Monthly Notices of the ROYAL ASTRONOMICAL SOCIETY

MNRAS **476,** L69–L72 (2018) doi:10.1093/mnrasl/sly035
Advance Access publication 2018 March 6

Theoretical implications of the galactic radial acceleration relation of McGaugh, Lelli, and Schombert

Robert K. Nesbet★

IBM Almaden Research Center, 650 Harry Road, San Jose, CA 95120-6099, USA

Accepted 2018 February 23. Received 2018 January 19; in original form 2017 August 23

ABSTRACT

Velocities in stable circular orbits about galaxies, a measure of centripetal gravitation, exceed the expected Kepler/Newton velocity as orbital radius increases. Standard Λ cold dark matter (ΛCDM) attributes this anomaly to galactic dark matter. McGaugh et al. have recently shown for 153 disc galaxies that observed radial acceleration is an apparently universal function of classical acceleration computed for observed galactic baryonic mass density. This is consistent with the empirical modified Newtonian dynamics (MOND) model, not requiring dark matter. It is shown here that suitably constrained ΛCDM and conformal gravity (CG) also produce such a universal correlation function. ΛCDM requires a very specific dark matter distribution, while the implied CG non-classical acceleration must be independent of galactic mass. All three constrained radial acceleration functions agree with the empirical baryonic v^4 Tully–Fisher relation. Accurate rotation data in the nominally flat velocity range could distinguish between MOND, ΛCDM, and CG.

Key words: gravitation – galaxies: kinematics and dynamics – dark energy – dark matter – cosmology: theory.

1 INTRODUCTION

Velocities of objects in stable circular orbits about galaxies measure radial gravitational acceleration. Kepler/Newton velocity falls below observed velocity as orbital radius increases. Standard Λ cold dark matter (ΛCDM) attributes observed excess velocity to centripetal acceleration due to galactic dark matter.

McGaugh, Lelli & Schombert (2016) have recently shown for 153 disc galaxies that observed acceleration a is effectively a universal function of Newtonian acceleration a_{N}, computed for the observed baryonic distribution. The empirical function has negligible observed scatter.

This radial acceleration relation (RAR) is compatible with the empirical modified Newtonian dynamics (MOND) model (Milgrom 1983, 2016; Sanders 2010; Famaey & McGaugh 2012), which does not invoke cold dark matter (CDM). It implies some very simple natural law. It is shown here that such a law is predicted by conformal theory (Weyl 1918; Mannheim 2006; Nesbet 2013, 2014), without dark matter. It does not exclude a specific CDM source density derived here. This is consistent with conclusions that current ΛCDM simulations do not imply observed galactic rotation curves (Wu & Kroupa 2015) nor the observed distribution of extragalactic matter (Kroupa 2012). Incremental non-classical conformal gravity (CG) radial acceleration Δa, constant except for a smooth cut-off in the very large spherical dark halo, is determined in the isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) metric (Nesbet 2015). Derivations of Δa, important only for large orbital radii, are simplified here by imposing spherical symmetry, valid for acceleration at large radii.

The empirical RAR (McGaugh et al. 2016) resolves a long-standing conflict between MOND (Milgrom 1983; Sanders 2010; Famaey & McGaugh 2012) and CG. Fitting CG to earlier less precise data (Mannheim 2006; Mannheim & O'Brien 2011), non-classical acceleration parameter γ has been inferred to depend on galactic mass, incompatible with negligible scatter of the observed RAR. Mass-independent γ would agree with the RAR and with constant MOND scale parameter a_0. This supports a recent conclusion that the assumed mass-dependent part $\gamma_{\mathrm{G}} = N^* \gamma^*$ of γ cannot be derived from current theory (Nesbet 2014).

Agreement of CG with the RAR supports a universal conformal symmetry postulate (Nesbet 2013) that all elementary physical fields satisfy local Weyl scaling symmetry (Weyl 1918), modifying Einstein–Hilbert general relativity (CG; Mannheim & Kazanas 1989; Mannheim 2006) and the electroweak scalar field model (conformal Higgs model; Nesbet 2010, 2011).

