Prog. Theor. Exp. Phys. **2020**, 013E01 (18 pages)
DOI: 10.1093/ptep/ptz143

Maximal efficiency of the collisional Penrose process with a spinning particle. II. Collision with a particle on the innermost stable circular orbit

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Received September 11, 2019; Revised October 31, 2019; Accepted November 1, 2019; Published January 8, 2020

We analyze the collisional Penrose process between a particle on the innermost stable circular orbit (ISCO) orbit around an extreme Kerr black hole and a particle impinging from infinity. We consider both cases with non-spinning and spinning particles. We evaluate the maximal efficiency, $\eta_{\max} = (\text{extracted energy})/(\text{input energy})$, for the elastic collision of two massive particles and for the photoemission process, in which the ISCO particle will escape to infinity after a collision with a massless impinging particle. For non-spinning particles, the maximum efficiency is $\eta_{\max} \approx 2.562$ for the elastic collision and $\eta_{\max} \approx 7$ for the photoemission process. For spinning particles we obtain the maximal efficiency $\eta_{\max} \approx 8.442$ for the elastic collision and $\eta_{\max} \approx 12.54$ for the photoemission process.

Subject Index E00

1. Introduction

Energy extraction from a rotating black hole is one of the most fundamental issues in black hole physics. It may also be important for relativistic astrophysics. The Penrose process, in which we can extract the rotational energy using an ergo region of a black hole plays a key role in the extraction. In particular, the collisional Penrose process may be more important because it may give more efficient energy extraction. Although it has been pointed out that the center of mass energy diverges in the collision at the horizon of an extreme Kerr back hole between two particles impinging from infinity (the Bañados–Silk–West (BSW) effect [1]), it does not mean that one can extract as much energy as one would like because the infinite energy near the horizon will be red-shifted away. Hence, it is important to calculate how much energy we can really extract to infinity. The maximal efficiency $\eta = (\text{extracted energy})/(\text{input energy})$ has been calculated in collisions between particles impinging from infinity by many authors [2–8]. As for BSW effect, there have also been many studies [9–24].

In general relativity, the equation of motion for a spinning test particle is totally different from a test particle, and the effect of spin is non-trivial. Hence, the effect of spin has also been discussed just for collisions between particles impinging from infinity [25,26]. In our previous paper [25] (paper I), we considered collisions between spinning test particles impinging from infinity and discussed the energy efficiency for various processes: elastic collision, Compton scattering, and inverse Compton scattering. In the cases of elastic collision and Compton scattering, we found that those extraction efficiencies are twice as large as in the collision of non-spinning particles.

However, we may have to tune the initial conditions for such collisions to be possible. In order to make the collisional process more realistic, we may discuss a collision between a particle on the innermost stable circular orbit (ISCO) and a particle impinging from infinity. Although it has been shown that the center of mass energy diverges for a collision with a particle on the ISCO in the extreme Kerr black hole [17], the efficiency has not so far been calculated in a collision with a particle on the ISCO. We then consider the collisional Penrose process between a spinning particle on its ISCO and a particle impinging from infinity, and calculate the energy efficiency of the ejected particle. This is the second paper of a series which provides the maximal efficiency of the collisional Penrose process with spinning particles.

In Sect. 2 we briefly summarize the properties of a spinning particle, especially of an ISCO particle, in the extreme Kerr spacetime, and provide the timelike condition of the orbit. In Sect. 3 we study the collision in the extreme Kerr geometry between a particle on the ISCO and a spinning particle plunging from radial infinity, and analyze the maximal efficiency. We also discuss the collision of one spinning massive particle and one massless particle (the photoemission process). Section 4 is devoted to concluding remarks. Throughout this paper we use the geometrical units of $c = G = 1$ and follow Ref. [27] for the notation.

2. Basic equations

2.1. *Equations of motion of a spinning particle*

First, we briefly summarize the equation of motion of a spinning particle moving on the equatorial plane in the Kerr spacetime. We consider only the particle motion on the equatorial plane in an extreme Kerr black hole [28].

Introducing the tetrad basis

$$e_\mu^{\ (a)} = \begin{pmatrix} \frac{r-M}{r} & 0 & 0 & -\frac{M(r-M)}{r} \\ 0 & \frac{r}{r-M} & 0 & 0 \\ 0 & 0 & r & 0 \\ -\frac{M}{r-M} & 0 & 0 & \frac{r^2+M^2}{r-M} \end{pmatrix},$$

we discuss the equation of motion using the tetrad components, which are described by latin indices with a bracket.

We define a specific spin vector $s^{(a)}$ by

$$s^{(a)} = -\frac{1}{2\mu}\epsilon^{(a)}{}_{(b)(c)(d)}u^{(b)}S^{(c)(d)},$$

where μ is the mass of a spinning particle, $u^{(a)}$ is the specific momentum defined by $u^{(a)} := p^{(a)}/\mu$ (the four-momentum $p^{(a)}$ divided by the mass of the spinning particle μ), and $S^{(a)(b)}$ is the spin tensor. $\epsilon_{(a)(b)(c)(d)}$ is the totally antisymmetric tensor with $\epsilon_{(0)(1)(2)(3)} = 1$.

For a spinning particle to move just on the equatorial plane, the direction of spin should be perpendicular to the equatorial plane. Hence we find only one component of $s^{(a)}$ is non-trivial:

$$s^{(2)} = -s.$$

When $s > 0$, the particle spin is parallel to the direction of the black hole rotation, while it is anti-parallel if $s < 0$. From the supplementary condition $S^{(a)(b)}p_{(b)} = 0$, which fixes the center of mass of the spinning particle, the spin tensor is described as

$$S^{(0)(1)} = -sp^{(3)}, \quad S^{(0)(3)} = sp^{(1)}, \quad S^{(1)(3)} = sp^{(0)}.$$

For a Killing vector $\xi_{(a)}$ we find a conserved quantity of the spinning particle,

$$Q_\xi = p^{(a)}\xi_{(a)} + \frac{1}{2}S^{(a)(b)}(w_{(a)(b)(c)}\xi^{(c)} + e^{\mu}{}_{(a)}\xi_{(b),\mu}),$$

where the Ricci rotation coefficient is defined by $w_{(a)(b)(c)} := e_{(c)\mu;\nu}e^{\mu}_{(b)}e^{\nu}_{(a)}$. Since there are two Killing vectors in the present spacetime, we have two conserved quantities: the energy E and the total angular momentum J along the world line of the spinning particle,

$$E = \frac{r-M}{r}p^{(0)} + \frac{M(r+s)}{r^2}p^{(3)}, \tag{1}$$

$$J = \frac{r-M}{r}(M+s)p^{(0)} + \frac{r(r^2+M^2)+Ms(r+M)}{r^2}p^{(3)}. \tag{2}$$

In what follows, we shall use normalized variables,

$$\tilde{E} = \frac{E}{\mu}, \quad \tilde{J} = \frac{J}{\mu M}, \quad \tilde{s} = \frac{s}{M},$$

$$\tilde{t} = \frac{t}{M}, \quad \tilde{r} = \frac{r}{M}, \quad \tilde{\tau} = \frac{\tau}{M},$$

where τ is the proper time. We will drop the tilde for brevity.

From Eqs. (1) and (2), we find that

$$u^{(0)} = \frac{[r^3+(1+s)r+s]E-(r+s)J}{r^2(r-1)\left(1-\frac{s^2}{r^3}\right)}, \tag{3}$$

$$u^{(3)} = \frac{J-(1+s)E}{r\left(1-\frac{s^2}{r^3}\right)}. \tag{4}$$

Since we have the normalization condition $u_{(a)}u^{(a)} = -1$, the radial component of the specific momentum is given by

$$u^{(1)} = \sigma\sqrt{(u^{(0)})^2-(u^{(3)})^2-1},$$

where $\sigma = \pm 1$ correspond to the outgoing and ingoing motions, respectively.

