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Back Reaction to the Spectrum of Magnetic Field in the Kinetic Dynamo Theory

—— *Modified Kulsrud-Anderson Equation* ——

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We take account of the lowest order back reaction on a fluid and modify the Kulsrud-Anderson equation, $\partial_t \mathcal{E}_M = 2\gamma \mathcal{E}_M$, obtained in the kinetic dynamo theory, where $\mathcal{E}_M$ is the energy density of the magnetic field. We apply our results to certain astrophysical stages where the magnetic field is expected to be amplified by the dynamo mechanism.

§1. Introduction

Magnetic fields have been observed on various astrophysical scales. [1] The origin of such fields is one of the important problems in cosmology. [2] Although an attractive mechanism in protogalaxies was proposed by Kulsrud et al. [3] for galactic magnetic fields, this cannot explain how the magnetic fields are produced in intergalactic and intercluster regions. [2] Thus, it is worth investigating the generation and evolution of primordial magnetic fields. Since the strength of these fields is too small, we expect that their amplification occurs due to the dynamo mechanism. It is well known that the mean magnetic field can be amplified enough in kinetic dynamo theory. [4] However, as Kulsrud and Anderson showed, [5] the growth rate of fluctuations around the mean magnetic field is much larger than that of the mean field in interstellar mediums. This implies that kinetic dynamo theory breaks down. Hence, one must investigate the effect of the back reaction on kinetic theory. Previously the back reaction on the mean field has been considered. [6] Setting apart the problem of the kinetic dynamo theory in interstellar media, it is obvious that the kinetic theory cannot hold near the equipartition state in general.

In this paper, we consider the back reaction on fluctuations and derive the evolution equation for the energy of the magnetic field, that is, a modified Kulsrud-Anderson equation. We then apply this equation to some examples.

The rest of the paper is organized as follows. In §2, we derive a modified Kulsrud-Anderson equation including the lowest order back reaction using a phenomenological

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assumption. In §3 we consider applications of this equation, and in §4, we make some concluding remarks.

§2. A modified Kulsrud-Anderson equation

In Fourier space the basic equations of an incompressible MHD are

$$\partial_t v_i(\boldsymbol{k},t) = -iP_{ijk}(\boldsymbol{k}) \int \frac{d^3q}{(2\pi)^3}[v_j(\boldsymbol{k}-\boldsymbol{q},t)v_k(\boldsymbol{q},t) - b_j(\boldsymbol{k}-\boldsymbol{q},t)b_k(\boldsymbol{q},t)] \tag{2·1}$$

and

$$\partial_t b_i(\boldsymbol{k},t) = ik_j \int \frac{d^3q}{(2\pi)^3}[v_i(\boldsymbol{k}-\boldsymbol{q},t)b_j(\boldsymbol{q},t) - v_j(\boldsymbol{k}-\boldsymbol{q},t)b_i(\boldsymbol{q},t)], \tag{2·2}$$

where $b_i(\boldsymbol{k},t) := B_i(\boldsymbol{k},t)/\sqrt{4\pi\rho}$, ρ is the energy density of the fluid and $P_{ijk}(\boldsymbol{k}) = k_j P_{ik}(\boldsymbol{k}) = k_j(\delta_{ik} - k_i k_k/|\boldsymbol{k}^2|)$. For simplicity, we have neglected the diffusion terms in the above equations. This simplification is justified by the fact that almost of astrophysical systems have high magnetic Reynolds number.

Following Kulsrud and Anderson [5] and considering a small time step as the parameter of expansion, we evaluate the time evolution of the magnetic field iteratively as

$$\begin{aligned} b_i(\boldsymbol{k},t) &= b_i(\boldsymbol{k},0) + b_i^{(1)}(\boldsymbol{k},t) + b_i^{(2)}(\boldsymbol{k},t) + \cdots \\ &= b_i^{(0)}(\boldsymbol{k}) + b_i^{(1)}(\boldsymbol{k},t) + b_i^{(2)}(\boldsymbol{k},t) + \cdots, \end{aligned} \tag{2·3}$$

