$

and

$$B(z,t)=\begin{bmatrix} u_1 & v_1 & -1 & & & & & & \\ & u_2 & v_2 & 0 & & & & & \\ & & u_3 & v_3 & -1 & & & & \\ & & & u_4 & v_4 & & & & \\ & & & & & \ddots & & & \\ & & & & & & v_{2n-1} & -1 & \\ & & & & & & u_{2n} & v_{2n} & 0 \\ -tz & & & & & & & u_{2n+1} & v_{2n+1} \\ v_0 z & 0 & & & & & & & u_{2n+2} \end{bmatrix},$$

where

$$\begin{aligned}
&\varphi_{2i-2}=x_{i-1}-x_i, \quad \varphi_{2i-1}=y_{i-1}-y_i \quad (i=1,\dots,n+1),\\
&u_{2i-1}=\frac{a_i}{1+\underline{x_i}y_{i-1}}, \quad u_{2i}=1+\underline{x_i}y_i, \quad v_{2i-1}=-y_i \quad (i=1,\dots,n+1),\\
&v_0=t\underline{x_{n+1}}, \quad v_{2i}=\underline{x_i} \quad (i=1,\dots,n).
\end{aligned}$$

The system q-$P_{(n+1,n+1)}$ admits the affine Weyl group symmetry of type $A^{(1)}_{2n+1}$.

THEOREM 2.3 Let $r_0,\dots,r_{2n+1}$ be birational transformations defined by

$$\begin{aligned}
&r_{2j-2}(a_j)=b_{j-1}, \quad r_{2j-2}(b_{j-1})=a_j, \quad r_{2j-2}(x_{j-1})=x_{j-1}, \quad r_{2j-2}(y_{j-1})=y_{j-1}-\frac{b_{j-1}-a_j}{x_{j-1}-x_j},\\
&r_{2j-2}(a_i)=a_i, \quad r_{2j-2}(b_{i-1})=b_{i-1}, \quad r_{2j-2}(x_{i-1})=x_{i-1}, \quad r_{2j-2}(y_{i-1})=y_{i-1} \quad (i\neq j)
\end{aligned} \tag{2.3}$$

for $j=1,\dots,n+1$, and

$$\begin{aligned}
&r_{2j-1}(a_j)=b_j \quad r_{2j-1}(b_j)=a_j, \quad r_{2j-1}(x_j)=x_j-\frac{a_j-b_j}{y_{j-1}-y_j}, \quad r_{2j-1}(y_j)=y_j,\\
&r_{2j-1}(a_i)=a_i, \quad r_{2j-1}(b_i)=b_i, \quad r_{2j-1}(x_i)=x_i, \quad r_{2j-1}(y_i)=y_i \quad (i\neq j)
\end{aligned} \tag{2.4}$$

for $j=1,\dots,n+1$. Also let π be a birational transformation defined by

$$\pi(a_i)=q^{-\rho_1}b_i,\quad \pi(b_i)=q^{-\rho_1}a_{i+1}\quad (i=1,\dots,n),$$
$$\pi(a_{n+1})=q^{-\rho_1}b_{n+1},\quad \pi(b_{n+1})=q^{-\rho_1-1}a_1,\quad \pi(\rho_1)=-\rho_1-\frac{1}{n+1},$$
$$\pi(x_i)=q^{-2\rho_1}t^{\rho_1}\underline{y_i},\quad \pi(y_i)=q^{\rho_1}t^{-\rho_1}\underline{x_{i+1}}\quad (i=1,\dots,n),$$
$$\pi(x_{n+1})=q^{-2\rho_1}t^{\rho_1}\underline{y_{n+1}},\quad \pi(y_{n+1})=q^{\rho_1+1}t^{-\rho_1-1}\underline{x_1},\quad \pi(t)=\frac{q^2}{t}, \tag{2.5}$$

where

$$q^{\rho_1}=(q^n a_1 b_1\dots a_{n+1}b_{n+1})^{1/(n+1)}.$$

Then the system q-$P_{(n+1,n+1)}$ is invariant under actions of the transformations $r_0,\dots,r_{2n+1}$ and π. Furthermore, the group of symmetries $\langle r_0,\dots,r_{2n+1},\pi\rangle$ is isomorphic to the extended affine Weyl group of type $A^{(1)}_{2n+1}$. Namely those transformations satisfy the fundamental relations

$$r_i^2=1,\quad (r_ir_j)^{2-a_{i,j}}=1\quad (i,j=0,\dots,2n+1;i\neq j),$$
$$\pi^{2n+2}=1,\quad \pi r_i=r_{i+1}\pi,\quad \pi r_{2n+1}=r_0\pi\quad (i=0,\dots,2n),$$

where

$$\begin{aligned}&a_{i,i}=2 && (i=0,\dots,2n+1),\\ &a_{i,i+1}=a_{2n+1,0}=a_{i+1,i}=a_{0,2n+1}=-1 && (i=0,\dots,2n),\\ &a_{i,j}=0 && (\text{otherwise}).\end{aligned}$$

The system q-$P_{(2,2)}$ can be reduced to q-P_{VI}.

THEOREM 2.4 ([16]) If, in the system q-$P_{(2,2)}$, we set

$$f=\frac{t(x_2-x_1)\xi_1}{\xi_2},\quad g=\frac{\underline{x_2}(qt+\underline{x_1}y_2)\psi_1}{(1+\underline{x_2}y_2)\psi_2}, \tag{2.6}$$

where

$$\xi_1=(x_1-x_2)(y_0-y_1)-(a_1-b_1),$$
$$\xi_2=(tx_2-x_1)(x_1-x_2)(y_0-y_1)+(a_1-b_1)x_1+\{(b_1-a_2)t-(a_1-a_2)\}x_2,$$
$$\psi_1=q^{1/2}(q^{1/2}-a_1b_1t)\underline{x_2}y_2+(1-q^{1/2}a_1b_1)t,$$
$$\psi_2=q^{1/2}a_2(q^{1/2}-a_1b_1t)\underline{x_1}\underline{x_2}y_2+a_1(1-q^{1/2}b_1a_2)t\underline{x_1}-(a_1-a_2)t\underline{x_2},$$

then they satisfy the q-Painlevé VI equation

$$\frac{f\overline{f}}{\alpha_3\alpha_4}=\frac{(\overline{g}-t\beta_1)(\overline{g}-t\beta_2)}{(\overline{g}-\beta_3)(\overline{g}-\beta_4)},\quad \frac{g\overline{g}}{\beta_3\beta_4}=\frac{(f-t\alpha_1)(f-t\alpha_2)}{(f-\alpha_3)(f-\alpha_4)},$$

with parameters

$$\alpha_1 = 1, \quad \alpha_2 = q^{1/2} a_1 b_1, \quad \alpha_3 = 1, \quad \alpha_4 = \frac{1}{q^{1/2} a_2 b_2},$$
$$\beta_1 = q^{1/2} b_1, \quad \beta_2 = q^{1/2} a_1, \quad \beta_3 = \frac{1}{q a_2}, \quad \beta_4 = \frac{1}{b_2}.$$