2 QUALITATIVE IMPLICATIONS

The observed correlation (McGaugh et al. 2016) between classical and non-classical centripetal acceleration puts a strong constraint

★ E-mail: rkn@earthlink.net

on any theoretical model. Observed radial acceleration a must be a unique function of Newtonian a_N, regardless of galactic structure or mass. Standard ΛCDM assumes that a pre-existing dark matter aggregate attracts baryonic matter, which forms the observable galaxy. This must correlate the baryonic distribution to the assumed dark matter with no interaction other than gravity.

a and a_N are functions of two variables, galactic mass M and radius r, even beyond the range of dependence on galactic structure. If unrelated functions $a(x, y)$ and $b(x, y)$ were plotted against each other the general result would be a two-dimensional smear, not the one-dimensional line plot found by McGaugh et al. (2016). This result requires $a(x, y) = F(b(x, y))$. Observed correlation function F depends on only a single variable, with negligible scatter.

Correlation function $a = F(a_N)$ is a basic postulate of MOND (Milgrom 1983, 2016). It will be shown here that CG (Mannheim 2006) and the depleted halo model (Nesbet 2015) produce such a function if the implied non-classical acceleration parameter is independent of galactic mass. A particular distribution of dark matter is derived here for which ΛCDM also produces such a function.

3 MOND BACKGROUND

MOND (Milgrom 1983; Sanders 2010; Famaey & McGaugh 2012) modifies the Newtonian force law for acceleration below an empirical scale a_0. Using $y = a_N/a_0$ as independent variable (McGaugh 2008; Milgrom 2016), for assumed universal constant a_0, MOND postulates an interpolation function $\nu(y)$ such that observed radial acceleration $a = F(a_N) = a_N\nu(y)$, which defines a correlation function.

For $a_N \gg a_0$, $\nu \to 1$ and for $a_N \ll a_0$, $\nu^2 \to 1/y$. This implies asymptotic limit $a^2 \to a_0 a_N$ for small a_N, which translates into an asymptotically flat rotational velocity function $v(r)$ for large galactic radius r (Milgrom 1983).

4 ΛCDM BACKGROUND

External Schwarzschild potential function $B(r)$ is determined for a static spherical galactic model by simplified second-order differential equation:

$$\partial_r^2(rB(r)) = rw(r) \tag{1}$$

for $w(r)$ determined by source energy–momentum. Centripetal radial acceleration for a stable circular orbit is

$$a(r) = \frac{v^2(r)}{r} = \frac{1}{2}c^2 B'(r). \tag{2}$$

Spherically averaged mass/energy density $w(r)$ is modelled by baryonic $w_0(r)$ within galactic radius r_G, embedded in dark matter $w_1(r)$ within halo radius $r_H \gg r_G$. Then $w(r) = w_0(r) + w_1(r)$ within r_G. Functions $y_0 = rB(r)$ and derivative y_1 satisfy differential equations:

$$\partial_r y_0 = y_1,\ \partial_r y_1 = rw(r). \tag{3}$$

Gravitational potential $B(r)$ is required to be differentiable and free of singularities. $B(r) = \alpha - 2\beta/r$ is the source-free solution. $y_0(0) = 0$ prevents a singularity at the origin. $y_1(0)$ can be chosen to match boundary condition $\alpha = 1$ at r_H.

A solution of equation (1) for $r \le r_H$ is

$$y_0(r) = rB(r) = -\int_0^r wq^2\,\mathrm{d}q + \alpha r - r\int_r^{r_H} wq\,\mathrm{d}q,$$

$$y_1(r) = B(r) + rB'(r) = \alpha - \int_r^{r_H} wq\,\mathrm{d}q. \tag{4}$$

The simple form $a = a_N + \Delta a$ defines RAR correlation function $F(a_N)$ if Δa is a universal constant. Dependence on r or M would produce scatter about such a function plotted as $a = F(a_N)$ (McGaugh et al. 2016). Equations (2) and (4) imply ΛCDM dark matter term $\Delta a = \frac{1}{2}\frac{c^2}{r^2}\int_0^r w_1 q^2\,\mathrm{d}q$. Constant Δa requires $w_1(r) = \mu/r$, where μ is a universal constant. Constant Δa is also implied by the quantized inertia model (McCulloch 2013, 2017).

5 CONFORMAL GRAVITY BACKGROUND

Conformal gravity (CG) modifies the metric field action integral of standard general relativity, replacing the Einstein–Hilbert Ricci scalar by a quadratic contraction of the conformal Weyl tensor (Mannheim & Kazanas 1989; Mannheim 2006). Together with the conformal Higgs model (Nesbet 2011) of dark energy, also without dark matter, this follows a postulate of universal conformal symmetry (Nesbet 2013).

