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Illusory Z_N Metastable States in Hot Gauge Theories

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$SU(N)$ gauge theories with fundamental representation fermions are studied at high temperature. The possibility of Z_N metastable states has been discussed previously. We reconsider a partition function with internal symmetries. It is shown that these states are not metastable, and there is a single vacuum.

§1. Introduction

$SU(N)$ gauge theories are expected to undergo a phase transition at some temperature T_c. The order parameter is the expectation value of the Polyakov loop,

$$L(\boldsymbol{x}) = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp\left[ig\int_0^\beta dt A_0(t,\boldsymbol{x})\right], \tag{1·1}$$

where $\beta = \frac{1}{T}$ is the inverse temperature, and $\mathcal{P}$ denotes path ordering. The theories are in a confined phase specified by $\langle L\rangle = 0$ under T_c, and they are in a deconfined phase specified by $\langle L\rangle \neq 0$ above T_c.[1]

When fermions in the fundamental representation are absent, there is a Z_N symmetry, which is a center symmetry of $SU(N)$. Above T_c, it is expected that $\langle L\rangle$ takes one of the values of Z_N, i.e.

$$\langle L\rangle \propto e^{\frac{2\pi i n}{N}}. \quad (n = 0, 1, \cdots, N-1)$$

N-fold degenerate vacua are specified by these Z_N values.[2),3)] It is interesting to consider cosmological processes in which the domains of these degenerate Z_N vacua exist. Their interface tension has been calculated.[4)] (However, there is a claim that there is one unique vacuum and that physical Z_N domains do not exist, because the true symmetry is not $SU(N)$ but $SU(N)/Z_N$.[5)])

If fermions in the fundamental representation are added, the Z_N symmetry is broken. The reason for this is that, although the Lagrangian density

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}\gamma_\mu D_\mu\psi \tag{1·2}$$

and the periodic boundary condition for the gauge fields

$$A_\mu(\beta,\boldsymbol{x}) = A_\mu(0,\boldsymbol{x}) \tag{1·3}$$

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are invariant under the Z_N transformation

$$\psi \to U\psi, \quad A_\mu \to UA_\mu U^{-1} - \frac{i}{g}U\partial_\mu U^{-1},$$
$$U(t) = \exp\left[i\phi\frac{t}{\beta}\mathrm{diag}(1,\cdots,1,-N+1)\right]$$

with $\phi = \frac{2\pi n}{N}$, the antiperiodic boundary condition for fermions

$$\psi(\beta,\boldsymbol{x}) = -\psi(0,\boldsymbol{x}) \tag{1·4}$$

changes to

$$\psi(\beta,\boldsymbol{x}) = -z_n\psi(0,\boldsymbol{x}), \quad z_n = e^{\frac{2\pi i n}{N}}. \tag{1·5}$$

Thus the above-mentioned degeneracy is lifted, and the state with $\arg\langle L\rangle = 0$ is the absolute minimum. However, if the number of fermions is not so large, the states with $\arg\langle L\rangle \neq 0$ may become local minima, i.e. metastable states. [6] The role of these metastable states in cosmology has been discussed. [6] Metastable states in the standard model were also studied. [7]

Although these are interesting states, they have unacceptable thermodynamic behaviour. [8),9] We review two of the relevant problems: (a) positive free energy and (b) complex fermion number density.

(a) *Positive free energy*

To describe the Z_N vacua, it is convenient to introduce a constant part of A_0 with the parametrization

$$a_q = \frac{2\pi q}{g\beta N}\mathrm{diag}(1,\cdots,1,-N+1). \tag{1·6}$$

Substituting (1·6) into (1·1), and choosing $q = n = 0, 1, \cdots, N-1$, we find

$$L \propto e^{\frac{2\pi i n}{N}}.$$

It is not difficult to calculate the free energy density in the one-loop approximation. The result is [8]

$$F(q) = \pi^2 T^4[V_b(q) + N_f V_f(q) + V_{\mathrm{base}}], \tag{1·7}$$

where N_f is the number of fermions in the fundamental representation. The contribution from gauge fields (and ghosts) V_b and that from fermions V_f are given by

$$V_b(q) = \frac{4}{3}(N-1)h(q), \tag{1·8}$$
$$V_f(q) = \frac{4}{3}\left[\frac{N}{16} - (N-1)h\left(\frac{q}{N}+\frac{1}{2}\right) - h\left(\frac{q}{N} - q + \frac{1}{2}\right)\right], \tag{1·9}$$
$$h(q) = (q_{\mathrm{mod1}})^2(1-q_{\mathrm{mod1}})^2. \tag{1·10}$$

The constant term

$$V_{\mathrm{base}} = -\frac{1}{45}\left[(N^2-1) + \frac{7}{4}NN_f\right] \tag{1·11}$$