REMARK 2.5 In [16] we defined the transformation π by

$$\pi(a_i) = b_i, \quad \pi(b_i) = a_{i+1} \quad (i = 1, \dots, n), \quad \pi(a_{n+1}) = b_{n+1}, \quad \pi(b_{n+1}) = \frac{a_1}{q},$$
$$\pi(x_i) = \underline{y_i}, \quad \pi(y_i) = \underline{x_{i+1}} \quad (i = 1, \dots, n), \quad \pi(x_{n+1}) = \underline{y_{n+1}}, \quad \pi(y_{n+1}) = \frac{q}{t}\underline{x_1}, \quad \pi(t) = \frac{q^2}{t}.$$

As a matter of fact, unless we replace constraint (2.1) with

$$\prod_{i=1}^{n+1} \frac{a_i^{1/2}}{b_i^{1/2}} \frac{1 + \underline{x_i} y_i}{1 + \underline{x_i} y_{i-1}} = q^{1/4},$$

the system q-$P_{(n+1,n+1)}$ is not invariant under an action of π. If we do so, then the system q-$P_{(2,2)}$ seems to reduce not to q-P_{VI} but to a q-analogue of the Painlevé V equation. This q-difference equation was derived in [7] from a binational representation of the extended affine Weyl group of type $A_1^{(1)} \times A_3^{(1)}$ given in [6]. Afterward that equation was found to be a subsystem of q-P_{VI} in [17].

The system q-$P_{(n+1,n+1)}$ admits a particular solution in terms of the q-hypergeometric function ${}_{n+1}\phi_n$ defined by the formal power series

$${}_{n+1}\phi_n \left[\begin{matrix} \alpha_1, \dots, \alpha_n, \alpha_{n+1} \\ \beta_1, \dots, \beta_n \end{matrix} ; q, t \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1; q)_k \dots (\alpha_n; q)_k (\alpha_{n+1}; q)_k}{(\beta_1; q)_k \dots (\beta_n; q)_k (q; q)_k} t^k,$$

where $(\alpha; q)_k$ stands for the q-shifted factorial

$$(\alpha; q)_0 = 1, \quad (\alpha; q)_k = (1 - \alpha)(1 - q\alpha) \dots (1 - q^{k-1}\alpha) \quad (k \geq 1).$$

THEOREM 2.6 ([16]) If, in the system q-$P_{(n+1,n+1)}$, we assume that

$$y_i = 0 \quad (i = 1, \dots, n+1), \quad \prod_{i=1}^{n+1} a_i = q^{-n/2},$$

then a vector of the variables $\mathbf{x} = {}^t[x_1, \dots, x_{n+1}]$ satisfies a system of linear q-difference equations

$$\overline{\mathbf{x}} = \left(A_0 + \frac{A_1}{1 - qt} \right) \mathbf{x}, \tag{2.7}$$

with $(n+1)\times(n+1)$ matrices

$$A_0 = \begin{bmatrix} b_1 & b_2-a_2 & b_3-a_3 & \dots & b_n-a_n & b_{n+1}-a_{n+1} \\ & b_2 & b_3-a_3 & \dots & b_n-a_n & b_{n+1}-a_{n+1} \\ & & & \ddots & \vdots & \vdots \\ & & & & b_n-a_n & b_{n+1}-a_{n+1} \\ & O & & & b_n & b_{n+1}-a_{n+1} \\ & & & & & b_{n+1} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} a_1-b_1 & a_2-b_2 & a_3-b_3 & \dots & a_n-b_n & a_{n+1}-b_{n+1} \end{bmatrix}.$$

Furthermore system (2.7) admits a solution

$$\mathbf{x} = t^{-\log_q a_1} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_{n+1} \end{bmatrix},$$

$$\phi_j = \prod_{i=1}^{j-1} \frac{b_i - a_1}{a_{i+1}-a_1} {}_{n+1}\phi_n \left[\begin{matrix} q\frac{a_1}{b_1}, \dots, q\frac{a_1}{b_{j-1}}, \frac{a_1}{b_j}, \dots, \frac{a_1}{b_n}, \frac{a_1}{b_{n+1}} \\ q\frac{a_1}{a_2}, \dots, q\frac{a_1}{a_j}, \frac{a_1}{a_{j+1}}, \dots, \frac{a_1}{a_n} \end{matrix} ; q, q^{(n+2)/2} b_1 \dots b_{n+1} t \right].$$

Therefore, we want to regard the system q-$P_{(n+1,n+1)}$ as a generalization of q-P_{VI} from a viewpoint of a particular solution in terms of the q-hypergeometric function. However, as is seen in Section 1, we have two problems. In [16], we derived q-P_{VI} by reducing linear system (2.2) to the one with 2×2 matrices given in [4] from q-$P_{(2,2)}$. We could not use a similar method in a general case, although we reduced linear system (2.2) to the one with $(n+1)\times(n+1)$ matrices in [2]. Definition of dependent variables (2.6) is complicated and hence unsuitable for a generalization. In this article, we choose a suitable set of $2n$ dependent variables (f_i, g_i) and reduce the system q-$P_{(n+1,n+1)}$ to a one of $2n$-th order. In the obtained system, the forward q-shifts $(\overline{f}_i, \overline{g}_i)$ is expressed as functions in (f_i, g_i).

3. Main result

The key to solving is the affine Weyl group symmetry of q-$P_{(n+1,n+1)}$. We can simplify definition of dependent variables (2.6) as

$$r_1 r_2(f) = -t\frac{x_1 - x_2}{t x_2 - x_1}, \quad r_1 r_2(g) = \frac{a_2 t}{q^{1/2}} \frac{\underline{x_2}(1+\underline{x_1} y_1)}{\underline{x_1}(1+\underline{x_2} y_1)}.$$

This fact suggests a choice of dependent variables of a $2n$-th order system.