In spherical geometry, the static source-free Schwarzschild potential (Mannheim & Kazanas 1989; Mannheim 2006) is $B(r) = -2\beta/r + \alpha + \gamma r - \kappa r^2$, where all coefficients are constants and $\alpha^2 = 1 - 6\beta\gamma$ (Mannheim & Kazanas 1991). This fourth-order CG equation adds two integration parameters γ and κ to the second-order ΛCDM equation. γ defines non-classical radial acceleration and κ determines a cut-off at the halo boundary (Nesbet 2015). Outside an assumed model spherical source mass, Schwarzschild potential function $B(r)$ determines circular geodesics such that $ra/c^2 = v^2/c^2 = \frac{1}{2}rB'(r) = \beta/r + \frac{1}{2}\gamma r - \kappa r^2$. The Kepler formula is $ra_N/c^2 = \beta/r$. Agreement with standard general relativity for subgalactic phenomena requires $\beta = GM/c^2$.

Observed orbital velocities for 138 galaxies are fitted assuming $\gamma = \gamma_0 + \gamma_G$, where $\gamma_G = N^*\gamma^*$ (Mannheim 1997, 2006) for $N^* = M/M_\odot$. Constants inferred from this rotation data are $\gamma_0 = 3.06 \times 10^{-28}\,\mathrm{m}^{-1}$, $\gamma^* = 5.42 \times 10^{-39}\,\mathrm{m}^{-1}$, and $\kappa = 9.54 \times 10^{-50}\,\mathrm{m}^{-2}$ (Mannheim 1997; Mannheim & O'Brien 2011, 2012; O'Brien & Mannheim 2012; O'Brien & Moss 2015; O'Brien, Chiarelli & Mannheim 2017).

Well inside a galactic halo boundary, $2\kappa r/\gamma$ can be neglected. For $\Delta a = \frac{1}{2}\gamma c^2$ this defines RAR correlation function $F(a_N) = a_N + \Delta a$ if constant γ is mass independent, as indicated by a recent study (Nesbet 2014).

CG fits to galactic orbital velocities (Mannheim 2006; Mannheim & O'Brien 2012) determine γ directly for galactic mass M after scaling by an assumed mass-to-light ratio Υ. Υ is adjusted for each galaxy to make assumed $\gamma = \gamma_0 + N^*\gamma^*$ as consistent as possible for a set of galaxies, with universal constants γ_0 and γ^*. This procedure has been remarkably successful for 138 galaxies (Mannheim 1997; Mannheim & O'Brien 2011, 2012; O'Brien & Mannheim 2012).

Replacing γ_0 by total γ and eliminating γ^* would retain the orbital rotation velocity function to good accuracy. The practical issue is whether or not mass-to-light parameters Υ could be adjusted to give mass-independent γ. The recent study by McGaugh et al. (2016) strongly indicates that this is possible. This study, designed to reduce observational error as much as possible, eliminates the need to adjust Υ for each galaxy. CG can be empirically correct only if γ is mass independent.

6 DETERMINATION OF PARAMETER γ

In the Schwarzschild metric, non-classical CG acceleration parameter γ for a galaxy has been assumed to take the form $\gamma = \gamma_0 + \gamma_G$, where $\gamma_G = N^*\gamma^*$ (Mannheim 1997), proportional to galactic mass

$M = N^* M_\odot$ in solar mass units. Mass-independent γ_0 is attributed to the Hubble flow.

The depleted halo model (Nesbet 2015) justifies this rationale for mass-independent γ_0. One might anticipate a second fundamental constant γ^*, as assumed by Mannheim et al. (Mannheim 1997, 2006; Mannheim & O'Brien 2012; O'Brien & Moss 2015). The RAR (McGaugh et al. 2016) requires $\gamma^* = 0$. If so, γ_0 must be identified with inferred universal constant total γ, in agreement with MOND constant a_0.

A galaxy of mass M can be modelled by spherically averaged mass density $\bar{\rho}_G/c^2$ within radius r_G, formed by condensation of primordial uniform, isotropic matter of mass density ρ_m/c^2 from a sphere of large radius r_H (Nesbet 2015).

The depleted halo model (Nesbet 2015) identifies the dark halo inferred from gravitational lensing and anomalous centripetal acceleration with this depleted sphere.