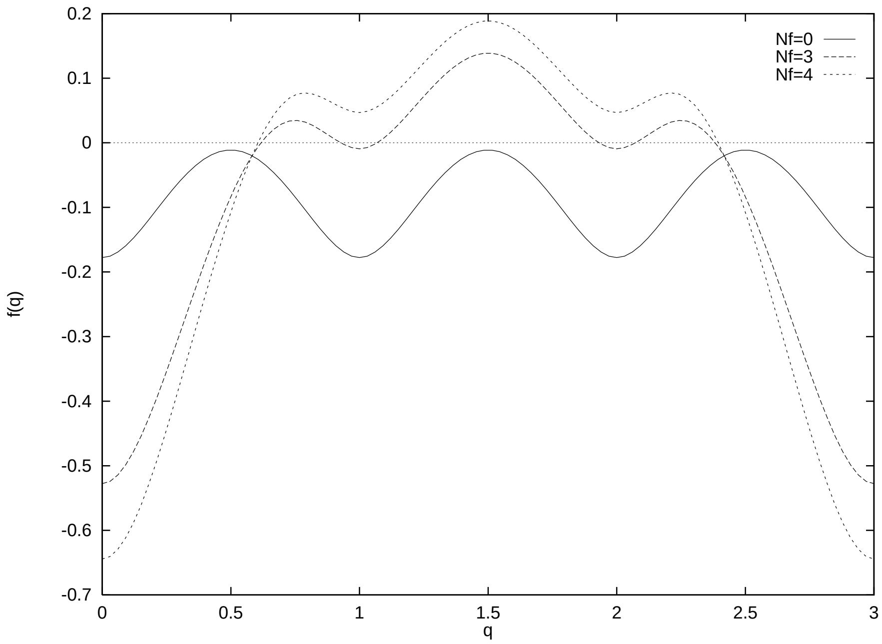

Fig. 1. Free energy density $f(q) = F(q)/(\pi^2 T^4)$ for $N = 3$. N_f is the number of fermions in the fundamental representation.

is required to reproduce the free energy density of an ideal gas of these particles.[8] The function $F(q)$ has an absolute minimum at $q = 0$ and may have local minima at $q = 1, 2, \cdots, N-1$. These local minima may become metastable states. In Fig. 1, $f(q) = F(q)/(\pi^2 T^4)$ is plotted for $N = 3$, $N_f = 0, 3, 4$ as an example.

However, as in the $N_f = 4$ case, these local minima may have positive free energy:

$$F = \gamma T^4, \quad \gamma > 0. \tag{1·12}$$

Equation (1·12) results in a negative pressure $p = -\gamma T^4$, negative entropy density $S = -4\gamma T^3$, negative internal energy density $U = -3\gamma T^4$, and negative specific heat $c = -12\gamma T^3$.[8]

(b) *Complex fermion number density*

Let us consider the Lagrangian density $\bar{\psi}\gamma_\mu D_\mu \psi$ with $D_\mu = \partial_\mu + igA_\mu$. If the gauge field A_0 contains the constant part a_q, the fermionic Lagrangian density gives the term

$$\psi^\dagger i g a_q \psi. \tag{1·13}$$

Thus iga_q plays the role of an imaginary chemical potential. From the free fermion propagator with (1·13), we find that the expectation value of the particle number density for a particular fermion flavor is[9]

$$\langle \psi^\dagger \psi \rangle = \int \frac{d^3p}{(2\pi)^3} 4N \left(\frac{1}{e^{\beta E_p} e^{2\pi i q/N} + 1} - \frac{1}{e^{\beta E_p} e^{-2\pi i q/N} + 1} \right). \tag{1·14}$$

Although $\psi^\dagger\psi$ is a Hermitian operator, its expectation value (1·14) is imaginary for $q \neq 0, N/2$.[9)]

From these problems, it was concluded that "although the Z_N domains are important in the Euclidean theory, they cannot be interpreted as physical domains in Minkowski space".[8),9)] However, the relation between the Z_N domains in the Euclidean theory and those in the Minkowski theory is not clear. In particular, it is not clear if the domains actually survive in the Euclidean theory. Furthermore, the origin of these problems is not known. We will study these points.

In general, the inequality $F(q) \geq F(0)$ holds. In the next section, we give a definition of a partition function with the background A_0. Based on this partition function, we review the essence of this inequality. In §3, the case with fundamental representation fermions is considered. We show that since fermions have a global $U(1)$ symmetry, the Z_N metastable states become unstable in this direction. Hence there is one vacuum and no metastable state. The necessity to take the $U(1)$ symmetry into account is discussed in §4. Section 5 is devoted to summary. In Appendix A, when the $U(1)$ symmetry is absent, we show that the partition function appearing in this article and that appearing in Ref. 10) are equivalent. The inequality used in the text is proved in Appendix B.