We should note that the four-velocity $v^{(a)} = dz^{(a)}/d\tau$, in which $z(\tau)$ is an orbit of a particle, is not always parallel to the specific four-momentum $u^{(a)}$. When we normalize the affine parameter τ as

$$u^{(a)}v_{(a)} = -1, \tag{5}$$

the difference between $v^{(a)}$ and $u^{(a)}$ is given by

$$v^{(a)} - u^{(a)} = \frac{S^{(a)(b)}R_{(b)(c)(d)(e)}u^{(c)}S^{(d)(e)}}{2(\mu^2+\frac{1}{4}R_{(b)(c)(d)(e)}S^{(b)(c)}S^{(d)(e)})}. \tag{6}$$

For the present setting, this relation between the four-velocity and the specific four-momentum is reduced to

$$v^{(0)} = \Lambda_s^{-1}u^{(0)}, \quad v^{(1)} = \Lambda_s^{-1}u^{(1)}, \quad v^{(3)} = \frac{\left(1+\frac{2s^2}{r^3}\right)}{\left(1-\frac{s^2}{r^3}\right)}\Lambda_s^{-1}u^{(3)},$$

where

$$\Sigma_s = r^2\left(1 - \frac{s^2}{r^3}\right), \quad \Lambda_s = 1 - \frac{3s^2 r[J-(1+s)E]^2}{\Sigma_s^3}.$$

Hence, we finally obtain the equations of motion of the spinning particle as

$$\Sigma_s\Lambda_s\frac{dt}{d\tau} = \left(1 + \frac{3s^2}{r\Sigma_s}\right)[J-(1+s)E] + \frac{r^2+1}{(r-1)^2}P_s,$$
$$\Sigma_s\Lambda_s\frac{dr}{d\tau} = \pm\sqrt{R_s},$$
$$\Sigma_s\Lambda_s\frac{d\phi}{d\tau} = \left(1 + \frac{3s^2}{r\Sigma_s}\right)[J-(1+s)E] + \frac{1}{(r-1)^2}P_s,$$

where

$$P_s = \left[r^2 + 1 + \frac{s}{r}(r+1)\right]E - \left(1 + \frac{s}{r}\right)J,$$
$$R_s = P_s^2 - (r-1)^2\left[\frac{\Sigma_s^2}{r^2} + [-(1+s)E + J]^2\right].$$

By using Σ_s and R_s, the radial component of the specific momentum is written as

$$u^{(1)} = \sigma\frac{r\sqrt{R_s}}{(r-1)\Sigma_s}. \tag{7}$$

2.2. *The innermost stable circular orbit*

Based on the analysis in Ref. [29], we consider the ISCO of a spinning particle; there are some analyses of this orbit in Refs. [30,31]. Since the orbit of a particle on the ISCO is circular, $v^{(1)}$ vanishes and then so does $u^{(1)}$. We then find a constraint on the conserved quantities E and J from Eqs. (3) and (4) as follows: We first write Eq. (7) as

$$(u^{(1)})^2 = A(E - U_{(+)})(E - U_{(-)}), \tag{8}$$

where

$$A(r,s) = \frac{r^2 B}{(r-1)^2\Sigma_s^2},$$
$$B(r,s) = \left[r^2 + 1 + s\left(1 + \frac{1}{r}\right)\right]^2 - (r-1)^2(1+s)^2,$$

and

$$U_{(\pm)}(r,J,s) = XJ \pm \sqrt{(X^2 - Y)J^2 - Z}, \tag{9}$$

with

$$X(r,s) = \frac{1}{B}\left[\left\{r^2 + 1 + s\left(1 + \frac{1}{r}\right)\right\}\left(1 + \frac{s}{r}\right) - (r-1)^2(1+s)\right],$$
$$Y(r,s) = \frac{1}{B}\left[\left(1 + \frac{s}{r}\right)^2 - (r-1)^2\right],$$
$$Z(r,s) = -\frac{1}{B}\frac{(r-1)^2\Sigma_s^2}{r^2}.$$

We can regard $U_{(\pm)}(r, J, s)$ as the effective potential of a particle on the equatorial plane. Since $U_{(-)}(r)$ usually does not have an extremum and is much less than unity, we shall consider only $U_{(+)}(r)$. For a circular orbit with the radius $r = r_0$, the following conditions must be satisfied:

$$U_{(+)}(r_0) = E \quad \text{and} \quad \frac{dU_{(+)}}{dr}(r_0) = 0.$$

In addition, for the stability of the orbit we have to impose

$$\frac{d^2U_+}{dr^2}(r_0) > 0.$$

Since the ISCO is the inner boundary of stable circular orbits, it must satisfy

$$U_{(+)}(r_{\rm ISCO}) = E, \quad \frac{dU_{(+)}}{dr}(r_{\rm ISCO}) = 0, \quad \frac{d^2U_+}{dr^2}(r_{\rm ISCO}) = 0, \tag{10}$$

where $r_{\rm ISCO}$ is the ISCO radius. From the three conditions in Eq. (10), we find the energy $E_{\rm ISCO}$ and angular momentum $J_{\rm ISCO}$ of the particle on the ISCO, and the ISCO radius $r_{\rm ISCO}$, in terms of the spin s. In the extreme Kerr spacetime, we find $r_{\rm ISCO} = 1$ (the horizon radius), and then

$$E_{\rm ISCO} = \frac{(1 - s^2)}{\sqrt{3(1 + 2s)}}, \tag{11}$$

$$J_{\rm ISCO} = 2E_{\rm ISCO}. \tag{12}$$

From Eq. (6), we find that

$$v^{(0)}_{\rm ISCO} = \frac{u^{(0)}_{\rm ISCO}}{\tilde{\delta}}\left(1 - \frac{s^2}{r^3}\right), \quad v^{(3)}_{\rm ISCO} = \frac{u^{(3)}_{\rm ISCO}}{\tilde{\delta}}\left(1 + \frac{2s^2}{r^3}\right),$$

where

$$\tilde{\delta} = 1 - \frac{s^2}{r^3}\left[1 + 3(u^{(3)}_{\rm ISCO})^2\right].$$

2.3. *Constraints on the orbits*

Here, we stress two important points:

(1) In order for a particle to reach the horizon $r_{\rm H} := 1$, the radial function R_s must be non-negative for $r \geq r_{\rm H}$.
(2) As seen in the previous section, the four-velocity $v^{(a)}$ is not always parallel to the specific four-momentum. Hence, we have to impose the timelike condition $v^{(a)}v_{(a)} < 0$ even though $u^{(a)}u_{(a)} = -1$ is imposed.

Before discussing the collisional Penrose process, we must treat these points properly. Since we have already discussed these points for particles plunging from infinity in Ref. [25], we shall discuss only the case for the ISCO particle. For the first condition, $R_s(r_{\rm ISCO}) = 0$ always holds by definition, so it is always satisfied.