Given constant mean density $\bar{\rho}_G$ within r_G, this model determines empty halo radius $r_H = r_G(\bar{\rho}_G/\rho_m)^{1/3}$. Empirical parameters γ and κ from Schwarzschild potential $B(r)$ imply halo radius $r_H = \frac{1}{2}\gamma/\kappa$ (Nesbet 2015).

For the Milky Way, $r_H = 33.28 \times 10^{20}$ m $= 107.8$ kpc, compared with $r_G \simeq 15.0$ kpc.

The conformal Friedmann equation (Nesbet 2015), with Friedmann weight parameters Ω_k and Ω_m set to zero (Nesbet 2011), fits observed Hubble function $h(t) = H(t)/H_0$, scaled by Hubble constant H_0, as accurately as ΛCDM, with only one free constant for redshifts $z \leq 1$ (7.33 Gyr) (Nesbet 2010, 2011). This determines Friedmann weights, at present time t_0, $\Omega_\Lambda = 0.732$, $\Omega_q = 0.268$, where acceleration weight $\Omega_q = \frac{\ddot{a}a}{\dot{a}^2}$ and $a(t)$ is the computed Friedmann scale factor (Nesbet 2010, 2011).

A geodesic passing into the empty halo from the surrounding cosmic background is deflected by acceleration proportional to incremental Hubble acceleration (Nesbet 2015) $\Delta\Omega_q = (1 - \Omega_\Lambda)(0) - (1 - \Omega_\Lambda - \Omega_m)(\rho_m) = \Omega_m(\rho_m) = \frac{2}{3}\frac{\bar{\tau}c^2\rho_m}{H_0^2}$ (Nesbet 2011).

Converted from Hubble units, this implies non-classical centripetal acceleration $\frac{1}{2}\gamma c^2 = -cH_0\Omega_m(\rho_m)$ (Nesbet 2015). $\Omega_m < 0$ because non-classical constant $\bar{\tau} < 0$ (Nesbet 2011). This is observed as gravitational lensing and in anomalous orbital rotation velocities.

This logic is equivalent to requiring continuous radial acceleration across halo radius r_H as a boundary condition:

$$\frac{1}{2}\gamma_H c^2 - cH_0\Omega_q(0) = -cH_0\Omega_q(\rho_m). \tag{5}$$

Notation γ_H is used here for the contribution to total acceleration parameter γ arising from the halo boundary. Signs here follow from the definition of Ω_q as centrifugal acceleration weight.

Comparison of conformal theory with observed data depends on exact solutions of the field equations in highly symmetric geometries characterized by two different relativistic metrics. The conformal Higgs model (Nesbet 2011, 2013) has an exact time-dependent, spatially uniform solution in the FLRW metric, which describes Hubble expansion. CG (Mannheim & Kazanas 1989; Mannheim 2006) has an exact solution for spherical symmetry in the static Schwarzschild metric, which describes anomalous galactic rotation.

The equations are decoupled (Nesbet 2014) by separating source mass/energy density ρ into uniform average density $\bar{\rho}$ for the conformal Higgs model and residual density $\hat{\rho} = \rho - \bar{\rho}$ for CG. These solutions must be made consistent by choice of parameters and boundary conditions (Nesbet 2014).

For a spherical solar mass isolated in a galactic halo, $\gamma^* = 0$ results from requiring continuous radial acceleration across boundary radius $r_\odot$ (Nesbet 2014). Mean internal mass density $\bar{\rho}_\odot$ within $r_\odot$ determines an exact solution of the conformal Higgs gravitational equation (Nesbet 2011, 2013), giving internal acceleration weight $\Omega_q(\bar{\rho}_\odot)$. For continuous radial acceleration across $r_\odot$,

$$\frac{1}{2}\gamma_{\odot,\mathrm{in}}c^2 - cH_0\Omega_q(\bar{\rho}_\odot) = \frac{1}{2}\gamma c^2 - cH_0\Omega_q(0), \tag{6}$$

constant $\gamma_{\odot,\mathrm{in}}$ is determined by local mean source density $\bar{\rho}_\odot$, valid inside $r_\odot$. γ is a constant of integration that cannot vary within the source-free halo. Equation (6) does not determine a mass-dependent increment.