§2. Inequality with background A_0

Let us consider a system described by a Hamiltonian H with a global symmetry G. If the rank of G is r, there are mutually commuting charges Q^a ($a = 1, \cdots, r$) satisfying

$$[H, Q^a] = 0, \quad [Q^a, Q^b] = 0.$$

A general generating function is defined by[11)]

$$\hat{Z}(\tilde{\lambda}^1, \cdots, \tilde{\lambda}^r) = \mathrm{Tr}\exp\left(-\beta H + i\sum_{a=1}^{r}\tilde{\lambda}^a Q^a\right). \tag{2·1}$$

Particular partition functions, e.g. a partition function for a specific representation, that for chargeless states, etc., are obtained from $\hat{Z}$.[11)]

In the case of the $SU(N)$ gauge theory without fermions, the Hamiltonian in the $A_0 = 0$ gauge is

$$H = \int d^3x \frac{1}{2}[(E_i^a)^2 + (B_i^a)^2], \tag{2·2}$$

where E_i^a is a conjugate momentum of A_i^a. As H possesses the global $SU(N)$ symmetry $A_i(\boldsymbol{x}) \to \Omega A_i(\boldsymbol{x})\Omega^{-1}$, $\hat{Z}$ contains the element of the Cartan subalgebra $G(\tilde{\lambda}) = \sum_{a=1}^{N-1}\tilde{\lambda}^a Q^a$. In addition, (2·2) has the local $SU(N)$ symmetry $A_i(\boldsymbol{x}) \to A_i^\Omega(\boldsymbol{x}) = \Omega(\boldsymbol{x})[A_i(\boldsymbol{x}) - \frac{i}{g}\partial_i]\Omega^{-1}(\boldsymbol{x})$. To make transition amplitudes gauge invariant, external states must be invariant under this transformation. For this purpose, we introduce a projection operator P_0 defined by

$$P_0|A_i(\boldsymbol{x})\rangle = \int_{\Omega(\infty)=1} D\mu(\Omega(\boldsymbol{x}))|A_i^\Omega(\boldsymbol{x})\rangle,$$

where $d\mu(\Omega)$ is the Haar measure of $SU(N)$, and $\Omega(\infty) = \lim_{|\boldsymbol{x}|\to\infty} \Omega(\boldsymbol{x})$.*) Some properties of P_0 are discussed in Appendix A. By inserting P_0, $\hat{Z}$ becomes

$$\begin{aligned}\hat{Z}(\tilde{\lambda}) &= \mathrm{Tr}(e^{-\beta H + iG(\tilde{\lambda})} P_0) \\ &= \int DA_i(\boldsymbol{x}) \langle A_i(\boldsymbol{x})| e^{-\beta H + iG(\tilde{\lambda})} P_0 |A_i(\boldsymbol{x})\rangle. \end{aligned} \tag{2·3}$$

Here we used the trace formula

$$\mathrm{Tr}\Xi = \int \langle \xi | \Xi | (-1)^{\zeta} \xi \rangle d\xi, \tag{2·4}$$

where $|\xi\rangle$ is a state in the coherent representation, and ζ is the Grassmann parity of ξ.

It is easy to show that (2·3) becomes

$$\hat{Z}(\tilde{\lambda}) = \int DA_i(\boldsymbol{x}) \langle (A_i)_{\mathrm{inv}} | e^{-\beta H + iG(\tilde{\lambda})} | (A_i)_{\mathrm{inv}} \rangle \tag{2·5}$$

with $|(\Psi)_{\mathrm{inv}}\rangle = P_0|\Psi\rangle$. Furthermore, we can show that (2·3) is derived from the partition function

$$\int_{\Gamma} DA_\mu(x) \delta_{\mathrm{gf}} \Delta_{\mathrm{gh}} e^{-S} \delta\left(\tilde{\lambda} - \frac{1}{V}\int d^3x \lambda(\boldsymbol{x})\right), \tag{2·6}$$

where Γ, δ_{gf} and Δ_{gh} are explained in Appendix A, and $\lambda(\boldsymbol{x}) = \sum_{a=1}^{N-1} \lambda^a(\boldsymbol{x}) T^a$ with diagonal and traceless $N \times N$ matrices T^a. Although fermions are included, these results are proved in Appendix A. The relation between $\tilde{\lambda}$ and a_q, which is (A·15), is also discussed there. Equation (2·6) is the partition function studied in Ref. 10). However, to see the essential point clearly, we use (2·5).

Now we show $F(q) \geq F(0)$. For convenience in this explanation, following Ref. 13) we introduce a complete set of the energy eigenstates $|k, \nu_\alpha\rangle$ of H ($H|k,\nu_\alpha\rangle = E_k|k,\nu_\alpha\rangle$), where α labels irreducible representations of $SU(N)$, $\nu_\alpha = (\nu_\alpha^1, \cdots, \nu_\alpha^{N-1})$ labels the values of $SU(N)$ charges Q^a $(a = 1, \cdots, N-1)$ of the states, and the α-dependence of E_k is included in the label k for simplicity. For the gauge invariant energy eigenfunctionals $\Phi_{(k,\nu_\alpha)}(A_i) = \langle A_i | P_0 | k, \nu_\alpha\rangle$, the normalization

$$\int DA_i \Phi^*_{(k,\nu_\alpha)}(A_i) \Phi_{(l,\zeta_\epsilon)}(A_i) = \delta_{kl}\delta_{\nu_\alpha \zeta_\epsilon} \tag{2·7}$$

is assumed.[13] Then (2·5) becomes

$$\hat{Z}(\tilde{\lambda}) = \sum_k \sum_{\nu_\alpha} e^{-\beta E_k} \langle k, \nu_\alpha | e^{iG(\tilde{\lambda})} | k, \nu_\alpha \rangle.$$