THEOREM 3.1 If, in the system q-$P_{(n+1,n+1)}$, we set

$$f_i = t\frac{x_i - x_{i+1}}{tx_{n+1} - x_1}, \quad g_i = a_{i+1}\frac{\underline{x_{i+1}}(1+\underline{x_i}y_i)}{\underline{x_i}(1+\underline{x_{i+1}}y_i)} \quad (i = 1,\dots,n), \tag{3.1}$$

then they satisfy a system of q-difference equations

$$f_i\overline{f}_i = qt\frac{F_iF_{i+1}\overline{g}_0(b_i - \overline{g}_i)(\overline{g}_i - a_{i+1})}{F_{n+1}F_1\overline{g}_i(b_0 - \overline{g}_0)(\overline{g}_0 - a_1)} \quad (i = 1,\dots,n), \tag{3.2}$$

$$g_i\overline{g}_i = \frac{F_{i+1}G_i}{F_iG_{i+1}} \quad (i = 1,\dots,n), \tag{3.3}$$

where

$$b_0 = qb_{n+1}, \quad g_0 = \frac{1}{q^{(n-2)/2}t}\prod_{i=1}^{n}\frac{1}{g_i} = \frac{qa_1}{t}\frac{\underline{x_1}(1+\underline{x_{n+1}}y_{n+1})}{\underline{x_{n+1}}(1+\underline{x_1}y_0)} = a_1\frac{\underline{x_1}(1+\underline{x_0}y_0)}{\underline{x_0}(1+\underline{x_1}y_0)},$$

and

$$F_i = \sum_{j=1}^{i-1}f_j + t\sum_{j=i}^{n}f_j + t,$$

$$G_i = \sum_{j=i}^{n}\prod_{k=i}^{j-1}b_ka_{k+1}\frac{\prod_{l=j+1}^{n}g_l}{\prod_{l=1}^{j-1}g_l}f_j + q^{n/2}t\prod_{k=i}^{n}b_ka_{k+1}$$
$$+ q^nt\sum_{j=1}^{i-1}\frac{b_{n+1}a_1\prod_{k=1}^{n}b_ka_{k+1}}{\prod_{k=j}^{i-1}b_ka_{k+1}}\frac{\prod_{l=j+1}^{n}g_l}{\prod_{l=1}^{j-1}g_l}f_j \quad (i = 1,\dots,n+1).$$

Furthermore, if we set $h = tx_{n+1} - x_1$, then it satisfies a linear q-difference equation

$$\overline{h} = -\frac{F_{n+1}F_1(b_0 - \overline{g}_0)(\overline{g}_0 - a_1)}{t(t-1)^2\overline{g}_0}h. \tag{3.4}$$

System (3.2), (3.3) is equivalent to q-P_{VI} in the case of $n = 1$. In this section, we prove this theorem. We also discuss the relationship between the dependent variables f_i, g_i, h $(i = 1,\dots,n)$ and the ones x_j, y_j $(j = 1,\dots,n+1)$ in more detail at the end of this section.

3.1 *Proof of equation* (3.2)

Definition of dependent variables (3.1) implies

$$g_i - a_{i+1} = -a_{i+1}\frac{\underline{x_i} - \underline{x_{i+1}}}{\underline{x_i}(1+\underline{x_{i+1}}y_i)} \quad (i = 0,1,\dots,n). \tag{3.5}$$

The first equation of q-$P_{(n+1,n+1)}$ can be rewritten to

$$b_i - g_i = \frac{1+\underline{x_i}y_i}{\underline{x_i}}(x_i - x_{i+1}) \quad (i = 0, 1, \ldots, n). \tag{3.6}$$

Combining them, we obtain

$$\begin{aligned} \frac{(b_0-g_0)(g_0-a_1)}{g_0} &= -\frac{(t\underline{x_{n+1}} - \underline{x_1})(tx_{n+1}-x_1)}{t\underline{x_{n+1}}\,\underline{x_1}}, \\ \frac{(b_i-g_i)(g_i-a_{i+1})}{g_i} &= -\frac{(\underline{x_i}-\underline{x_{i+1}})(x_i-x_{i+1})}{\underline{x_i}\,\underline{x_{i+1}}} \quad (i = 1, \ldots, n). \end{aligned} \tag{3.7}$$

Hence we can derive equation (3.2) by using

$$F_i = \frac{t(t-1)x_i}{tx_{n+1}-x_1} \quad (i = 1, \ldots .n+1).$$

3.2 *Proof of equations* (3.3) *and* (3.4)

Equation (3.5) can be rewritten to

$$\begin{aligned} y_0 &= -\frac{\underline{x_0}g_0 - a_1\underline{x_1}}{\underline{x_0}\,\underline{x_1}(g_0-a_1)} = -\frac{t\underline{x_{n+1}}g_0 - qa_1\underline{x_1}}{t\underline{x_{n+1}}\,\underline{x_1}(g_0-a_1)} = \frac{q}{t}y_{n+1}, \\ y_i &= -\frac{\underline{x_i}g_i - a_{i+1}\underline{x_{i+1}}}{\underline{x_i}\,\underline{x_{i+1}}(g_i-a_{i+1})} \quad (i = 1, \ldots, n). \end{aligned}$$

Substituting it to the second equation of q-$P_{(n+1,n+1)}$, we obtain

$$\begin{aligned} \underline{y_{i-1}} - \underline{y_i} &= \frac{\underline{x_{i-1}}g_{i-1}}{\underline{x_i}(\underline{x_{i-1}}-\underline{x_i})} - \frac{a_i}{\underline{x_{i-1}}-\underline{x_i}} + \frac{b_i a_{i+1}\underline{x_{i+1}}}{\underline{x_i}(\underline{x_i}-\underline{x_{i+1}})g_i} - \frac{b_i}{\underline{x_i}-\underline{x_{i+1}}} \quad (i = 1, \ldots, n), \\ \underline{y_n} - \underline{y_{n+1}} &= \frac{\underline{x_n}g_n}{\underline{x_{n+1}}(\underline{x_n}-\underline{x_{n+1}})} - \frac{a_{n+1}}{\underline{x_n}-\underline{x_{n+1}}} + \frac{qb_{n+1}a_1\underline{x_1}}{\underline{x_{n+1}}(t\underline{x_{n+1}}-q\underline{x_1})g_0} - \frac{b_{n+1}t}{t\underline{x_{n+1}}-q\underline{x_1}}. \end{aligned}$$