As for the timelike condition $v^{(a)}v_{(a)} < 0$, we find that

$$E^2_{\rm ISCO} < \frac{(1 - s)^2(1 + s)^4}{3s^2(2 + s^2)}. \tag{13}$$

Since the conserved energy $E_{\rm ISCO}$ is a function of the spin s, this inequality is reduced to

$$(1-s^2)^2(s^4-2s^3-3s^2-4s-1)<0.$$

From this inequality, we obtain the constraint of the spin s for the ISCO particle, i.e.,

$$s_{\rm min}^{\rm ISCO} \le s \le s_{\rm max}^{\rm ISCO}, \tag{14}$$

where $s_{\rm min}^{\rm ISCO} \approx -0.302776$ and $s_{\rm max}^{\rm ISCO}=1$. The energy of the ISCO particle is bounded as

$$0 \le E_{\rm ISCO} \le E_{\rm ISCO}^{\rm max} \approx 0.8349996.$$

Note that for a particle plunging from infinity we have the constraint that

$$s_{\rm min} \le s \le s_{\rm max},$$

where $s_{\rm min} \approx -0.2709$ and $s_{\rm max} \approx 0.4499$ are the solutions of $s^6+2s^5-4s^4-4s^3-7s^2+2s+1=0$ with the condition $-1 \le s_{\rm min} < s_{\rm max} \le 1$.

3. Maximal efficiency of collision of particles

Now, we discuss the collision of two particles, 1 and 2, whose four-momenta are $p_1^{(a)}$ and $p_2^{(a)}$. We then assume that particle 1 is on the ISCO, while particle 2 is impinging from infinity. In Appendix A we show that the center of mass energy can take arbitrary values in the collision between these two spinning particles just as in the usual BSW effect. We will then analyze the maximal efficiency of the energy extraction, i.e. how much energy we can extract from an extreme black hole. For this purpose we shall follow the same procedure as in paper I. The difference is that particle 1 is moving on the ISCO orbit with the radius $r_{\rm ISCO}$. In the extreme Kerr spacetime, the ISCO radius is $r_{\rm ISCO}=1$. Hence, the collision must take place very close to the horizon ($r_{\rm H}=1$). We then assume that the collisional point is given by $r_{\rm c}=1/(1-\epsilon)$ ($0<\epsilon \ll 1$).

At the collisional point $r_{\rm c}$, we impose the following conservations:

$$\begin{aligned} E_1+E_2 &= E_3+E_4,\\ J_1+J_2 &= J_3+J_4,\\ s_1+s_2 &= s_3+s_4,\\ p_1^{(1)}+p_2^{(1)} &= p_3^{(1)}+p_4^{(1)}. \end{aligned}$$

After the collision, we assume that particle 3 with the four-momentum $p_3^{(a)}$ is going away to infinity, while particle 4 with the four-momentum $p_4^{(a)}$ falls into the black hole. In order for particle 2 to reach the horizon from infinity, it must be subcritical ($J_2<2E_2$) and ingoing ($\sigma_2=-1$). So we assume that $\sigma_4=\sigma_2=-1$.

We then expand the radial component of the four-momentum $p^{(1)}$ in terms of ϵ as

$$p^{(1)} \approx \sigma \frac{|2E-J|}{\epsilon(1-s)}+\cdots$$

The conservation of the radial components of the momenta ($p_1^{(1)}+p_2^{(1)}=p_3^{(1)}+p_4^{(1)}$) yields

$$-\frac{|2E_2-J_2|}{1-s_2}=\sigma_3\frac{|2E_3-J_3|}{1-s_3}-\frac{|2E_4-J_4|}{1-s_4}+O(\epsilon), \tag{15}$$

in which we use $J_1 = 2E_1$. Here, we also assume $s_2 = s_4$ for simplicity. This means that particle 1 becomes particle 3 without change of the spin angular momentum after the collision. Then, the above conservation is written as

$$\left[\sigma_3 \frac{\mathrm{sign}[2E_3 - J_3]}{1 - s_3} + \frac{1}{1 - s_4}\right](2E_3 - J_3) = O(\epsilon).$$

The above setting gives

$$J_3 = 2E_3(1 + \alpha_3\epsilon + \beta_3\epsilon^2 + \cdots), \tag{16}$$

where α_3 and β_3 are parameters of $O(\epsilon^0)$. Since particle 2 is subcritical ($J_2 < 2E_2$), the angular momentum J_2 is written as

$$J_2 = 2E_2(1 + \zeta), \tag{17}$$

where $\zeta < 0$ with $\zeta = O(\epsilon^0)$. From the conservation laws, we find that

$$E_4 = E_1 + E_2 - E_3, \quad J_4 = J_1 + J_2 - J_3, \tag{18}$$

giving

$$J_4 = 2E_4\left(1 + \frac{E_2}{E_4}\zeta + \cdots\right).$$

Hereafter, we consider two cases:

[A] collision of two massive particles (MMM), and
[B] collision of massless and massive particles (MPM),

where we use the symbols MMM and MPM following Ref. [5]. In the case of [A] (MMM), we assume all the masses of the particles are the same, i.e. $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu$. In the case of [B] (MPM), particles 2 and 4 are massless and non-spinning, because a photon has no stable circular orbit, while particles 1 and 3 are massive with the same mass, i.e. $\mu_1 = \mu_3 = \mu$. From the conservation of spin, we obtain $s_1 = s_3$. Since the massive particle is ejected by the incoming photon, we shall call process [B] (MPM) photoemission. Note that in the first paper we called the process inverse Compton scattering because a massive particle with large energy is going out after the collision.

Next, we evaluate E_2 and E_3 for the cases [A] and [B] separately.

3.1. *Maximal efficiency in case [A] (MMM: collision of two massive particles)*

For the massive particle, the radial component of the specific four-momentum is given as

$$\begin{aligned} u^{(1)} &= \sigma\frac{r\sqrt{R_s}}{\Sigma_s\sqrt{\Delta}} \\ &= \frac{\sigma\sqrt{r^2[(r^3 + (1+s)r + s)E - (r+s)J]^2 - (r-1)^2[(r^3 - s^2)^2 + r^4(J - (1+s)E)^2]}}{(r-1)(r^3 - s^2)}. \end{aligned} \tag{19}$$

By expanding $u^{(1)}$ with the conditions in Eqs. (16) and (17) in terms of ϵ, we find Eqs. (3.15)–(3.17) in paper I. Note that for particle 1 on the ISCO, $u_1^{(1)} = 0$.

Since $u_1^{(1)} + u_2^{(1)} = u_3^{(1)} + u_4^{(1)}$, we find that the leading order of ϵ^{-1} is trivial. From the next to leading order, i.e. $O(\epsilon^0)$, we find

$$\sigma_3 \frac{f(s_1, E_3, \alpha_3)}{1 - s_1^2} = \frac{[E_1(2 + s_2) - E_3 g_1(s_2, \alpha_3)]}{1 - s_2^2},$$

where

$$f(s, E, \alpha) := \sqrt{E^2[3 - 2\alpha(1 + s)][1 + 2s - 2\alpha(1 + s)] - (1 - s^2)^2},$$
$$g_1(s, \alpha) := 2 + s - 2\alpha(1 + s).$$

This equation is reduced to

$$\mathcal{A}E_3^2 - 2\mathcal{B}E_3 + \mathcal{C} = 0, \tag{20}$$

where

$$\mathcal{A} = -[3 - 2\alpha_3(1 + s_1)][1 + 2s_1 - 2\alpha_3(1 + s_1)] + \frac{(1 - s_2^2)^2}{(1 - s_1^2)^2} g_1^2(s_2, \alpha_3), \tag{21}$$

$$\mathcal{B} = g_1(s_2, \alpha_3) \frac{(1 - s_2^2)^2}{(1 - s_1^2)^2}(2 + s_2)E_1, \tag{22}$$

$$\mathcal{C} = \frac{(1 - s_2^2)^2}{(1 - s_1^2)^2}(2 + s_2)^2 E_1^2 + (1 - s_1^2)^2, \tag{23}$$

with the condition that $E_3 \leq E_{3,\mathrm{cr}}$ for $\sigma_3 = 1$, or $E_3 \geq E_{3,\mathrm{cr}}$ for $\sigma_3 = -1$, where

$$E_{3,\mathrm{cr}} := \frac{2 + s_2}{g_1(s_2, \alpha_3)} E_1.$$

In the case of $\sigma_3 = 1$, we do not expect large efficiency since the energy E_3 has the upper bound $E_{3,\mathrm{cr}}$, whose magnitude is of the order of E_1. In fact, we find that the efficiency for the case of $\sigma_3 = 1$ is not so high in Appendix B. Hence, we will focus on the case of $\sigma_3 = -1$, i.e. particle 3 is assumed to be ingoing after the collision. For particle 3 to go back to infinity the orbit must be supercritical ($J_3 > 2E_3$), which means either $\alpha_3 > 0$ or $\alpha_3 = 0$ with $\beta_3 > 0$.