*) The restriction $\Omega(\infty) = 1$[12] is important. To keep $A_i(\boldsymbol{x})$ absolutely integrable, i.e. to insure the inequality $\int d^3x |A_i(\boldsymbol{x})| < \infty$, $\Omega(\infty)$ must be constant. Furthermore, unless $\Omega(\infty) = 1$, P_0 does not satisfy $P_0^2 = P_0$. If the projection operator P in (A·16), which does not have this restriction, is used, there is a problem (see §5).

Using (B·3), we find

$$\langle k,\nu_\alpha|e^{iG(\tilde{\lambda})}|k,\nu_\alpha\rangle = \left\langle k,\nu_\alpha|e^{i\sum_{a=1}^{N-1}\tilde{\lambda}^a\nu_\alpha^a}|k,\nu_\alpha\right\rangle \\ \leq \langle k,\nu_\alpha|k,\nu_\alpha\rangle = 1,$$

where, from (B·4), the equality holds iff

$$e^{iG(\tilde{\lambda})}|k,\nu_\alpha\rangle = |k,\nu_\alpha\rangle. \qquad (2\cdot 8)$$

Since there is now Z_N symmetry, this condition is satisfied for all states only when $e^{iG(\tilde{\lambda})}$ generates the Z_N transformation.*) Thus

$$\hat{Z}(\tilde{\lambda}) \leq \hat{Z}(\tilde{\lambda}_n) = \sum_k \sum_{\nu_\alpha} e^{-\beta E_k},$$

where $\tilde{\lambda}_n$ $(n = 0,1,\cdots,N-1)$ satisfies $e^{i\tilde{\lambda}_n} \in Z_N$. Using the relation $\hat{Z}(\tilde{\lambda}) = e^{-\beta V F(\tilde{\lambda})}$, we find [10)]

$$F(0) = F(\tilde{\lambda}_n) \leq F(\tilde{\lambda}).$$

We thus see that, because of the phase factor $e^{iG(\tilde{\lambda})}$, states with $a_q \neq 0$ are unfavorable.

We note that, although the Z_N vacua are shown to be the absolute minima, this argument does not exclude the possibility of local minima (metastable states).

§3. The case with fermions

The argument of §2 is applicable to the case with fermions in the fundamental representation. However, as the function $\hat{Z}(\tilde{\lambda})$, which is defined by (A·1) and is equivalent to (A·2), now has no Z_N symmetry, (2·8) holds only when $\tilde{\lambda} = 0$. Therefore

$$\hat{Z}(\tilde{\lambda}) \leq \hat{Z}(0).$$

Although this expression implies that $\tilde{\lambda}_0 = 0$ gives an absolute minimum, local minima (metastable states) may exist. In fact, as we saw in §1, the values $\tilde{\lambda}_1,\cdots,\tilde{\lambda}_{(N-1)}$ may give the Z_N local minima at the one-loop level, though they may have unacceptable thermodynamical properties.

To understand the meaning of problem (b) raised in §1, we reconsider a partition function with fermions. The Hamiltonian in the $A_0 = 0$ gauge,

$$H = \int d^3x \left\{\frac{1}{2}[(E_i^a)^2 + (B_i^a)^2] + \bar{\psi}\gamma_i D_i \psi\right\}, \qquad (3\cdot 1)$$

possesses global and local $SU(N)$ symmetries. The former introduces the factor $e^{iG(\tilde{\lambda})} = e^{i\sum_{a=1}^{N-1}\tilde{\lambda}^a Q^a}$, and the latter introduces the projection operator P_0 into

*) Equation (2·5) shows this result clearly. It realizes maxima if $e^{iG(\tilde{\lambda})}|(A_i)_{\rm inv}\rangle = |(A_i^{\tilde{\Lambda}})_{\rm inv}\rangle = |(A_i)_{\rm inv}\rangle$, where $A_i^{\tilde{\Lambda}} = \tilde{\Lambda}A_i\tilde{\Lambda}^{-1}$ with $\tilde{\Lambda} = e^{i\tilde{\lambda}}$. As in (B·5), this condition is satisfied when $\tilde{\Lambda} \in Z_N$.

the partition function. In addition, the fermionic part has a global $U(1)$ symmetry (fermion number),[14] which we call $U_f(1)$. It introduces the factor $e^{i\theta Q_f}$ into $\hat{Z}$, where Q_f is the charge of $U_f(1)$. Thus it is natural to consider the function

$$\begin{aligned}\hat{Z}(\tilde{\lambda},\theta) &= \mathrm{Tr}(e^{-\beta H+iG(\tilde{\lambda})+i\theta Q_f}P_0)\\ &= \int DA_i D\psi D\bar{\psi}\langle A_i,\psi,\bar{\psi}|e^{-\beta H+iG(\tilde{\lambda})+i\theta Q_f}P_0|A_i,-\psi,-\bar{\psi}\rangle. \end{aligned} \tag{3·2}$$