It implies

$$\begin{aligned} & b_i(x_{i-1}-x_i) + a_i(x_i - x_{i+1}) + (x_{i-1}-x_i)(x_i-x_{i+1})(y_{i-1}-y_i) \\ & = \frac{x_{i-1}}{x_i}(x_i - x_{i+1})\overline{g_{i-1}} + b_i a_{i+1}\frac{x_{i+1}}{x_i}(x_{i-1}-x_i)\frac{1}{\overline{g_i}} \quad (i = 1, \ldots, n), \\ & b_{n+1}t(x_n - x_{n+1}) + a_{n+1}(tx_{n+1}-x_1) + (x_n - x_{n+1})(tx_{n+1}-x_1)(y_n - y_{n+1}) \\ & = \frac{x_n}{x_{n+1}}(tx_{n+1}-x_1)\overline{g_n} + b_{n+1}a_1\frac{x_1}{x_{n+1}}(x_n - x_{n+1})\frac{1}{\overline{g_0}}. \end{aligned} \tag{3.8}$$

On the other hand, equation (3.6) can be rewritten to

$$y_0 = -\frac{g_0 - b_0}{x_0 - x_1} - \frac{1}{\underline{x_0}} = -\frac{g_0 - qb_{n+1}}{tx_{n+1} - x_1} - \frac{q}{t\underline{x_{n+1}}} = \frac{q}{t}y_{n+1},$$

$$y_i = -\frac{g_i - b_i}{x_i - x_{i+1}} - \frac{1}{\underline{x_i}} \quad (i = 1, \dots, n).$$

Combining it with equation (3.7), we obtain

$$\begin{aligned} y_{i-1} - y_i &= \frac{g_i - b_i}{x_i - x_{i+1}} + \frac{1}{\underline{x_i}} - \frac{g_{i-1} - b_{i-1}}{x_{i-1} - x_i} - \frac{1}{\underline{x_{i-1}}} \\ &= \frac{g_i}{x_i - x_{i+1}} - \frac{b_i}{x_i - x_{i+1}} + \frac{b_{i-1}a_i}{(x_{i-1} - x_i)g_{i-1}} - \frac{a_i}{x_{i-1} - x_i} \quad (i = 1, \dots, n), \\ y_n - y_{n+1} &= \frac{tg_0 - qb_{n+1}t}{q(tx_{n+1} - x_n)} + \frac{1}{\underline{x_{n+1}}} - \frac{g_n - b_n}{x_n - x_{n+1}} - \frac{1}{\underline{x_n}} \\ &= \frac{tg_0}{q(tx_{n+1} - x_1)} - \frac{b_{n+1}t}{tx_{n+1} - x_1} + \frac{b_n a_{n+1}}{(x_n - x_{n+1})g_n} - \frac{a_{n+1}}{x_n - x_{n+1}}. \end{aligned}$$

We can rewrite it to

$$\begin{aligned} &b_i(x_{i-1} - x_i) + a_i(x_i - x_{i+1}) + (x_{i-1} - x_i)(x_i - x_{i+1})(y_{i-1} - y_i) \\ &= (x_{i-1} - x_i)g_i + b_{i-1}a_i(x_i - x_{i+1})\frac{1}{g_{i-1}} \quad (i = 1, \dots, n), \\ &b_{n+1}t(x_n - x_{n+1}) + a_{n+1}(tx_{n+1} - x_1) + (x_n - x_{n+1})(tx_{n+1} - x_1)(y_n - y_{n+1}) \\ &= \frac{t}{q}(x_n - x_{n+1})g_0 + b_n a_{n+1}(tx_{n+1} - x_1)\frac{1}{g_n}. \end{aligned} \tag{3.9}$$

Combining equations (3.8) and (3.9), we obtain

$$\begin{aligned} tf_1\frac{F_{n+1}}{F_1}\overline{g}_0 + tb_1a_2\frac{F_2}{F_1}\frac{1}{\overline{g}_1} &= tg_1 + qb_{n+1}a_1f_1\frac{1}{g_0}, \\ f_i\frac{F_{i-1}}{F_i}\overline{g}_{i-1} + b_ia_{i+1}f_{i-1}\frac{F_{i+1}}{F_i}\frac{1}{\overline{g}_i} &= f_{i-1}g_i + b_{i-1}a_if_i\frac{1}{g_{i-1}} \quad (i = 2, \dots, n), \\ t\frac{F_n}{F_{n+1}}\overline{g}_n + b_{n+1}a_1f_n\frac{F_1}{F_{n+1}}\frac{1}{\overline{g}_0} &= \frac{t}{q}f_ng_0 + b_na_{n+1}t\frac{1}{g_n}. \end{aligned} \tag{3.10}$$

Since equation (3.10) is equivalent to (3.3) in the case of $n = 1$, we consider the case of $n \geq 2$. Then equation (3.10) is transformed to

$$
\begin{aligned}
G_{n+1} - q^{n-1}t\left(t\frac{g_0g_1}{f_1} + qb_{n+1}a_1\right)G_1 + q^{n-1}b_1a_2t^2\frac{g_0g_1}{f_1}G_2 = 0,\\
G_{i-1} - \left(\frac{f_{i-1}g_{i-1}g_i}{f_i} + b_{i-1}a_i\right)G_i + b_ia_{i+1}\frac{f_{i-1}g_{i-1}g_i}{f_i}G_{i+1} = 0 \quad (i = 2, \ldots, n),\\
G_n - \left(\frac{1}{q}f_ng_ng_0 + b_na_{n+1}\right)G_{n+1} + q^{n-1}b_{n+1}a_1tf_ng_ng_0G_1 = 0,
\end{aligned} \tag{3.11}
$$

via a transformation

$$
\begin{aligned}
\overline{g}_i &= \frac{F_{i+1}G_i}{g_iF_iG_{i+1}} \quad (i = 1, \ldots, n),\\
\overline{g}_0 &= \frac{1}{q^{n/2}t}\prod_{i=1}^{n}\frac{1}{\overline{g}_i} = \frac{1}{q^{n-1}t^2}\frac{F_1G_{n+1}}{g_0F_{n+1}G_1}.
\end{aligned}
$$