From Eq. (21) we obtain the larger output energy E_3 in terms of s_1, s_2, and α_3:

$$E_3 = E_{3,+} := \frac{\mathcal{B} + \sqrt{\mathcal{B}^2 - \mathcal{A}\mathcal{C}}}{\mathcal{A}}. \tag{24}$$

From the next to leading order terms, we obtain

$$\mathcal{P}E_2 = (1 - s_2)^3 (E_3 - E_1)^2, \tag{25}$$

where

$$\begin{aligned}\mathcal{P} :&= 2(E_3 - E_1)(1 - s_2)^3 \\ &\quad + 4\zeta \Bigg\{ 2(1 + s_2)E_3[\alpha_3(2 + s_2) - \beta_3(1 - s_2^2)] - s_2(2 + s_2)^2(E_3 - E_1) \\ &\quad - \sigma_3 \frac{(1 - s_2^2)^2}{(1 - s_1^2)^2} \Bigg[\frac{E_3^2}{f(s_1, E_3, \alpha_3)} \Big(h(s_1) - 2(1 + s_1)^2(2 + s_1) g_2(s_1, \alpha_3) \\ &\quad + 2\beta_3(1 + s_1)(1 - s_1^2) g_1(s_1, \alpha_3) \Big) \Bigg] \Bigg\},\end{aligned} \tag{26}$$

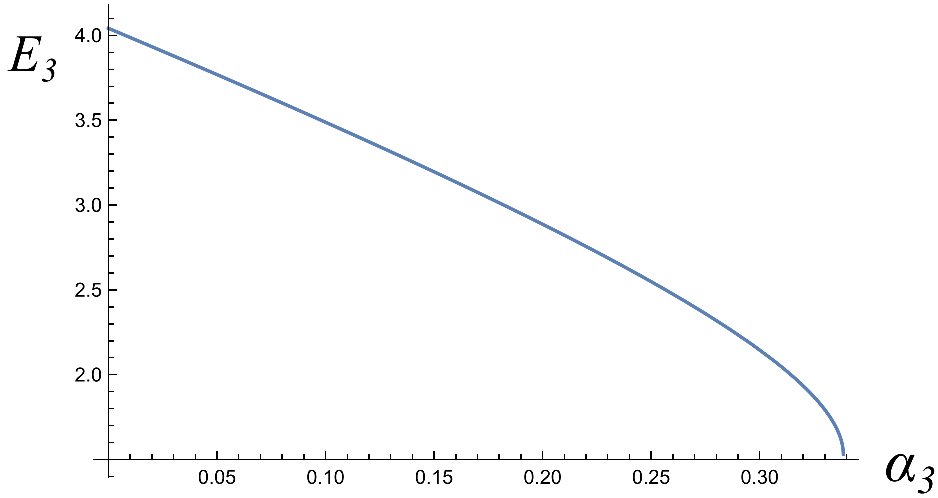

Fig. 1. The energy of particle 3 (E_3) in terms of α_3. The maximal value of this is given at $\alpha_3 = 0$.

in which

$$g_2(s, \alpha) := \alpha(2 + s - 2\alpha).$$

Since this fixes the value of E_2, we obtain the efficiency by

$$\eta = \frac{E_3}{E_1 + E_2}$$

when α_3, β_3, and ζ are given.

3.1.1. Non-spinning particles

We first consider the collision of non-spinning particles: $s_1 = s_2 = 0$. In this case, the energy of particle 1 (E_1) is $1/\sqrt{3}$. Since particle 2 is plunging from infinity, we have the constraint of $E_2 \geq 1$ for a massive particle. Hence, in order to obtain the maximal efficiency, we need to obtain the maximal energy of particle 3 and the minimal energy of particle 2, i.e. $E_2 = 1$.

The energy of particle 3 is given by

$$E_3 = \frac{1}{\sqrt{3}}\left[4(1 - \alpha_3) + \sqrt{4(3 - 2\alpha_3)(1 - 2\alpha_3) - 3}\right].$$

Then we find that from Fig. 1, the maximal value of the energy of particle 3 is given at $\alpha_3 = 0+$.

Next, when $\alpha_3 = 0+$ the energy of particle 2 is given by

$$E_2 = \frac{36\sqrt{3}}{36 + 7\zeta(4\beta_3 + 7)}, \tag{27}$$

which becomes unity when β_3 is taken to be

$$\beta_3 = \frac{1}{4}\left(\frac{36(\sqrt{3} - 1)}{7\zeta} - 7\right).$$

As a result, we obtain the maximal efficiency for the collision of non-spinning particles, $\eta_{\max} = 7(\sqrt{3} - 1)/2 \approx 2.562$.

3.1.2. Spinning particles

Next, we consider the collision of spinning particles. In our approach, the energy of particle 1 (E_1) is a function of s_1, and the energy of particle 2 (E_2) is a function of s_1, s_2, α_3, β_3, and ζ. On the other hand, the energy of particle 3 (E_3) is a function of s_1, s_2, and α_3. When particle 1 comes from infinity, as in the previous paper, we can look on E_1 as a parameter and treat E_1 and E_3 separately. However, in the current setup we must modify how to derive the maximal efficiency since E_1 is a function of s_1. This is the main difference from the analysis in the previous paper.

When the energy of particle 1 (E_1) is fixed for a given value of s_1, it is clear that the larger efficiency is obtained for larger E_3 as well as smaller E_2. Hence, we expect that the efficiency for an elastic scattering could take the maximum value as $E_3/(E_1+1)$, which depends on s_1, s_2, and α_3, because we have the constraint for particle 2, $E_2 \geq 1$, as for the non-spinninig particles. In order to justify this expectation, it is necessary to show that $E_2 = 1$ is possible for some choices of the remaining parameters (ζ and β_3) after finding s_1, s_2, and α_3 giving the maximal efficiency of $E_3/(E_1+1)$.

As for the energy of particle 3 (E_3), since the orbit of particle 3 is near critical, in order to determine E_3 from Eq. (24) properly we have to consider two constraints: $E_3 \geq E_{3,\mathrm{cr}}$ for $\sigma_3 = -1$ and the timelike condition in Eq. (13). Due to the timelike condition, in order to find the large value of E_3, the spin magnitude s_3 ($= s_1$) must be small. For a small value of s_1, we find $\alpha_3 \approx 0$ gives the largest efficiency of $E_3/(E_1+1)$.

Hence, setting $\alpha_3 = 0+$, we analyze the maximal efficiency in terms of s_1 and s_2. In Fig. 2 we show a contour map of $E_3/(E_1+1)$ in terms of s_1 and s_2. Form this figure we find that the red point, which is $(s_1, s_2) \approx (0.03196, s_{\min})$, gives the maximal value of $E_3/(E_1+1)$.