where $b_i^{(0)}(\boldsymbol{k})$ is the initial field. For the fluid velocity, we take account of the back reaction from the magnetic field (Lorentz force) as

$$v_i(\boldsymbol{k},t) = v_i^{(1)}(\boldsymbol{k},t) + \delta v_i(\boldsymbol{k},t), \tag{2·4}$$

where the $v_i^{(1)}(\boldsymbol{k},t)$ are statistically homogeneous and isotropic components satisfying

$$\begin{aligned} \langle v_i^{(1)*}(\boldsymbol{k},t')v_j^{(1)}(\boldsymbol{q},t)\rangle &= (2\pi)^3[J_1(k)P_{ij}(\boldsymbol{k}) + iJ_2(k)\epsilon_{ikj}k_k]\delta^3(\boldsymbol{k}-\boldsymbol{q})\delta(t-t') \\ &= (2\pi)^3 V_{ij}(\boldsymbol{k})\delta^3(\boldsymbol{k}-\boldsymbol{q})\delta(t-t'). \end{aligned} \tag{2·5}$$

This relation holds in regions sufficiently far from the boundary. $J_1(k)$ and $J_2(k)$ denote the velocity dispersion and the mean helicity of the fluid,

$$\langle \boldsymbol{v}^{(1)}(\boldsymbol{x},t)\cdot\boldsymbol{v}^{(1)}(\boldsymbol{x},t)\rangle = 2\int \frac{d^3k}{(2\pi)^3}J_1(k)\delta(0)$$

and

$$\langle \boldsymbol{v}^{(1)}(\boldsymbol{x},t)\cdot\nabla\times\boldsymbol{v}^{(1)}(\boldsymbol{x},t)\rangle = -2\int \frac{d^3k}{(2\pi)^3}k^2 J_2(k)\delta(0).$$

The second term on the right-hand side of Eq. (2·4), $\delta v_i(\boldsymbol{k},t)$, is determined by Eq. (2·1), and this corresponds to the back reaction term from the magnetic field.

To each order, the MHD equation becomes

$$\partial_t b_i^{(1)}(\boldsymbol{k},t) = 2ik_j \int \frac{d^3q}{(2\pi)^3} v_{[i}(\boldsymbol{k}-\boldsymbol{q},t) b_{j]}^{(0)}(\boldsymbol{q}), \tag{2·6}$$

$$\partial_t b_i^{(2)}(\boldsymbol{k},t) = 2ik_j \int \frac{d^3q}{(2\pi)^3} v_{[i}(\boldsymbol{k}-\boldsymbol{q},t) b_{j]}^{(1)}(\boldsymbol{q},t) \tag{2·7}$$

and

$$\partial_t v_i(\boldsymbol{k},t) = 2iP_{ijk}(\boldsymbol{k}) \int \frac{d^3q}{(2\pi)^3} b_{(j}^{(0)}(\boldsymbol{k}-\boldsymbol{q}) b_{k)}^{(1)}(\boldsymbol{q},t). \tag{2·8}$$

The last equation contains the effect of the lowest order back reaction on the fluid, and it yields an explicit expression of $\delta v_i(\boldsymbol{k},t)$

$$\delta v_i(\boldsymbol{k},t) \simeq 2iP_{ijk}(\boldsymbol{k}) \int_0^t dt' \int \frac{d^3q}{(2\pi)^3} b_{(j}^{(0)}(\boldsymbol{k}-\boldsymbol{q}) b_{k)}^{(1)}(\boldsymbol{q},t'). \tag{2·9}$$