Furthermore equation (3.11) is reduced to a system of linear equations

$$
\begin{bmatrix}
1 & -\alpha_1 & \beta_1 & & & & \\
 & 1 & -\alpha_2 & \beta_2 & & & \\
 & & 1 & -\alpha_3 & \beta_3 & & \\
 & & & & \ddots & & \\
 & & & & & -\alpha_{n-2} & \beta_{n-2} \\
 & & & & & 1 & -\alpha_{n-1} \\
\beta_n & & & & & & 1
\end{bmatrix}
\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \\ G_{n-2} \\ G_{n-1} \\ G_n \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\beta_{n-1}G_{n+1} \\ \alpha_nG_{n+1} \end{bmatrix}, \tag{3.12}
$$

where

$$
\begin{aligned}
\alpha_i &= \frac{f_ig_ig_{i+1}}{f_{i+1}} + b_ia_{i+1}, \quad \beta_i = b_{i+1}a_{i+2}\frac{f_ig_ig_{i+1}}{f_{i+1}} \quad (i = 1, \ldots, n-1),\\
\alpha_n &= \frac{1}{q}f_ng_ng_0 + b_na_{n+1}, \quad \beta_n = q^{n-1}b_{n+1}a_1tf_ng_ng_0.
\end{aligned}
$$

In fact equation (3.11) can be rewritten to

$$
\begin{aligned}
G_1 - b_1a_2G_2 &= \frac{qf_1}{tg_0g_1}\left(\frac{1}{q^nt}G_{n+1} - b_{n+1}a_1G_1\right),\\
G_i - b_ia_{i+1}G_{i+1} &= \frac{f_i}{f_{i-1}g_{i-1}g_i}(G_{i-1} - b_{i-1}a_iG_i) \quad (i = 2, \ldots, n),\\
\frac{1}{q^nt}G_{n+1} - b_{n+1}a_1G_1 &= \frac{1}{q^{n-1}tf_ng_ng_0}(G_n - b_na_{n+1}G_{n+1}),
\end{aligned} \tag{3.13}
$$

and the first equation of (3.13) is derived from a combination of the other equations as follows.

$$\begin{aligned}\frac{1}{q^n t}G_{n+1} - b_{n+1}a_1 G_1 &= \frac{1}{q^{n-1}tf_n g_n g_0}(G_n - b_n a_{n+1}G_{n+1})\\ &= \frac{1}{q^{n-1}tf_{n-1}g_{n-1}g_n^2 g_0}(G_{n-1} - b_{n-1}a_n G_n)\\ &\vdots\\ &= \frac{1}{q^{n-1}tf_1 g_1 g_2^2\cdots g_n^2 g_0}(G_1 - b_1 a_2 G_2)\\ &= \frac{tg_0 g_1}{qf_1}(G_1 - b_1 a_2 G_2).\end{aligned}$$

We will solve system (3.12) for G_i $(i = 1, \ldots, n)$ in order to derive equation (3.3). For this purpose we introduce the following lemma.

LEMMA 3.2 The determinant of the coefficient matrix of system (3.12) is given by

$$\begin{vmatrix} 1 & -\alpha_1 & \beta_1 & & & & \\ & 1 & -\alpha_2 & \beta_2 & & & \\ & & 1 & -\alpha_3 & \beta_3 & & \\ & & & & \ddots & & \\ & & & & & -\alpha_{n-2} & \beta_{n-2} \\ & & & & & 1 & -\alpha_{n-1} \\ \beta_n & & & & & & 1 \end{vmatrix} = 1 + q^{n/2}\sum_{j=1}^{n} b_{n+1}a_1 \prod_{k=1}^{j-1} b_k a_{k+1} \frac{\prod_{l=j+1}^{n} g_l}{\prod_{l=1}^{j-1} g_l} f_j.$$

Proof. We obtain

$$\begin{vmatrix} 1 & -\alpha_1 & \beta_1 & & & & \\ & 1 & -\alpha_2 & \beta_2 & & & \\ & & 1 & -\alpha_3 & \beta_3 & & \\ & & & & \ddots & & \\ & & & & & -\alpha_{n-2} & \beta_{n-2} \\ & & & & & 1 & -\alpha_{n-1} \\ \beta_n & & & & & & 1 \end{vmatrix} = 1 + (-1)^{n-1}\beta_n \Delta_{n-1}.$$

where

$$\Delta_i = \begin{vmatrix} -\alpha_1 & \beta_1 & & & & \\ 1 & -\alpha_2 & \beta_2 & & & \\ & 1 & -\alpha_3 & \beta_3 & & \\ & & & \ddots & & \\ & & & & -\alpha_{i-1} & \beta_{i-1} \\ & & & & 1 & -\alpha_i \end{vmatrix},$$

via a cofactor expansion. The determinant of the tridiagonal matrix Δ_i satisfies a recurrence relation

$$\Delta_0 = 1, \quad \Delta_1 = -\alpha_1, \quad \Delta_i = -\alpha_i \Delta_{i-1} - \beta_{i-1}\Delta_{i-2} \quad (i = 2, 3, \ldots).$$

By solving this relation, we obtain

$$\Delta_i = (-1)^i \sum_{j=1}^{i+1} \prod_{k=1}^{j-1} b_k a_{k+1} \prod_{l=j}^{i} g_l g_{l+1} \frac{f_j}{f_{i+1}} \quad (i = 2, 3, \ldots).$$

It implies

$$\begin{aligned}(-1)^{n-1}\beta_n \Delta_{n-1} &= q^{n-1} b_{n+1} a_1 t f_n g_n g_0 \sum_{j=1}^{n} \prod_{k=1}^{j-1} b_k a_{k+1} \prod_{l=j}^{n-1} g_l g_{l+1} \frac{f_j}{f_n} \\ &= q^{n/2} \sum_{j=1}^{n} b_{n+1} a_1 \prod_{k=1}^{j-1} b_k a_{k+1} \frac{\prod_{l=j+1}^{n} g_l}{\prod_{l=1}^{j-1} g_l} f_j.\end{aligned}$$