Thus galactic γ consists entirely of constant γ_H determined at halo boundary r_H. It is constant and spatially uniform in the source-free space because it depends only on uniform cosmic background density ρ_m and on Hubble constant $H_0 = 2.197 \times 10^{-18}$ s^{-1} (Planck Collaboration XIII 2016).

7 THE TULLY–FISHER RELATION

Static spherical geometry defines Schwarzschild potential $B(r)$. For a test particle in a stable exterior circular orbit with velocity v the centripetal acceleration is $a = v^2(r)/r = \frac{1}{2}B'(r)c^2$. Given $\beta = GM/c^2$, Newtonian $B(r) = 1 - 2\beta/r$ for sufficiently large r, so that $a_N = \beta c^2/r^2 = GM/r^2$.

MOND postulate $a^2 \to a_N a_0$ as $a_N \to 0$ (Milgrom 1983; McGaugh 2011) implies $v^4 = a^2r^2 \to GMa_0$. This supports the empirical baryonic Tully–Fisher relation(Tully & Fisher 1977; McGaugh 2005, 2011).

CG function $B(r)$ determines orbital velocity in the source-free halo $v^2/c^2 = ra/c^2 = \beta/r + \frac{1}{2}\gamma r - \kappa r^2$. For r in a range outside r_G such that Newtonian $ra_N/c^2 \simeq \beta/r$, while $2\kappa r/\gamma$ can be neglected, the slope of $v^2(r)$ vanishes at $r_{TF}^2 = 2\beta/\gamma$. This implies that $v^4(r_{TF})/c^4 = (\beta/r_{TF} + \frac{1}{2}\gamma r_{TF})^2 = 2\beta\gamma$ (Mannheim 1997; Nesbet 2014). This is the Tully–Fisher relation, exact at stationary point r_{TF} of the $v(r)$ function. Given $\beta = GM/c^2$, $v^4 \simeq 2GM\gamma c^2$, for relatively constant $v(r)$ centred at r_{TF}. This derivation holds for CG neglecting κ and for equivalent ΛCDM with source density μ/r. MOND $v^4 = GMa_0$ would be identical if $a_0 = 2\gamma c^2$ (Mannheim 1997). r_{TF} can be defined as the outermost crossing point of the Newtonian and non-classical acceleration functions.

8 DATA FOR MILKY WAY

Given kpc $= 0.30857 \times 10^{20}$ m, $G = 6.674 \times 10^{-11}$ m^3 kg s^{-2}, $c^2 = 8.982 \times 10^{16}$ m^2 s^{-2}, $\gamma = 6.35 \times 10^{-28}$ m^{-1}, and $M = 1.207 \times 10^{41}$ kg (Mannheim 1997, 2006; McGaugh 2008; Mannheim & O'Brien 2011; O'Brien & Moss 2015), then $\beta c^2 = GM = 8.056 \times 10^{30}$ m^3 s^{-2}.

Milky Way Tully–Fisher radius $r_{TF} = 17.2$ kpc, halo radius $r_H = 107.8$ kpc (Nesbet 2015, 2014) for $r_G \simeq 15.0$ kpc. Implied MOND constant $a_0 = 2\gamma c^2 = 1.14 \times 10^{-10}$ m s^{-2}. Outside r_G, $a_N \simeq \beta c^2/r^2$. a (CDM) $= a_N + \frac{1}{2}\gamma c^2$, using empirical CG Δa, a (CG) $= a_N + \frac{1}{2}\gamma c^2(1 - r/r_H)$, including parameter κ, and a (McGaugh et al. 2016) $\simeq a_N/(1 - e^{-\sqrt{a_N/a_0}})$, just MOND with a particular interpolation function and $a_0 = 1.20 \times 10^{-10}$ m s^{-2}. The CDM function is generic for any model with universal constant Δa.

Table 1 compares detailed predictions for the implied nearly flat external orbital velocity curve for the Milky Way galaxy. The CDM curve rises gradually, the CG curve remains remarkably flat, while the MOND (McGaugh et al. 2016) curve falls gradually towards a definite asymptotic velocity.