From Eqs. (2·6) $\sim$ (2·9), the time derivative of the energy becomes

$$\begin{aligned}
\partial_t\langle|\boldsymbol{b}(\boldsymbol{k},t)|^2\rangle &= \langle b_i^{(1)*}(\boldsymbol{k},t)\dot{b}_i^{(1)}(\boldsymbol{k},t)\rangle + b_i^{(0)*}\langle \dot{b}_i^{(2)}(\boldsymbol{k},t)\rangle + \text{c.c.} \\
&= 4\int_0^t dt' \int \frac{d^3q d^3p}{(2\pi)^6} k_j k_k b_{[j}^{(0)*}(\boldsymbol{q}) \langle v_{i]}^*(\boldsymbol{k}-\boldsymbol{q},t') v_{[i}(\boldsymbol{k}-\boldsymbol{p},t)\rangle b_{k]}^{(0)}(\boldsymbol{p}) \\
&\quad -4\int_0^t dt' \int \frac{d^3q d^3p}{(2\pi)^6} k_j q_\ell b_i^{(0)*}(\boldsymbol{k}) \langle v_{[i}^*(\boldsymbol{q}-\boldsymbol{k},t) v_{[j]}(\boldsymbol{q}-\boldsymbol{p},t')\rangle b_{\ell]}^{(0)}(\boldsymbol{p}) \\
&\quad + \text{c.c.},
\end{aligned} \tag{2·10}$$

where

$$\begin{aligned}
\langle v_i^*(\boldsymbol{k},t') v_j(\boldsymbol{q},t)\rangle &\simeq \langle v_i^{(1)*}(\boldsymbol{k},t') v_j^{(1)}(\boldsymbol{q},t)\rangle \\
&\quad + \langle v_i^{(1)*}(\boldsymbol{k},t') \delta v_j(\boldsymbol{q},t)\rangle + \langle \delta v_i^*(\boldsymbol{k},t') v_j^{(1)}(\boldsymbol{q},t)\rangle \\
&= (2\pi)^3 V_{ij}(\boldsymbol{k}) \delta^3(\boldsymbol{k}-\boldsymbol{q}) \delta(t-t') \\
&\quad -4\int_0^t dt'' \int \frac{d^3p}{(2\pi)^3} P_{jk\ell}(\boldsymbol{q}) p_m b_{[m}^{(0)}(\boldsymbol{p}-\boldsymbol{k}) V_{i(\ell]}(\boldsymbol{k}) b_{k)}^{(0)}(\boldsymbol{q}-\boldsymbol{p}) \\
&\quad -4\int_0^{t'} dt'' \int \frac{d^3p}{(2\pi)^3} P_{ik\ell}(\boldsymbol{k}) p_m b_{[m}^{(0)*}(\boldsymbol{p}-\boldsymbol{q}) V_{(\ell]|j|}(\boldsymbol{q}) b_{k)}^{(0)*}(\boldsymbol{k}-\boldsymbol{p}) \\
&=: (2\pi)^3 V_{ij}(\boldsymbol{k}) \delta^3(\boldsymbol{k}-\boldsymbol{q}) \delta(t-t') + \delta\langle v_i^*(\boldsymbol{k},t') v_j(\boldsymbol{q},t)\rangle.
\end{aligned} \tag{2·11}$$

Equation (2·10), along with Eq. (2·11), constitutes the formal relation expressing the effect of the back reaction.

Let us consider a simple example with the initial conditions

$$b_i^{(0)}(\boldsymbol{x}) = b_0 \delta_{iz} \quad \text{or} \quad b_i^{(0)}(\boldsymbol{k}) = b_0 (2\pi)^3 \delta^3(\boldsymbol{k}) \delta_{iz}. \tag{2·12}$$

These conditions hold approximately as long as the spatial scale of the magnetic field is much larger than the typical scale of eddies. In this case, Eq. (2·10) becomes

$$\partial_t \langle |\boldsymbol{b}(\boldsymbol{k},t)|^2\rangle = 2(2\pi)^3\delta(\mathbf{0})k_z^2 V_{ii}(\boldsymbol{k}) + 2\int_0^t dt' k_z^2 \delta\langle v_i^*(\boldsymbol{k},t')v_i(\boldsymbol{k},t)\rangle b_0^2$$
$$= 4(2\pi)^3 k_z^2 J_1(k) b_0^2 \delta^3(\mathbf{0}) - 6(2\pi)^3 k_z^4 b_0^4 (\Delta t)_k^2 J_1(k)\delta^3(\mathbf{0}), \qquad (2\cdot13)$$