□

Thanks to Lemma 3.2, we can find that system (3.12) admits only one solution for G_i $(i = 1, \ldots, n)$. Hence we only have to verify that

$$G_i = \sum_{j=i}^{n} \prod_{k=i}^{j-1} b_k a_{k+1} \frac{\prod_{l=j+1}^{n} g_l}{\prod_{l=1}^{j-1} g_l} f_j + q^{n/2} t \prod_{k=i}^{n} b_k a_{k+1} + q^n t \sum_{j=1}^{i-1} \frac{b_{n+1} a_1 \prod_{k=1}^{n} b_k a_{k+1}}{\prod_{k=j}^{i-1} b_k a_{k+1}} \frac{\prod_{l=j+1}^{n} g_l}{\prod_{l=1}^{j-1} g_l} f_j,$$

satisfy system (3.12). It is shown since

$$\begin{aligned} G_i - b_i a_{i+1} G_{i+1} &= \left(1 - q^n t b_{n+1} a_1 \prod_{k=1}^{n} b_k a_{k+1}\right) \frac{\prod_{l=i+1}^{n} g_l}{\prod_{l=1}^{i-1} g_l} f_i \quad (i = 1, \ldots, n), \\ \frac{1}{q^n t} G_{n+1} - b_{n+1} a_1 G_1 &= \frac{1}{q^{n/2}} \left(1 - q^n t b_{n+1} a_1 \prod_{k=1}^{n} b_k a_{k+1}\right), \end{aligned}$$

satisfy equation (3.13). From the above, we have derived equation (3.3).

In the last, we prove equation (3.4). The first equation of (3.7) is rewritten to

$$\overline{t x_{n+1} - x_1} = -\frac{(b_0 - \overline{g}_0)(\overline{g}_0 - a_1)}{\overline{g}_0} \frac{t x_{n+1} x_1}{t x_{n+1} - x_1},$$

from which we obtain equation (3.4) immediately.

3.3 *Relationship between two types of dependent variables*

In equation (3.1), we define the dependent variables (f_i, g_i, h) as rational functions in (x_j, y_j) with constraint (2.1). Conversely, the dependent variables $(\underline{x}_j, y_j)$ are given as rational functions in (f_i, g_i, h). We obtain

$$\frac{1}{\underline{x}_1} - \frac{1}{t\underline{x}_{n+1}} = -\frac{(b_0 - g_0)(g_0 - a_1)}{g_0 h},$$

from equation (3.4) and

$$\frac{1}{\underline{x}_{i+1}} - \frac{1}{\underline{x}_i} = -\frac{t(b_i - g_i)(g_i - a_{i+1})}{f_i g_i h}. \quad (i = 1, \ldots, n),$$

from equation (3.7). They imply

$$\begin{aligned}\frac{1}{\underline{x}_i} &= \frac{t}{q-t}\frac{(b_0 - g_0)(g_0 - a_1)}{g_0 h} \\ &+ \frac{t}{q-t}\sum_{j=1}^{i-1}\frac{t(b_j - g_j)(g_j - a_{j+1})}{f_j g_j h} + \frac{q}{q-t}\sum_{j=i}^{n}\frac{t(b_j - g_j)(g_j - a_{j+1})}{f_j g_j h} \quad (i = 1, \ldots, n+1). \quad (3.14)\end{aligned}$$

We also obtain

$$y_i = -\frac{g_i/\underline{x}_{i+1} - a_{i+1}/\underline{x}_i}{g_i - a_{i+1}} \quad (i = 1, \ldots, n),$$

from equation (3.5). The rest variable y_{n+1} is given by constraint (2.1).

REMARK 3.3 Equation (3.14) allows us to express the backward q-shifts $\underline{f}_1, \ldots, \underline{f}_n$ as rational functions in (f_i, g_i). Also we can express the backward q-shifts $\underline{g}_1, \ldots, \underline{g}_n$ as rational functions in (f_i, g_i) by solving equation (3.10) for $g_1, \ldots, g_n$. We don't give its detail here.

We next state a relationship between the variables (f_i, g_i, h) and the ones (x_j, y_j). Equation (3.9) is rewritten to

$$\begin{aligned}\frac{q}{t}y_{n+1} - y_1 &= t(g_1 - b_1)\frac{1}{f_1 h} + a_1\left(qb_{n+1}\frac{1}{g_0} - 1\right)\frac{1}{h} \\ y_{i-1} - y_i &= t(g_i - b_i)\frac{1}{f_i h} + a_i t\left(b_{i-1}\frac{1}{g_{i-1}} - 1\right)\frac{1}{f_{i-1}h} \quad (i = 2, \ldots, n), \\ y_n - y_{n+1} &= \frac{t}{q}(g_0 - qb_{n+1})\frac{1}{h} + a_{n+1}t\left(b_n\frac{1}{g_n} - 1\right)\frac{1}{f_n h}. \end{aligned} \quad (3.15)$$

Recall that

$$tx_{n+1} - x_1 = h, \quad x_i - x_{i+1} = \frac{f_i h}{t} \quad (i = 1, \ldots, n).$$

Hence we can give the variables (x_j, y_j) as rational functions in (f_i, g_i, h). Conversely, since equation (3.15) admits only one solution for $g_1, \ldots, g_n$, the variables (f_i, g_i, h) are given as rational functions in (x_j, y_j). Then one constraint between the variables (x_j, y_j) is obtained together. We don't give their explicit formulas here. Those facts allow us to express the backward q-shifts $(\underline{x_j}, \underline{y_j})$ as functions in (x_j, y_j).

4. Affine Weyl group symmetry

As is seen in Section 2, the system q-$P_{(n+1,n+1)}$ is invariant under the action of the group of symmetries $\langle r_0, \ldots, r_{2n+1}, \pi\rangle \simeq \widetilde{W}(A^{(1)}_{2n+1})$. This action can be restricted to systems (3.2) and (3.3).