Then, we have to confirm that $E_2 = 1$ is possible for certain values of the remaining parameters (ζ and β_3). If the condition $E_2 = 1$ is satisfied, we obtain the relation between ζ and β_3 from Eq. (25), which is a linear equation in β_3. As for ζ, since the particle 2 also has spin s_2, we have to consider the timelike condition for the particle 2 orbit. This gives the constraint on ζ of $\zeta_{\min} < \zeta$, where

$$\zeta_{\min} := -\frac{(1-s_2)}{2}\left[1 + \frac{(1-s_2)(1+s_2)^2}{\sqrt{3s_2^2(2+s_2^2)}}\right].$$

For the parameters giving the maximal value of $E_3/(E_1+1)$ we obtain $\zeta_{\min} \approx -1.271$. Under this constraint, we find in Fig. 3 the relation giving the condition $E_2 = 1$ between ζ and β_3.

Thus, we find the maximal efficiency is given by

$$\eta_{\max} = \frac{E_3}{E_1+1} \approx 8.442. \tag{28}$$

3.2. Maximal efficiency in case [B] (MPM: photoemission)

For the collision of a massless particle (photon) and a massive particle, we should assume that particle 1 (the ISCO particle) is massive and the incoming particle is massless because there is no ISCO for massless particles. Hence, we consider the photoemission process, i.e. a massive particle is emitted via a collision process by an incoming photon.

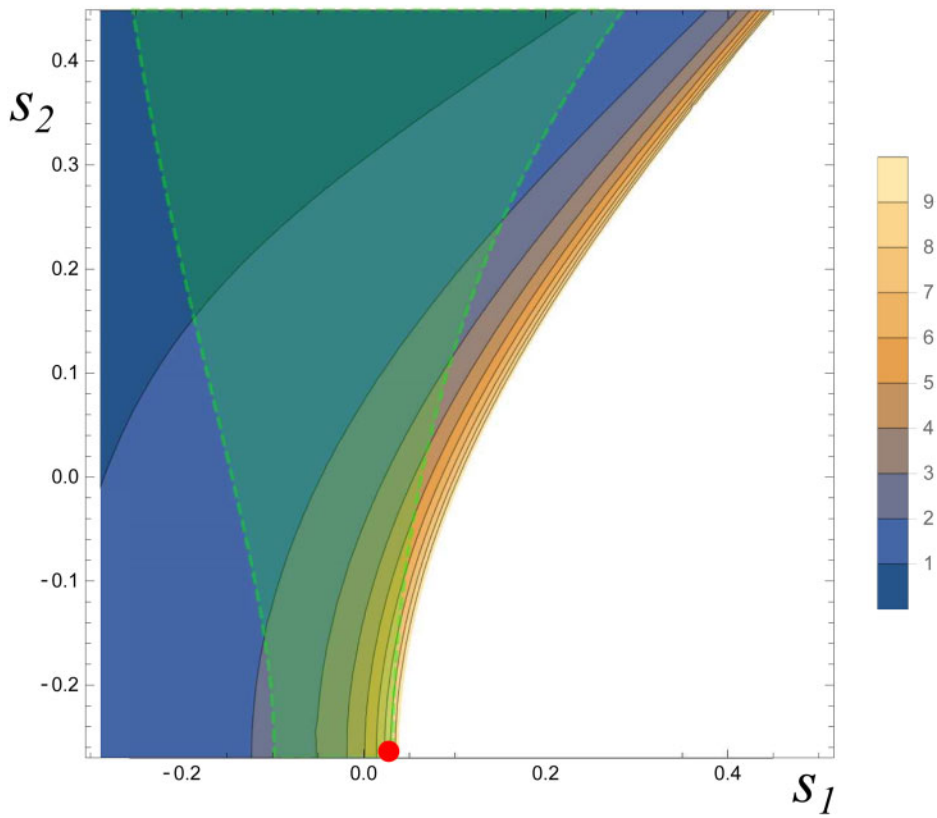

Fig. 2. The contour map of $E_3/(E_1+1)$ in terms of s_1 and s_2 when we set $\alpha_3 = 0+$. In the light green region, the timelike condition for the particle 3 orbit is satisfied. The maximal value of $E_3/(E_1+1) \approx 8.442$ is obtained when $s_2 = s_{\min} \approx -0.2709$ and $s_1 \approx 0.03196$ (the red point in the figure).

For the momenta of the massive particles 1 and 3, the radial components of the four-momentua do not change, while for the massless particles 2 and 4, we find

$$p_2^{(1)} = 2\epsilon^{-1}E_2\zeta - 2E_2(1+2\zeta) - \epsilon\frac{E_2(1-4\zeta^2)}{4\zeta} + O(\epsilon^2), \tag{29}$$

$$p_4^{(1)} = 2\epsilon^{-1}E_2\zeta - 2\left[E_4 + 2E_2\zeta + E_3\alpha_3\right] - \epsilon\frac{E_4^2 - 8E_2E_3(2\alpha_3-\beta_3)\zeta - 4E_2^2\zeta^2}{4E_2\zeta} + O(\epsilon^2), \tag{30}$$

where $E_4 = E_1 + E_2 - E_3$.

From the conservation of the radial components of the four-momenta, we find

$$E_3 = \left.\frac{\mathcal{B} + \sqrt{\mathcal{B}^2 - \mathcal{A}\mathcal{C}}}{\mathcal{A}}\right|_{s_2=0}, \tag{31}$$

and

$$E_2 = \left.\frac{(E_3-E_1)^2}{\mathcal{P}}\right|_{s_2=0}, \tag{32}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{P}$ are given by Eqs. (21), (22), (23), and (26), evaluated with $s_2 = 0$. Note that the representations for E_2 and E_3, i.e. Eqs. (31) and (32), respectively coincide with Eqs. (24) and (25) for the collision of a spinning massive particle and a non-spinning massive particle.

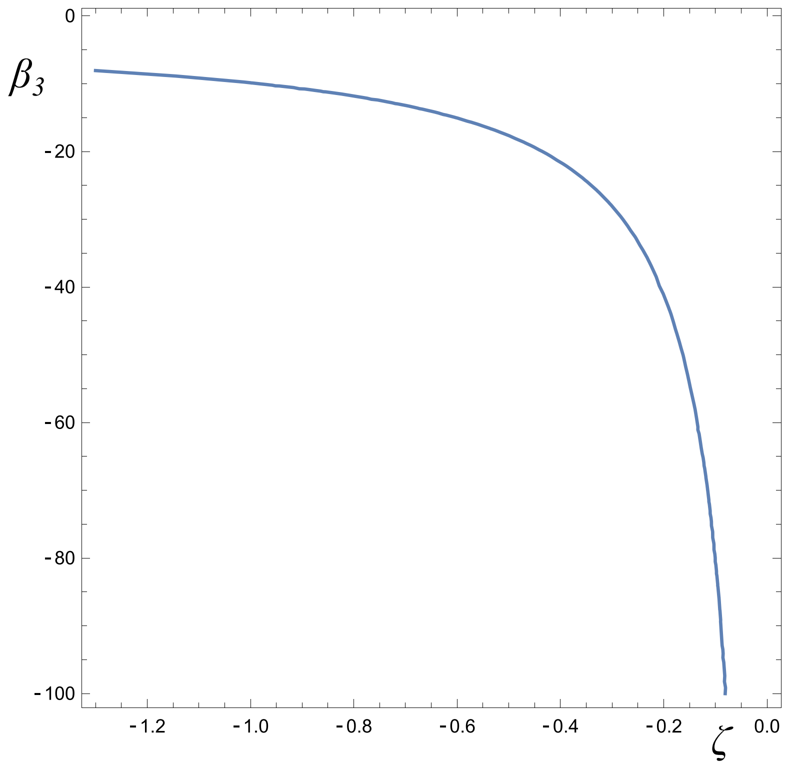

Fig. 3. The relation giving $E_2 = 1$ between ζ and β_3. The other parameters (s_1, s_2, and α_3) are chosen to give the maximal value of $E_3/(E_1 + 1)$. The timelike condition for the particle 2 orbit gives the constraint of $\zeta_{\rm min} < \zeta < 0$, with $\zeta_{\rm min} \approx -1.271$.