Now using $e^{i\theta Q_f}|\psi\rangle = |e^{-i\theta}\psi\rangle$, we can show that (3·2) can be written in the form

$$\hat{Z}(\tilde{\lambda},\theta) = \int_{\Gamma_\theta} DA_\mu(x)D\psi(x)D\bar{\psi}(x)\delta_{\rm gf}\Delta_{\rm gh}e^{-S}\delta\left(\tilde{\lambda}-\frac{1}{V}\int d^3x\lambda(\boldsymbol{x})\right), \tag{3·3}$$

where the boundary condition Γ_θ implies (1·3) for A_μ and

$$\psi(\beta,\boldsymbol{x}) = -e^{i\theta}\psi(0,\boldsymbol{x}) \tag{3·4}$$

for fermions. In addition to the usual minus sign, which is a result of the trace formula (2·4), $e^{i\theta Q_f}$ introduces the factor $e^{i\theta}$.

From (3·3), we can calculate the free energy density $F(q,s) = -\frac{T}{V}\ln\hat{Z}(\tilde{\lambda},\theta)$ in the one-loop approximation as

$$F(q,s) = \pi^2T^4[V_b(q) + N_f V_f(q,s) + V_{\rm base}], \tag{3·5}$$

where $s = \frac{\theta}{2\pi}$, and $V_b(q)$ and $V_{\rm base}$ are given in (1·8) and (1·11), respectively. The fermionic part becomes

$$V_f(q,s) = \frac{4}{3}\left[\frac{N}{16} - (N-1)h\left(\frac{q}{N}-s+\frac{1}{2}\right) - h\left(\frac{q}{N}-q-s+\frac{1}{2}\right)\right], \tag{3·6}$$

which coincides with (1·9) if $s = 0$ (mod 1).

Now let us consider

$$\frac{T}{V}\frac{\partial\ln\hat{Z}}{\partial\theta}(\tilde{\lambda},\theta=0) = i\langle\psi^\dagger\psi\rangle.$$

At the one-loop level, this expression becomes

$$2\pi\frac{\partial F}{\partial s}(q,s=0) = -i\langle\psi^\dagger\psi\rangle$$

with $\langle\psi^\dagger\psi\rangle$ in (1·14). The implication of problem (b) here is that $\langle\psi^\dagger\psi\rangle$ is purely imaginary at $(\tilde{\lambda},\theta) = (\tilde{\lambda}_n,0)$, except for $n = 0, N/2$. Thus $\frac{\partial F}{\partial\theta}(n,0)$ is real and nonzero for $n \neq 0, N/2$. This implies that F is unstable in the θ-direction at these points.

Furthermore, from (3·5), we can show

$$\frac{\partial^2 F}{\partial s^2}\left(\frac{N}{2},0\right) < 0.$$

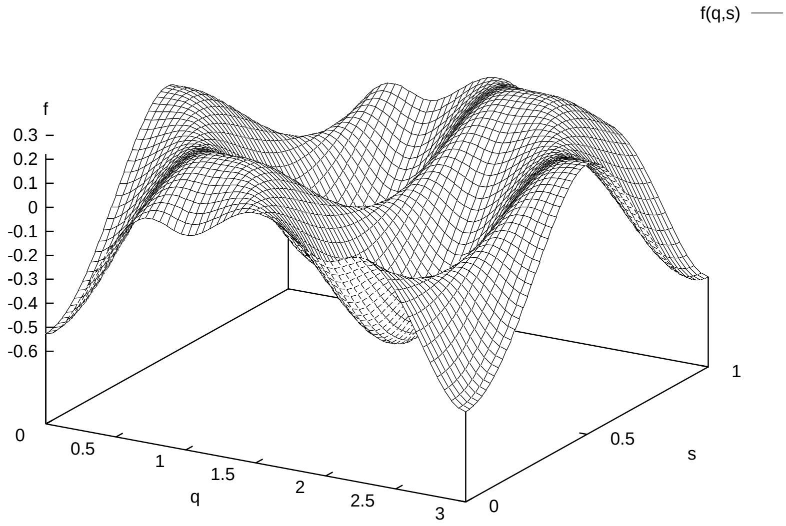

Fig. 2. Free energy density $f(q,s) = F(q,s)/(\pi^2 T^4)$ for $N = 3, N_f = 3$.

Namely, although $\frac{\partial F}{\partial s} = 0$ at $(\tilde{\lambda}_{N/2}, 0)$, this position is not a local minimum but a local maximum in the θ-direction. As an example, $f(q,s) = F(q,s)/(\pi^2 T^4)$ is plotted for $N = 3, N_f = 3$ in Fig. 2.