THEOREM 4.1 The birational transformations $r_0, \ldots, r_{2n+1}, \pi$ defined by (2.3), (2.4) and (2.5) act on the dependent variables f_i, g_i $(i = 1, \ldots, n)$ as follows.

$$r_{2j-2}(f_i) = f_i, \quad r_{2j-2}(g_i) = g_i, \tag{4.1}$$

for $j = 1, \ldots, n+1$,

$$\begin{aligned}
&r_1(f_1) = f_1\frac{R_1^{a,a,a}}{R_1^{b,a,b}}, \quad r_1(g_1) = g_1\frac{R_1^{b,a,a}}{R_1^{b,b,b}}, \quad r_1(f_i) = f_i\frac{R_1^{b,a,a}}{R_1^{b,a,b}}, \quad r_1(g_i) = g_i \quad (i \neq 1),\\
&r_{2j-1}(f_{j-1}) = f_{j-1}\frac{R_j^{b,a,b}}{R_j^{b,a,a}}, \quad r_{2j-1}(g_{j-1}) = g_{j-1}\frac{R_j^{b,b,b}}{R_j^{b,a,a}}, \quad r_{2j-1}(f_j) = f_j\frac{R_j^{a,a,a}}{R_j^{b,a,a}}, \quad r_{2j-1}(g_j) = g_j\frac{R_j^{b,a,a}}{R_j^{b,b,b}},\\
&r_{2j-1}(f_i) = f_i, \quad r_{2j-1}(g_i) = g_i \quad (i \neq j-1, j),\\
&r_{2n+1}(f_n) = f_n\frac{R_{n+1}^{b,a,b}}{R_{n+1}^{a,a,a}}, \quad r_{2n+1}(g_n) = g_n\frac{R_{n+1}^{b,b,b}}{R_{n+1}^{b,a,a}}, \quad r_{2n+1}(f_i) = f_i\frac{R_{n+1}^{b,a,a}}{R_{n+1}^{a,a,a}}, \quad r_{2n+1}(g_i) = g_i \quad (i \neq n)
\end{aligned} \tag{4.2}$$

for $j = 2, \ldots, n$ and

$$\begin{aligned}
&\pi(f_i) = \frac{q^2}{t}\frac{(g_iR_i^* - b_{i+1}R_{i+1}^*)(b_{i+1} - g_{i+1})(R_{i+1}^* + 1 - \frac{t}{q})f_1}{(g_0R_0^* - b_1R_1^*)(b_1 - g_1)(R_1^* + 1 - \frac{t}{q})f_{i+1}}, \quad \pi(g_i) = \frac{a_{i+1}}{q^{\rho_1}}\frac{b_{i+1}R_{i+1}^*}{g_iR_i^*} \quad (i \neq n),\\
&\pi(f_n) = q\frac{(g_nR_n^* - b_{n+1}R_0^*)(b_0 - g_0)(R_0^* + 1 - \frac{t}{q})f_1}{(g_0R_0^* - b_1R_1^*)(b_1 - g_1)(R_1^* + 1 - \frac{t}{q})f_0}, \quad \pi(g_n) = \frac{a_{n+1}}{q^{\rho_1}}\frac{b_{n+1}R_0^*}{g_nR_n^*},
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
R_j^{\alpha,\beta,\gamma} &= (g_j - \alpha_j)\frac{1}{f_j} + \left(\beta_jb_{j-1}\frac{1}{g_{j-1}} - \gamma_j\right)\frac{1}{f_{j-1}} \quad (j \neq n+1),\\
R_{n+1}^{\alpha,\beta,\gamma} &= \frac{1}{q}(g_0 - q\alpha_{n+1}) + \left(\beta_{n+1}b_n\frac{1}{g_n} - \gamma_{n+1}\right)\frac{1}{f_n},
\end{aligned}$$

and

$$R_i^* = -\frac{f_i}{b_i - g_i}\left(\frac{t}{q}\sum_{j=1}^{i} R_j^{b,a,a} + \sum_{j=i+1}^{n+1} R_j^{b,a,a}\right)$$
$$= \frac{f_i}{b_i - g_i}\left(\sum_{j=0}^{i-1}\frac{\frac{t}{q}(1-\frac{a_{j+1}}{g_j})(b_j-g_j)}{f_j} + \frac{(\frac{t}{q}-\frac{a_{i+1}}{g_i})(b_i-g_i)}{f_i} + \sum_{j=i+1}^{n}\frac{(1-\frac{a_{j+1}}{g_j})(b_j-g_j)}{f_j}\right).$$

Here we set $f_0 = t$.

Recall that

$$b_0 = qb_{n+1}, \quad g_0 = \frac{1}{q^{(n-2)/2}tg_1\ldots g_n}, \quad q^{\rho_1} = (q^n a_1 b_1 \ldots a_{n+1}b_{n+1})^{1/(n+1)}.$$

In this section, we prove this theorem.

4.1 *Proof of action* (4.1)

The action of r_{2j-2} on the dependent variables with the exception of y_{j-1} is trivial for any $j = 1,\ldots,n+1$. Hence we only have to verify that $r_{2j-2}(g_{j-1}) = g_{j-1}$ is satisfied. And it is shown by using the first equation of q-$P_{(n+1,n+1)}$ as follows

$$r_{2j-2}(g_{j-1}) = b_{j-1}\frac{\underline{x_j}}{\underline{x_{j-1}}}\frac{1+\underline{x_{j-1}}y_{j-1} - \frac{b_{j-1}-a_j}{x_{j-1}-x_j}\underline{x_{j-1}}}{1+\underline{x_j}y_{j-1} - \frac{b_{j-1}-a_j}{x_{j-1}-x_j}\underline{x_j}}$$
$$= b_{j-1}\frac{\underline{x_j}}{\underline{x_{j-1}}}\frac{a_j(1+\underline{x_{j-1}}y_{j-1})}{b_{j-1}(1+\underline{x_j}y_{j-1})}$$
$$= g_{j-1}.$$

4.2 *Proof of action* (4.2)

The action of r_{2j-1} on the dependent variables with the exception of x_j is trivial for any $j = 1,\ldots,n+1$. Hence we have to investigate the following actions:

$$r_1(f_1), \quad r_1(g_1), \quad r_1(f_i) \quad (i \neq 1),$$
$$r_{2j-1}(f_{j-1}), \quad r_{2j-1}(g_{j-1}), \quad r_{2j-1}(f_j), \quad r_{2j-1}(g_j),$$
$$r_{2n+1}(f_n), \quad r_{2n+1}(g_n), \quad r_{2n+1}(f_i) \quad (i \neq n).$$

We first consider the action $r_{2j-1}(f_j)$. It is described in terms of the variables x_i, y_i as

$$r_{2j-1}(f_j) = t\frac{x_j - \frac{a_j-b_j}{y_{j-1}-y_j} - x_{j+1}}{tx_{n+1}-x_1} = t\frac{(x_j-x_{j+1})(y_{j-1}-y_j)-(a_j-b_j)}{(tx_{n+1}-x_1)(y_{j-1}-y_j)}. \tag{4.4}$$

Substituting equation (3.15) to (4.4), we obtain

$$r_{2j-1}(f_j) = f_j \frac{(g_j - a_j)\frac{1}{f_j} + a_j(b_{j-1}\frac{1}{g_{j-1}} - 1)\frac{1}{f_{j-1}}}{(g_j - b_j)\frac{1}{f_j} + a_j(b_{j-1}\frac{1}{g_{j-1}} - 1)\frac{1}{f_{j-1}}} \quad (j = 1, \ldots, n+1).$$

The other actions on the variables $f_1, \ldots, f_n$ can be shown in a similar way.