3.2.1. Non-spinning particles

We consider the collision between a non-spinning massive particle and a massless particle: $s_1 = s_2 = 0$. As the non-spinning particles in the elastic collision, the energy of particle 1 (E_1) is given by $1/\sqrt{3}$ and the maximal value of the energy of particle 3 is given at $\alpha_3 = 0+$. On the other hand, the energy of particle 2 is given by Eq. (27). Since particle 2 plunges from infinity, we have the constraint for the energy of particle 2 that $E_2 \geq 0$. When we take the limit $\zeta\beta_3 \to \infty$ in Eq. (27), for the minimal energy of particle 2 $E_2 \to 0$ is obtained. Thus, we obtain the maximal efficiency as $\eta_{\rm max} = 7$.

3.2.2. Spinning particle + massless particle

As describe before, when the energy of the particle E_1 is fixed, larger efficiency is obtained for larger E_3 as well as smaller E_2. Hence, we first discuss E_3/E_1, which depends on α_3 and s_1, because we cannot treat E_1 and E_3 separately and have the constraint $E_2 \geq 0$ for a massless particle. After finding these parameters to give the maximum value of E_3/E_1, we need to confirm that the condition $E_2 \to 0$ is possible for the remaining parameters (ζ and β_3). If the condition $E_2 \to 0$ is possible for the remaining parameters, the maximal efficiency is really given by $\eta_{\rm max} = E_3/E_1$.

The energy ratio E_3/E_1 depends on only the two parameters α_3 and s_1, and we can directly draw the contour map as in Fig. 4. From this figure we find that the red point, which is $(\alpha_3, s_1) = (0, 0.06360)$, gives the maximal value of E_3/E_1.

Next, we confirm that $E_2 \to 0$ is possible for the remaining parameters (ζ and β_3) after we set $(\alpha_3, s_1) = (0+, 0.06360)$ giving the maximal value of E_3/E_1. As for the remaining parameters, ζ is

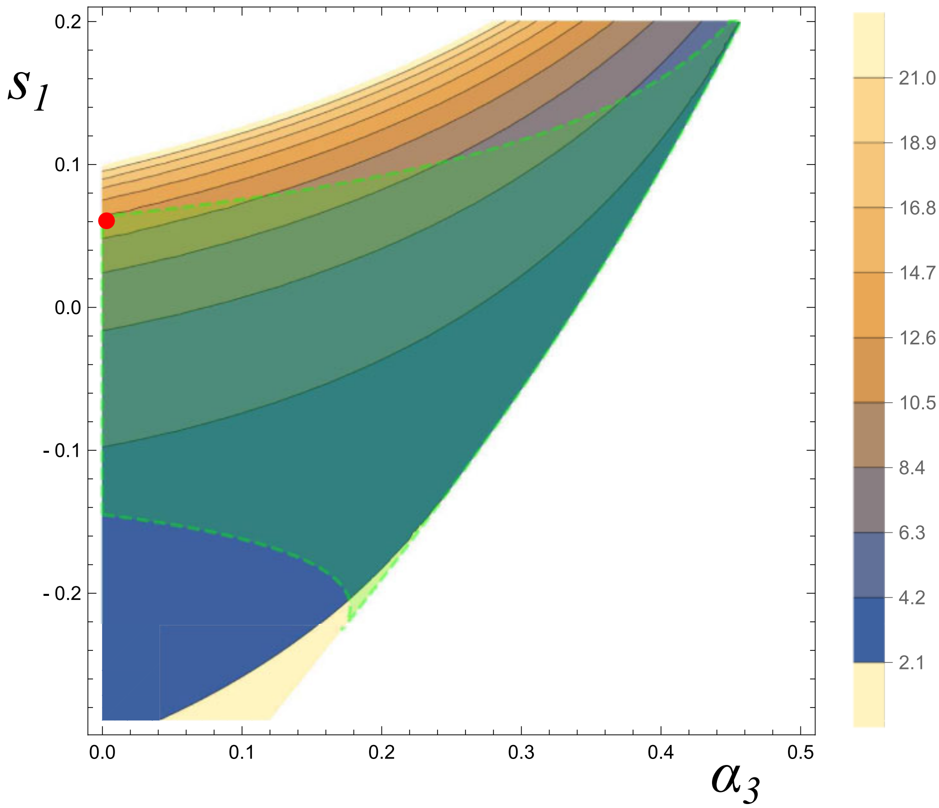

Fig. 4. The contour map of E_3/E_1 in terms of α_3 and s_1. In the light-green region, the timelike condition for particle 3 is satisfied. The maximum value of $E_3/E_1 = 12.54$ is obtained at the red point $(\alpha_3, s_1) = (0+, 0.06360)$.

constrained as $-\infty < \zeta < 0$ because particle 2 is non-spinning, while β_3 takes an arbitrary value as long as $\alpha_3 > 0$ is satisfied. Under these constraints, we find the asymptotic behavior of $\mathcal{P}$ from Eq. (26) when $\zeta\beta_3 \to \infty$:

$$\mathcal{P} \approx 8E_3\zeta\beta_3\left[\frac{E_3(2+s_1)}{(1-s_1)f(s_1, E_3, 0)} - 1\right].$$

To take this limit, β_3 must be negative. From this asymptotic representation it is easy to find that $E_2 \to 0$ when $\zeta\beta_3 \to \infty$.

Thus, we find the maximum efficiency $\eta_{\max} \approx 12.54$ for the photoemission process.

4. Concluding remarks

We have studied the collisional Penrose process for non-spinning and spinning particles around an extreme Kerr black hole. For the collision between a particle on the ISCO orbit and a particle impinging from infinity, we have evaluated the maximal efficiency of the energy extraction. We summarize our present result as well as the previous one from paper I [25] in Table 1.

In the non-spinning case we find that the maximal efficiency is 2.562 for the elastic collision, and 7 for the photoemission process. For spinning particles, we obtain the maximal efficiency in the elastic scattering (MMM0) as $\eta_{\max} \approx 8.442$, and $\eta \approx 12.54$ for the photoemission process (MPM0). When we take spin into account, we find that the efficiency becomes larger both in the elastic collision and for the photoemission process.

Table 1. The maximal efficiencies and energies for three collisions between a particle in its ISCO and a particle impinging from infinity. We include the previous result for the non-ISCO orbit case. Following Ref. [5], we use the symbols of MMM0, PMP0, MPM0 for each process, where "0" means a collision with a particle on the ISCO orbit. We also include the cases of three collisions with particles impinging from infinity discussed in paper I [25] as a reference. The maximal efficiencies and maximal energies are always enhanced when the spin effect is taken into account.