The minima of $F(q,s)$ are found easily. The function $h(q)$ has minima at $q = k$ and maxima at $q = k + \frac{1}{2}$ with integer k. Therefore the bosonic part $V_b(q)$ has minima at $q = n = 0, 1, \cdots, N-1 \pmod N$. Inserting $q = n$, the fermionic part $V_f(q,s)$ has minima at $s = \frac{n}{N}$. Thus the free energy density $F(q,s)$ has degenerate minima at

$$q = n, \quad s = \frac{n}{N}, \quad n = 0, 1, \cdots, N-1 \pmod N, \tag{3·7}$$

and the minimum value is $F(n, \frac{n}{N}) = \pi^2 T^4 V_{\text{base}}$. There is no local minimum with $F(q,s) > \pi^2 T^4 V_{\text{base}}$. We note that (3·4) with (3·7) becomes (1·5). This means that the minima with $n \neq 0$ are related to the minimum with $n = 0$ by the Z_N transformation. In other words, they are gauge equivalent, and there is one vacuum with $n = 0$ essentially.

We note that the above minima, i.e. the maxima of $\hat{Z}(\tilde{\lambda}, \theta)$, are determined without using the one-loop potential. From (3·2), it is evident that $\hat{Z}(\tilde{\lambda}, \theta)$ realizes maxima if $e^{iG(\tilde{\lambda})} e^{i\theta Q_f} |A_i, -\psi, -\bar{\psi}\rangle = |A_i, -\psi, -\bar{\psi}\rangle$. As in (B·7), this condition is satisfied by the values in (3·7).

§4. The necessity of θ

In the previous section, we showed that the positions specified by the values $(q,s) = (n, 0)$ with $n \neq 0$ are not metastable in the θ-direction. The usual partition function $Z(\tilde{\lambda})$ in (A·2) is the case with $\theta = 0$ in $\hat{Z}(\tilde{\lambda}, \theta)$. Our insistence is that, although θ yields the unfamiliar boundary condition (3·4), $\theta = 0$ should not be assumed from the outset.

Usually an antiperiodic boundary condition is imposed. This follows from the partition function

$$Z = \sum_k e^{-\beta E_k}, \tag{4·1}$$

which is written, by using a complete set of energy eigenstates ($H|k\rangle = E_k|k\rangle$), as

$$\begin{aligned} Z &= \sum_k \langle k|e^{-\beta H}|k\rangle \\ &= \mathrm{Tr} e^{-\beta H}. \end{aligned} \tag{4·2}$$

Since fermions possess the anticommuting property, there is a minus sign in the trace formula (2·4) for fermions. This minus sign is the origin of the antiperiodic boundary condition. In other words, an antiperiodic boundary condition is required to reproduce (4·1).

If H has a global symmetry G, states with energy E_k are degenerate, because $U(g)|k\rangle$, where $U(g)$ is the unitary representation of $g \in G$, have the same energy as $|k\rangle$. Hence the general transition amplitude with energy E_k is $\langle k|e^{-\beta H}U(g)|k\rangle$. We should take all the possible transition amplitudes into account, and a vacuum should be determined from them. Thus it is natural to consider the function

$$\hat{Z}(g) = \mathrm{Tr}[e^{-\beta H}U(g)].$$

Since H and $U(g)$ are simultaneously diagonalizable, this form of $\hat{Z}(g)$ can be rewritten as (2·1).[11]

We can treat $U(g)$ in two ways. One way is to put it into an action, which introduces the constant mode of A_0. The other way is to operate with it on external states, which changes the usual (anti)periodic boundary conditions. In the present case, there are the global symmetries $SU(N)$ and $U_f(1)$. The contribution of the former is put into the action, and that of the latter gives the unusual boundary condition (3·4).

In spite of the unfamiliar boundary condition, $\hat{Z}(\tilde{\lambda}, \theta)$ reproduces the original partition function (4·1) at its extremum point $(\tilde{\lambda}_n, \theta_n = \frac{2\pi n}{N})$. Contrastingly, if we set $\theta = 0$ from the outset, there are "metastable" states with $\tilde{\lambda} = \tilde{\lambda}_n$ $(n = 1, \cdots, N-1)$. However, in spite of the antiperiodic boundary condition, the original form (4·1) is not reproduced. Roughly speaking, we obtain $\sum_k e^{-\beta E_k} \cos(\mathrm{tr}(\tilde{\lambda}_n))$, which has unacceptable thermodynamical properties.

This situation is understood as follows. For the energy eigenstate $|k\rangle$, there is the degenerate state $e^{i\theta Q_f}|k\rangle$. When the operator $e^{-\beta H + iG(\tilde{\lambda}_n)}$ operates on $|k\rangle$, the general transition amplitude with energy E_k is $\langle k|e^{i\theta Q_f} \cdot e^{-\beta H + iG(\tilde{\lambda}_n)}|k\rangle$. As $e^{-\beta H + iG(\tilde{\lambda}_n)}|k\rangle = e^{-\beta E_k} e^{-i\theta_n Q_f}|k\rangle$, the diagonal element $\langle k|e^{-\beta H + iG(\tilde{\lambda}_n)}|k\rangle$ does not have the largest transition probability. Instead, transition to the state $\langle k|e^{i\theta_n Q_f}$ has the largest probability $\langle k|e^{i\theta_n Q_f} \cdot e^{-\beta H + iG(\tilde{\lambda}_n)}|k\rangle = \langle k|e^{-\beta H}|k\rangle = e^{-\beta E_k}$. Summing the largest amplitudes with respect to k, we obtain (4·1).