We next consider the action $r_{2j-1}(g_j)$. By using the system q-$P_{(n+1,n+1)}$ twice, we can describe it in terms of the variables x_i, y_i as

$$\begin{aligned} r_{2j-1}(g_j) &= a_{j+1} \frac{\underline{x_{j+1}}\left(1 + \underline{x_j} y_j - \frac{a_j - b_j}{\underline{y_{j-1}} - \underline{y_j}} y_j\right)}{\left(\underline{x_j} - \frac{a_j - b_j}{\underline{y_{j-1}} - \underline{y_j}}\right)(1 + \underline{x_{j+1}} y_j)} \\ &= g_j \frac{a_j \underline{x_j}(y_{j-1} - y_j)}{b_j(1 + \underline{x_j} y_{j-1}) - a_j(1 + \underline{x_j} y_j)} \\ &= g_j \frac{(x_{j-1} - x_j)(y_{j-1} - y_j)}{(x_{j-1} - x_j)(y_{j-1} - y_j) - (a_j - b_j)(b_{j-1}\frac{1}{g_{j-1}} - 1)}. \end{aligned} \tag{4.5}$$

Then, substituting equation (3.15) to (4.5), we obtain

$$r_{2j-1}(g_j) = g_j \frac{(g_j - b_j)\frac{1}{f_j} + a_j(b_{j-1}\frac{1}{g_{j-1}} - 1)\frac{1}{f_{j-1}}}{(g_j - b_j)\frac{1}{f_j} + b_j(b_{j-1}\frac{1}{g_{j-1}} - 1)\frac{1}{f_{j-1}}} \quad (j = 1, \ldots, n+1).$$

The other actions on the variables $g_1, \ldots, g_n$ can be shown in a similar way.

4.3 *Proof of action* (4.3)

The action $\pi(f_i)$ is rewritten by the second equation of q-$P_{(n+1,n+1)}$ and equation (3.6) to

$$\begin{aligned} \pi(f_i) &= \frac{q^2}{t} \frac{\underline{y_i} - \underline{\underline{y_{i+1}}}}{\underline{y_0} - \underline{y_1}} = \frac{q^2}{t} \frac{\underline{x_1}}{\underline{x_{i+1}}} \frac{\frac{g_i f_i}{b_i - g_i} y_i - \frac{b_{i+1} f_{i+1}}{b_{i+1} - g_{i+1}} y_{i+1}}{\frac{g_0 t}{b_0 - g_0} y_0 - \frac{b_1 f_1}{b_1 - g_1} y_1} \quad (i = 1, \ldots, n-1), \\ \pi(f_n) &= \frac{q^2}{t} \frac{\underline{y_n} - \underline{\underline{y_{n+1}}}}{\underline{y_0} - \underline{y_1}} = q \frac{\underline{\underline{x_1}}}{\underline{x_0}} \frac{\frac{g_n f_n}{b_n - g_n} y_n - \frac{b_{n+1} t}{b_0 - g_0} y_0}{\frac{g_0 t}{b_0 - g_0} y_0 - \frac{b_1 f_1}{b_1 - g_1} y_1}. \end{aligned} \tag{4.6}$$

On the other hand, we obtain

$$\frac{(\frac{q}{t} - 1)(x_0 - x_1)}{q} \frac{1}{\underline{x_i}} = \sum_{j=0}^{i-1} \frac{\frac{t}{q}(1 - \frac{a_{j+1}}{g_j})(b_j - g_j)}{f_j} + \sum_{j=i}^{n} \frac{(1 - \frac{a_{j+1}}{g_j})(b_j - g_j)}{f_j} \quad (i = 0, \ldots, n), \tag{4.7}$$

from equation (3.7) and

$$\frac{(\frac{t}{q}-1)(x_0-x_1)}{t}y_i = \sum_{j=0}^{i-1}\frac{\frac{t}{q}(1-\frac{a_{j+1}}{g_j})(b_j-g_j)}{f_j} + \frac{(\frac{t}{q}-\frac{a_{i+1}}{g_i})(b_i-g_i)}{f_i} + \sum_{j=i+1}^{n}\frac{(1-\frac{a_{j+1}}{g_j})(b_j-g_j)}{f_j} \quad (i=0,\ldots,n), \tag{4.8}$$

from equation (3.15). Substituting equations (4.7) and (4.8) to (4.6), we can show the action $\pi(f_i)$.

The action $\pi(g_i)$ is rewritten by equation (3.6) to

$$\pi(g_i) = \frac{b_{i+1}}{q^{\rho_1}}\frac{y_{i+1}(1+\underline{x_i}y_i)}{y_i(1+\underline{x_{i+1}}y_{i+1})} = \frac{a_{i+1}}{q^{\rho_1}}\frac{\frac{b_{i+1}f_{i+1}}{b_{i+1}-g_{i+1}}y_{i+1}}{\frac{g_if_i}{b_i-g_i}y_i} \quad (i=1,\ldots,n-1),$$

$$\pi(g_n) = \frac{b_{n+1}}{q^{\rho_1}}\frac{y_{n+1}(1+\underline{x_n}y_n)}{y_n(1+\underline{x_{n+1}}y_{n+1})} = \frac{a_{n+1}}{q^{\rho_1}}\frac{\frac{b_{n+1}t}{b_0-g_0}y_0}{\frac{g_nf_n}{b_n-g_n}y_n}. \tag{4.9}$$

Substituting equation (4.8) to (4.9), we can show the action $\pi(g_i)$.

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