where $(\Delta t)_k$ is the time scale of the eddy turnover, whose expression is given below. We have assumed here that the time integral should be estimated as $\int_0^t dt'[\cdots] \sim (\Delta t)_k[\cdots]$ in the second line on the right-hand side of Eq. (2·13) because the back reaction is effective only on the time scale of the eddy turnover. Here we assume Kolmogoroff spectrum for the inertial range*) $k_0 < k < k_{\rm max} \sim R^{3/4}k_0$,[7] where k_0 is the wave number of the largest eddy and R is the Reynolds number. From the definition of the velocity dispersion

$$\langle v^2\rangle = 2\int \frac{d^3k}{(2\pi)^3} J_1(k)\delta(0) =: \int_{k_0}^{k_{\rm max}} dk I(k), \qquad (2\cdot14)$$

we obtain the relation

$$I(k) = \frac{1}{\pi^2}k^2 J_1(k)(\Delta t)_k^{-1} \simeq \frac{2}{3}v_0^2 \frac{k_0^{2/3}}{k^{5/3}}, \qquad (2\cdot15)$$

where v_0 is the typical velocity ($v_0 \sim \sqrt{\langle v^2\rangle}$) and we have used $\delta(0) \sim (\Delta t)_k^{-1}$. The expression of $(\Delta t)_k$ is given by the estimation of the order of magnitude in Eq. (2·14), that is, $(1/k(\Delta t)_k)^2 \sim kI(k)$.

Integrating Eq. (2·13) over $\boldsymbol{k}$, we obtain the modified Kulsrud-Anderson equation

$$\partial_t \rho_M = 2\gamma\rho_M - 2\zeta\rho_M^2, \qquad (2\cdot16)$$

where

$$\rho_M := \frac{\mathcal{E}_M}{4\pi\rho} := \frac{1}{V}\int \frac{d^3k}{(2\pi)^3}\langle |\boldsymbol{b}(\boldsymbol{k},t)|^2\rangle, \qquad (2\cdot17)$$

$$\gamma := 2\int \frac{d^3k}{(2\pi)^3} k_z^2 J_1(k) \qquad (2\cdot18)$$

and

$$\zeta := 3\int \frac{d^3k}{(2\pi)^3} k_z^4 J_1(k)(\Delta t)_k^2. \qquad (2\cdot19)$$

In the above derivation, we have used $\delta^3(\mathbf{0}) \sim V$, where V is the characteristic volume of the system.

*) The inertial range is defined by a scale which is smaller than the largest eddy ($\sim k_0^{-1}$) and larger than a small scale ($\sim R^{-3/4}k_0^{-1}$) below which the viscosity term is dominant. In this range, a transfer of the energy takes place from large eddies to small eddies without dissipation of the energy. This leads to a sort of 'equilibrium state' characterized by Kolmogoroff spectrum[7] (Kolmogoroff theory).

Now we evaluate the coefficients γ and ζ. The results are

$$\gamma \simeq \int_{k_0}^{k_{\rm max}} dk k^2 I(k)(\Delta t)_k \simeq \int_{k_0}^{k_{\rm max}} dk[kI(k)]^{1/2} \sim v_0 k_0^{1/3} k_{\rm max}^{2/3} \sim R^{1/2} v_0 k_0 \tag{2·20}$$

and

$$\zeta \simeq \int_{k_0}^{k_{\rm max}} dk k^4 I(k)(\Delta t)_k^3 \simeq \int_{k_0}^{k_{\rm max}} dk[kI(k)]^{-1/2} \sim \frac{k_{\rm max}^{4/3}}{v_0 k_0^{1/3}} \sim R\frac{k_0}{v_0}. \tag{2·21}$$

Defining the dimensionless quantity $\mu_M := \rho_M/v_0^2$, we can see that Eq. (2·16) becomes

$$\partial_t \mu_M = 2\gamma\mu_M - 2\zeta' \mu_M^2, \tag{2·22}$$

where $\zeta' = \zeta v_0^2 \sim Rk_0 v_0$. The second term on the right-hand side of Eq. (2·22) comes from the effect of the back reaction. One can easily see from the above equation that the back reaction gives an effect opposite to the original kinetic term and causes the energy of the magnetic field to balance with the energy of the fluid.