Collisional process		Spin $(s_{\rm ISCO}\ (\text{or}\, s_1), s_2)$	Input energy $(E_{\rm ISCO}, E_2)$	Output energy (E_3)	Maximal efficiency
Collision of two massive particles	MMM0	non-spinning	$(0.5773\mu, \mu)$	4.041μ	2.562
	ISCO	$(0.03196\mu M, -0.2709\mu M)$	$(0.5591\mu, \mu)$	13.16μ	8.442
	MMM+	non-spinning	(μ, μ)	12.66μ	6.328
	non-ISCO	$(0.01379\mu M, -0.2709\mu M)$		30.02μ	15.01
Photoemission	MPM0	non-spinning	$(0.5773\mu, 0)$	4.041μ	7
	ISCO	$(0.06360\mu M, 0)$	$(0.5416\mu, 0)$	6.791μ	12.54
Compton scattering	PMP+	non-spinning	$(+\infty, \mu)$	$+\infty$	13.93
	non-ISCO	$(0, -0.2709\mu M)$		$+\infty$	26.85
Inverse Compton scattering	MPM+	non-spinning	$(\mu, 0)$	12.66μ	12.66
	non-ISCO	$(0.02679\mu M, 0)$		15.64μ	15.64

Note that for the collision between particles impinging from infinity, the maximal efficiency becomes the largest in the Compton scattering (PMP+) when the energy of particle 1 (E_1) takes $E_1 \to \infty$. This result does not change even if the spin is taken into account. In the present case, however, the particle on the ISCO should be massive, which means that the PMP process is not possible.

In the current analysis, compared with the non-spinning case, the maximal efficiencies for both the elastic scattering and the photoemission process become twice as large as the non-spinning case. We can conclude that spin also plays an important role in the collision with the ISCO particle. Note that the efficiency does not change significantly in the case of the photoemission process. This is because the absorbed massless particle is non-spinning.

Our analysis was performed for an extreme Kerr black hole. Since the existence of an extreme black hole may not be likely [32], we will extend the present analysis into the case for a non-extreme black hole.

Acknowledgements

We would like to thank Hirotada Okawa, Shingo Suzuki, and Keigo Shimada for useful information and comments. This work was supported in part by Japan Society for the Promotion of Science (JSPS) KAKENHI Grant Numbers JP17H06359 (KM) and JP19K03857 (KM).

Appendix A. The BSW effect for the collision with a spinning particle on the ISCO

The center of mass energy $E_{\rm cm}$ is defined as

$$E_{\rm cm}^2 = -(p_{1(a)} + p_{2(a)})(p_1^{(a)} + p_2^{(a)}).$$

We consider a collision between spinning particles which have the same mass μ. If particle 1 is on the ISCO, $p_1^{(1)} = 0$ holds. Hence, we find

$$\frac{E_{\rm cm}^2}{2\mu^2} = 1 - \frac{r^4 A_s(s_1, E_1, J_1) A_s(s_2, E_2, J_2)}{(r^3 - s_1^2)(r^3 - s_2^2)} + \frac{r^2 B_s(s_1, E_1, J_1) B_s(s_2, E_2, J_2)}{(r-1)^2 (r^3 - s_1^2)(r^3 - s_2^2)},$$

where

$$A_s(s, E, J) = (1 + s)E - J,$$
$$B_s(s, E, J) = (r^3 + (1 + s)r + s)E - J(r + s).$$

In addition, the relation between the angular momentum J_1 and the energy E_1 is given by $J_1 = 2E_1$. Assuming the collision takes place near the horizon, i.e. $r = 1 + \epsilon$, the above equation becomes

$$\frac{E_{\rm cm}^2}{2\mu^2} = \frac{(2 + s_1)}{(1 - s_1^2)(1 - s_2)} \frac{E_1(2E_2 - J_2)}{\epsilon} + O(\epsilon^0).$$

Thus, we find that the center of mass energy $E_{\rm cm}$ diverges at the horizon ($\epsilon \to 0$).

Appendix B. The case for particle 3 with $\sigma_3 = 1$

B.1. Case [A] (MMM: collision of two massive particles)

In this case, the condition $E_3 \le E_{3,(\rm cr)}$ must be satisfied. Hence, the larger root of Eq. (20), $E_{3,+}$, is excluded, and we obtain the energy of particle 3 (E_3) as

$$E_3 = E_{3,-} := \frac{\mathcal{B} - \sqrt{\mathcal{B}^2 - \mathcal{A}\mathcal{C}}}{\mathcal{A}}.$$

From the condition $E_3 \le E_{3,(\rm cr)}$, the energy of particle 3 ($E_{3,-}$) is less than or equal to $E_{3,(\rm cr)}$. $E_{3,(\rm cr)}$ is a function of α_3, s_1, and s_2 which increases monotonically for $\alpha_3 < \alpha_{3,\infty} := \frac{2+s_2}{2(1+s_2)}$. In this region, $E_{3,\rm cr}$ is positive and $E_{3,\rm cr} \to \infty$ as $\alpha_3 \to \alpha_{3,\infty}$. For $\alpha_3 > \alpha_{3,\infty}$, $E_{3,\rm cr}$ becomes negative and this case should be excluded. As α_3 increases in $0 < \alpha_3 < \alpha_{3,\infty}$, the energy of particle 3 ($E_{3,-}$) also increases but faster than $E_{3,\rm cr}$ and reaches the upper bound $E_{3,\rm cr}$ at some value of $\alpha_3 = \alpha_{3,\rm cr}$. Hence, for given values of s_1 and s_2, it is sufficient to find α_3 satisfying $E_3 = E_{3,\rm cr}$. From the condition $E_3 = E_{3,\rm cr}$ and Eq. (20), we find a quadratic equation in α_3. Solving this equation, we obtain

$$\alpha_3 = 0 \quad \text{or} \quad \alpha_3(\mathrm{s}_1, \mathrm{s}_2),$$
$$\alpha_3(s_1, s_2) := \frac{(2 + s_2)(1 - s_2 - 3s_1 s_2 + s_1^2(2 + s_2))}{(1 + 2s_1)(1 - 2s_2 - 2s_2^2) + s_1^2(2 + s_2)}.$$

Thus, we obtain the largest value of E_3 or $E_{3,\rm cr}$ by inserting the solutions. For $\alpha_3 = 0$, E_3 is equivalent to E_1. Hence, we find the maximal value of $E_3/(E_1 + 1)$:

$$\frac{E_3}{E_1 + 1} = \frac{1}{1 + 1/E_1} \le 0.4550 \quad (s_1 = s_{\rm min}^{\rm ISCO}).$$

On the other hand, for $\alpha_3 = \alpha_3(s_1, s_2)$, we show the contour map of $E_3/(E_1 + 1)$ in Fig. B.1. The maximum efficiency of $E_3/(E_1 + 1) \approx 0.06587$ is obtained at the red point $(s_1, s_2) \approx (0.08230, 0.449)$.

Comparing these results, we find $E_3/(E_1 + 1) \approx 0.4550$ ($s_1 = s_{\rm min}^{\rm ISCO}$) as the maximum value for arbitrary s_2. Since we find that $E_2 = 1$ is possible from Fig. B.2, $\eta_{\rm max} \approx 0.4550$ is obtained as the maximum efficiency.