Table 1. Milky Way: radial acceleration(10^{-10} m s^{-2}).

r (kpc)	a_N	CDM a	$10^3 \frac{v}{c}$	CG a	$10^3 \frac{v}{c}$	MOND a	$10^3 \frac{v}{c}$
15	0.376	0.661	0.584	0.621	0.566	0.877	0.672
20	0.212	0.497	0.584	0.444	0.552	0.617	0.650
25	0.135	0.420	0.601	0.354	0.551	0.475	0.638
30	0.094	0.379	0.625	0.300	0.556	0.385	0.630
35	0.069	0.354	0.652	0.261	0.560	0.324	0.624
40	0.053	0.338	0.682	0.232	0.565	0.279	0.619
45	0.042	0.327	0.717	0.208	0.567	0.246	0.616
50	0.034	0.319	0.740	0.187	0.566	0.219	0.613

9 CONCLUSIONS

ΛCDM, restricted to CDM source density μ/r; CG, restricted to mass-independent non-classical acceleration parameter γ; and MOND, with a particular implied interpolation function (McGaugh et al. 2016), are consistent with the recent RAR (McGaugh et al. 2016) and with other qualitative features of observed stellar-dominated galactic orbital velocities. Velocities exceed the Newtonian value but remain nearly constant for a large range of radii extending into the galactic dark halo. This constant velocity is characterized by the baryonic Tully–Fisher relation (McGaugh 2005, 2011), with v^4 proportional to baryonic galactic mass M. Note that the integrated CDM source density produces a mass-independent constant, consistent with CG non-classical acceleration γ.

If γ is independent of galactic mass, CG is compatible with the RAR (McGaugh et al. 2016). This supports the conclusion that CG determines only mass-independent γ (Nesbet 2014). Dark matter source density μ/r would determine constant Δa in ΛCDM, with the same implications as CG except at large radii, where CG implies effects not described by ΛCDM or MOND. CG orbital velocity drops to zero at an outer boundary (Mannheim & O'Brien 2011), identified as the dark halo radius (Nesbet 2015). CG parameter κ, consistent with the halo radius, does not have a counterpart in ΛCDM or MOND. Distinction between ΛCDM, CG, and MOND requires accurate rotational data at large galactic radii.

ACKNOWLEDGEMENTS

The author is grateful to colleagues Barbara Jones and John Baglin for helpful comments.

REFERENCES

Famaey B., McGaugh S. S., 2012, Living Rev. Relativ., 15, 10
Kroupa P., 2012, preprint (arXiv:1610.03854)
McCulloch M. E., 2013, Europhys. Lett., 101, 59001
McCulloch M. E., 2017, Ap&SS, 362, 149
McGaugh S. S., 2005, Phys. Rev. Lett., 95, 171302
McGaugh S. S., 2008, ApJ, 683, 137
McGaugh S. S., 2011, Phys. Rev. Lett., 106, 121303
McGaugh S. S., Lelli F., Schombert J. M., 2016, Phys. Rev. Lett., 117, 201101
Mannheim P. D., 1997, ApJ, 479, 659
Mannheim P. D., 2006, Progress Part. Nucl. Phys., 56, 340
Mannheim P. D., Kazanas D., 1989, ApJ, 342, 635
Mannheim P. D., Kazanas D., 1991, Phys. Rev. D, 44, 417
Mannheim P. D., O'Brien J. G., 2011, Phys. Rev. Lett., 106, 121101
Mannheim P. D., O'Brien J. G., 2012, Phys. Rev. D, 85, 124020
Milgrom M., 1983, ApJ, 270, 571
Milgrom M., 2016, preprint (arXiv:1609.06642)
Nesbet R. K., 2010, preprint (arXiv:1004.5097)
Nesbet R. K., 2011, Modern Phys. Lett. A, 26, 893
Nesbet R. K., 2013, Entropy, 15, 152
Nesbet R. K., 2014, preprint (arXiv:1410.8076)
Nesbet R. K., 2015, Europhys. Lett., 109, 59001
O'Brien J. G., Mannheim P. D., 2012, MNRAS, 421, 1273
O'Brien J. G., Moss R. J., 2015, J. Phys. Conf., 615, 012002
O'Brien J. G., Chiarelli T. L., Mannheim P. D., 2017, preprint (arXiv:1704.03921)
Planck Collaboration XIII, 2016, A&A, 594, A13
Sanders R. H., 2010, The Dark Matter Problem: A Historical Perspective. Cambridge Univ. Press, New York
Tully R. B., Fisher J. R., 1977, A&A, 54, 661
Weyl H., 1918, Math. Z., 2, 384
Wu X., Kroupa P., 2015, MNRAS, 446, 330

This paper has been typeset from a TEX/LATEX file prepared by the author.