Although we know from the procedure used here that Eq. (2·22) holds only during a small time step ($\mu_M \ll 1$), we now extrapolate this result. We thereby find the solution

$$\mu_M = \frac{\gamma}{\zeta'} \frac{1}{1 - \left(1 - \frac{1}{\mu_M(0)}\frac{\gamma}{\zeta'}\right) e^{-2\gamma t}}. \tag{2·23}$$

One can easily see from this expression that the magnetic 'energy' goes toward the terminal value $\mu_M^* = \gamma/\zeta' \sim R^{-1/2}$ on a time scale $\sim \gamma^{-1}$. This corresponds to the saturation value, which is estimated naively on the assumption that the drain by the magnetic field is comparable to the turbulent power. [3), 5)]

§3. Applications

In this section we apply Eq. (2·22) to two examples in which the magnetic field is amplified by the dynamo mechanism. First, we treat the time evolution of the magnetic field during the first order phase transition in the very early universe. We also consider briefly the amplification of the magnetic field in interstellar media.

3.1. *Electroweak plasma*

There are attractive mechanisms involved in the generation of the primordial magnetic field during the course of cosmological phase transitions. [8)] In these scenarios the strong magnetic field is expected to be amplified by MHD turbulence during the first order phase transition. The details of the amplification are studied by using the Kulsrud-Anderson equation in Ref. 9).

We reconsider the amplification of the magnetic field during such phase transition using the modified Kulsrud-Anderson equation derived in the previous section. The time scale for the equipartition is $t_{\rm equi} \sim \gamma^{-1} \sim R^{-1/2} v_0^{-1} k_0^{-1}$. Since the Reynolds number is $R \sim 10^2$, [9] we can see that it is of the same order as the time scale of the phase transition. Thus, the magnetic field can be sufficiently amplified, and the final energy is given by

$$\mathcal{E}_M^* \sim R^{-1/2} \mathcal{E}_v \sim 0.1 \times \mathcal{E}_v, \qquad (3\cdot1)$$

where $\mathcal{E}_v$ is the energy of the plasma fluid.

3.2. *Interstellar media*

As stated in the Introduction, kinetic dynamo theory breaks down in interstellar media. [5] For interstellar media, typical values of key quantities are $2\pi/k_0 \sim 100$ pc, $v_0 \sim 10^6$ cm/s and $R \sim v_0/k_0\nu \sim 10^8$, where ν denotes the kinetic ion viscosity($\nu \sim 10^{18}$ cm^{2}s^{-1}. [3]) Then the characteristic time scale is given by $t_{\rm ISM} \sim \gamma^{-1} \sim 10^2$ yr. Since the time scale of the mean field is $\sim 10^{10}$ yr, [5] we realize again the mean field theory is meaningless in the present perturbative approach. The final energy of the magnetic field is given by $\mathcal{E}_M^* \sim 10^{-4} \times \mathcal{E}_v$.

§4. Concluding remark

In this paper, we have considered the lowest order back reaction to kinetic dynamo theory and modified the equation for the energy of the magnetic field. As a result we obtained an equation that successfully describes time evolution of the energy of the magnetic field. That is, the terminal value of the magnetic energy obtained from Eq. (2·23) agree with the previous qualitative estimation of the saturation energy. [3), 5] We have also presented an expression depending on k (Eq. (2·13)), with which we can evaluate the evolution of the magnetic field for various scales. Since the present formalism is general, our equation is useful in other situations, for example, in the fireball model for γ-ray bursts. [10]

Finally, we should comment on our assumption regarding the initial conditions (Eq. (2·12)) and the extrapolation of Eq. (2·22). We chose these initial conditions in order to obtain a simple result like Eq. (2·22). Although this assumption holds approximately in some cases, it may not be correct in general. We should also note that we have considered only the effect of lowest order back reaction. Properly speaking, if one wishes to analyze the vicinity of the equipartition, one must take into account higher order effects of the back reaction. A study near the equipartition might be possible by using something like the renormalization group approach. A study assuming more general initial conditions and taking into account higher order effects of the back reaction should be carried out in the future. At the same time, the spatial structure as the typical coherence length of the magnetic field should also be discussed.

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