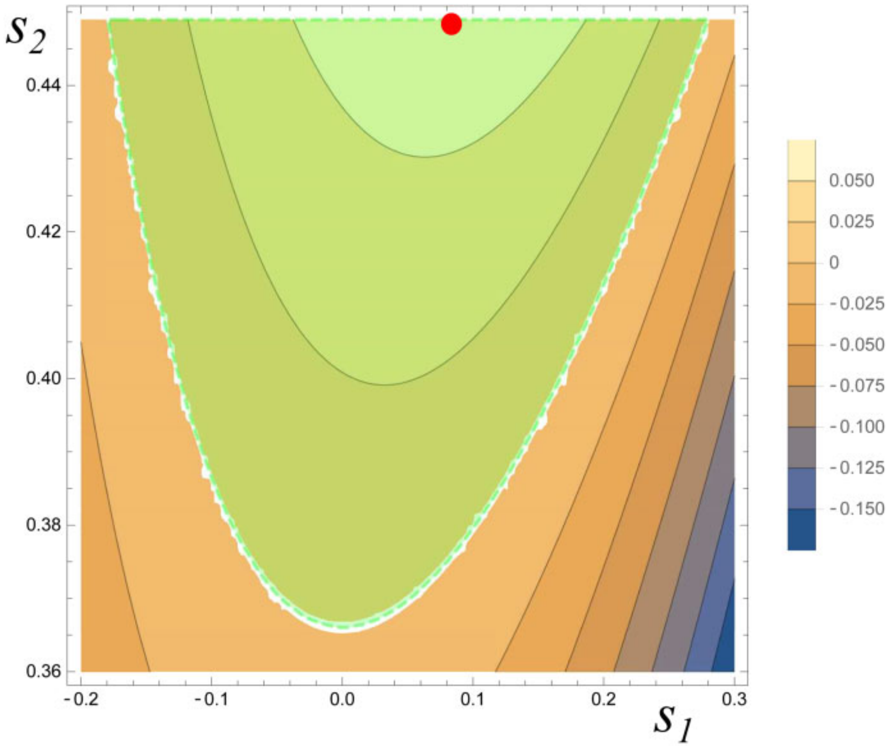

Fig. B.1. The contour map of $E_3/(E_1+1)$ for $\alpha_3 = \alpha_3(s_1, s_2)$ in terms of s_1 and s_2. The timelike condition of particle 3 is satisfied in the light-green shaded region. In this case, $E_3/(E_1+1) \approx 0.06587$ is obtained as the maximum value at the red point $(s_1, s_2) \approx (0.08230, 0.449)$.

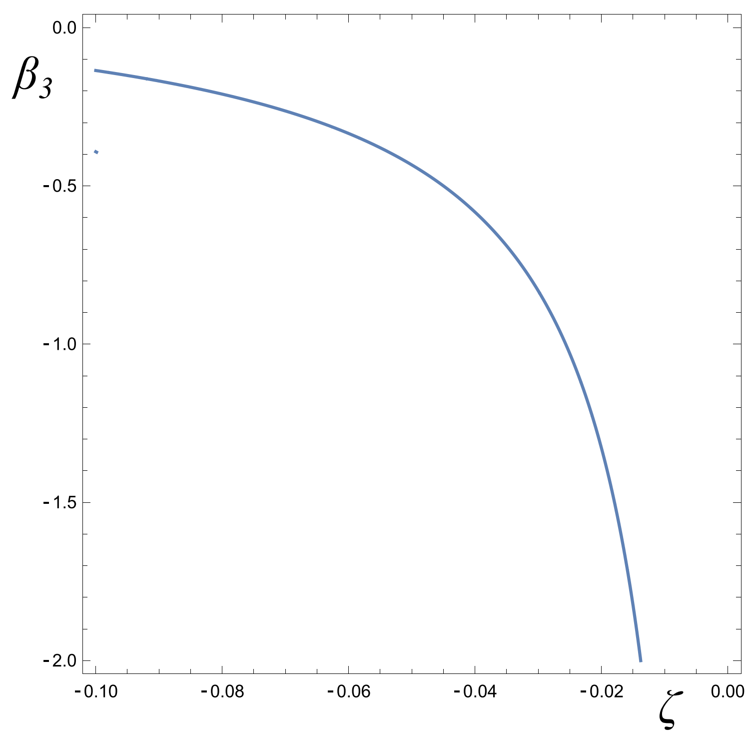

Fig. B.2. The contour of $E_2 = 1$ in terms of ζ and β_3. Here, $s_1 = s_{\rm min}^{\rm ISCO}$ and we set $s_2 = 0$ as example. In this case, the timelike condition for particle 2 is trivial and ζ can take an arbitrary value.

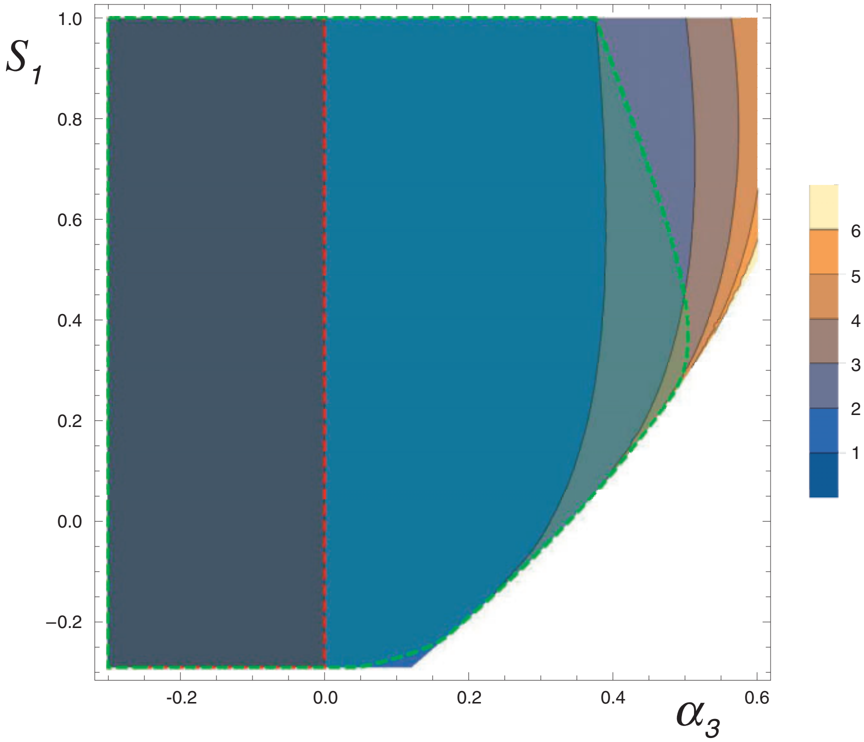

Fig. B.3. The contour map of E_3/E_1 in terms of α_3 and s_1. In the light-green region, the timelike condition for the particle 3 orbit is satisfied. In the light-red region, the condition $E_3 \leq E_{3,(\mathrm{cr})}$ is satisfied. The maximum efficiency, $E_3/E_1 \approx 1.000$, is obtained at $\alpha_3 \approx 0$ for any s_1.

B.2. Case [B] (MPM: photoemission)

In this case, the condition $E_3 \leq E_{3,(\mathrm{cr})}$ must be satisfied. We show the contour map of E_3/E_1 in terms of α_3 and s_1 in Fig. B.3. As the maximum efficiency, $E_3/E_1 \approx 1.000$ is obtained at $\alpha_3 \approx 0$ for any s_1.

This result has physical meaning when $E_2 = 0$ is possible in terms of ζ and β_3. To see this, from Eq (26) we find that the asymptotic behavior of $\mathcal{P}$ becomes

$$\mathcal{P} \approx -8E_3\zeta\beta_3\left[\frac{E_3(2+s_1)}{(1-s_1)f(s_1,E_3,0)}+1\right]$$

when we take the limit $\zeta\beta_3 \to -\infty$. ζ is constrained as $-\infty < \zeta < 0$ because particle 2 is non-spinning, while β_3 is arbitrary. As a result, we obtain $E_2 \to 0$ in the limit $\zeta\beta_3 \to -\infty$. Hence, we find the maximum efficiency $\eta_{\mathrm{max}} \approx 1.000$ for the inverse Compton scattering. For $s_1 = 0$, the maximum efficiency also becomes $\eta_{\mathrm{max}} \approx 1.000$ since it does not depend on s_1.

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