Thus, θ is necessary to describe the degeneracy allowed by the symmetry. Its value is determined from the minima of the free energy. At the minima, the original partition function (4·1) is recovered.

§5. Summary

The possibility of Z_N metastable states has been discussed in hot $SU(N)$ gauge theory with fundamental representation fermions. Although these states seem to be local minima of the effective potential, they have unphysical properties.

To study the problem of the metastable states, we reconsidered the partition function with the background A_0. Although the function $Z(\tilde{\lambda})$ in (A·2) was used in Ref. 10), functions based on (2·1) have also been used previously.*) We showed that $Z(\tilde{\lambda})$ is equivalent to $\hat{Z}(\tilde{\lambda})$ in (A·1). In $\hat{Z}(\tilde{\lambda})$, $e^{iG(\tilde{\lambda})}$ is related to the global $SU(N)$ symmetry, and P_0 makes external states gauge invariant.

Based on $\hat{Z}(\tilde{\lambda})$, the relation $F(0) \leq F(q)$ was reviewed. The origin of this inequality was found to be the phase factor $e^{iG(\tilde{\lambda})}$.

When fermions in the fundamental representation exist, there is the additional $U(1)$ symmetry related to the fermion number. Thus it is natural to put the factor $e^{i\theta Q_f}$ into $\hat{Z}$, though it has not previously been taken into account. We showed that the problem of the complex fermion number implies that the Z_N "metastable" states are unstable in the θ-direction. Since the transition from $|k\rangle$ to $\langle k|e^{i\theta Q_f}$ is possible, it is unreasonable that these states remain metastable. By using $\hat{Z}(\tilde{\lambda},\theta)$ in (3·2), we showed that there is essentially one vacuum and no Z_N metastable state.

Finally, we note the importance of using P_0 in $\hat{Z}$. Instead of P_0, one might insert the operator P defined by

$$P|\Psi\rangle = \int D\mu\left(\Omega(\boldsymbol{x})\right)|\Psi^{\Omega}\rangle.$$

$\Omega(\boldsymbol{x})$ in P_0 is restricted as $\lim_{|\boldsymbol{x}|\to\infty}\Omega(\boldsymbol{x}) = 1$, although that in P is not. In fact, the function

$$\mathrm{Tr}\left(e^{-\beta H}U(g)P\right) \tag{5·1}$$

with $U(g) = e^{iG(\tilde{\lambda})}$ was discussed in Ref. 10). However this function is incorrect. As $e^{iG(\tilde{\lambda})}P = P$, which is discussed in (A·17), (5·1) becomes

$$\mathrm{Tr}(e^{-\beta H}P).$$

That is, $\tilde{\lambda}$ disappears from the partition function, and, as a result, the potential for the Polyakov loop becomes trivial.[10)]

Appendix A
—— *The Relation of Partition Functions with Background A_0* ——

In this appendix, we derive

$$\hat{Z}(\tilde{\lambda}) = \mathrm{Tr}(e^{-\beta H + iG(\tilde{\lambda})}P_0), \tag{A·1}$$

*) See the references in Ref. 10).

which is studied in the main text, from the partition function [5),10)]

$$Z(\tilde{\lambda}) = \int_{\Gamma} DA_\mu(x) D\psi(x) D\bar{\psi}(x) \delta_{\rm gf} \varDelta_{\rm gh} e^{-S} \delta\left(\tilde{\lambda} - \frac{1}{V}\int d^3x \lambda(\boldsymbol{x})\right) \quad (\text{A}\cdot 2)$$
$$= e^{-\beta\mathcal{F}(\tilde{\lambda})},$$

where $S = \int \mathcal{L} dx$ with (1·2), and Γ represents the usual boundary conditions (1·3) and (1·4). Also, $\delta_{\rm gf}$ and $\varDelta_{\rm gh}$ are a gauge-fixing term and a corresponding ghost term, respectively. The element of the Cartan sub-algebra $\lambda(\boldsymbol{x})$ is obtained by diagonalizing the Wilson line

$$W(\boldsymbol{x}) = \mathcal{P} e^{ig\int_0^\beta A_0(t,\boldsymbol{x})dt}$$

as*)

$$W(\boldsymbol{x}) = U(\boldsymbol{x})\varLambda(\boldsymbol{x})U^\dagger(\boldsymbol{x}), \quad \varLambda(\boldsymbol{x}) = e^{i\lambda(\boldsymbol{x})}.$$

A "constraint" effective potential is defined by $\mathcal{V}(\tilde{\lambda}) = \mathcal{F}(\tilde{\lambda})/V$. When the spatial volume V becomes infinity, $\mathcal{V}$ becomes the usual effective potential. [16)] The limit $V \to \infty$ is assumed below.

Let us write a charged (off-diagonal) component of A_0 as $A_0^{\rm ch}$. In (A·2), the gauge $\partial_0 A_0 = 0, A_0^{\rm ch} = 0$ is chosen, and A_0 is transformed away from the action as

$$\tilde{\varOmega}^{1-\frac{t}{\beta}}(\boldsymbol{x})\left(A_0(\boldsymbol{x}) - \frac{i}{g}\partial_0\right)\tilde{\varOmega}^{\frac{t}{\beta}-1}(\boldsymbol{x}) = 0.$$

This equation gives

$$A_0 = -\frac{i}{g\beta}\ln\tilde{\varOmega}. \quad (\text{A}\cdot 3)$$

Substituting this into the definition of W, we find $W = \tilde{\varOmega}$. Since $\tilde{\varOmega}$ as well as A_0 is diagonal, this relation implies $U(\boldsymbol{x}) = 1$ and

$$\tilde{\varOmega}(\boldsymbol{x}) = \varLambda(\boldsymbol{x}) \quad (\text{A}\cdot 4)$$

in this gauge. Now we can show [13)] that (A·2) can be rewritten as [10)]

$$Z(\tilde{\lambda}) = \int D\bar{\mu}\left(\varLambda(\boldsymbol{x})\right)\int DA_i(\boldsymbol{x})D\psi(\boldsymbol{x})D\bar{\psi}(\boldsymbol{x})$$
$$\langle A_i, \psi, \bar{\psi}| e^{-\beta H} | A_i^\varLambda, -\psi^\varLambda, -\bar{\psi^\varLambda}\rangle \delta\left(\tilde{\lambda} - \frac{1}{V}\int d^3x\lambda(\boldsymbol{x})\right), \quad (\text{A}\cdot 5)$$

where H is the Hamiltonian in the $A_0 = 0$ gauge given in (3·1), $d\bar{\mu}(\varLambda)$ is the reduced Haar measure for the diagonal element $\varLambda$ in the Cartan subgroup $U(1)^{N-1} \in SU(N)$, and

$$\psi^\varLambda = \varLambda\psi, \quad A_i^\varLambda = \varLambda\left(A_i - \frac{i}{g}\partial_i\right)\varLambda^\dagger.$$

*) $U(\boldsymbol{x})$ may become singular at some points, where some eigenvalues of $\lambda(\boldsymbol{x})$ are degenerate. Since this is a variant of the maximal Abelian gauge, magnetic monopoles appear at these points. [15)] However, we do not consider such a case here, as we are interested in the vacuum at high T. For the same reason, configurations with topological charges are not taken into account.

The reduced Haar measure comes from the relation $D\bar{\mu}(\Lambda(\boldsymbol{x})) = DA_0(\boldsymbol{x})\delta_{\rm gf}\Delta_{\rm gh}$ in this gauge.[13)]

Under the constraint $\tilde{\lambda} = \frac{1}{V}\int d^3x\lambda(\boldsymbol{x})$ with $V \to \infty$, we put $\lim_{|\boldsymbol{x}|\to\infty}\lambda(\boldsymbol{x}) = \tilde{\lambda}$,[10)] and introduce $U(\boldsymbol{x}) \in SU(N)/U(1)^{N-1}$, which satisfies $\lim_{|\boldsymbol{x}|\to\infty}U(\boldsymbol{x}) = 1$. Using the invariance of H under the gauge transformation with $U^\dagger$, i.e.

$$\langle\Psi|e^{-\beta H}|\Psi^{U^\dagger \Lambda U}\rangle = \langle\Psi^U|e^{-\beta H}|(\Psi^U)^\Lambda\rangle,$$

and changing the integration variable as $\Psi^U \to \Psi(D\Psi = D\Psi^U)$, we obtain[13)]

$$\int DU\int D\Psi\langle\Psi|e^{-\beta H}|\Psi^{U^\dagger\Lambda U}\rangle = \text{const}\int D\Psi\langle\Psi|e^{-\beta H}|\Psi^\Lambda\rangle, \tag{A·6}$$

where the integration over the coset $\int DU$ becomes trivial, yielding an irrelevant constant. By using (A·6), (A·5) becomes

$$\begin{aligned} Z(\tilde{\lambda}) =& \int_{\Omega(\infty)=\Omega_\infty} D\mu\left(\Omega(\boldsymbol{x})\right)\int DA_i(\boldsymbol{x})D\psi(\boldsymbol{x})D\bar{\psi}(\boldsymbol{x}) \\ & \langle A_i,\psi,\bar{\psi}|e^{-\beta H}|A_i^\Omega,-\psi^\Omega,-\bar{\psi}^\Omega\rangle, \end{aligned} \tag{A·7}$$

where $\Omega = U^\dagger\Lambda U$ satisfies $\lim_{|\boldsymbol{x}|\to\infty}\Omega(\boldsymbol{x}) = \Omega_\infty$ with $\Omega_\infty = e^{i\tilde{\lambda}}$, and $d\mu(\Omega(\boldsymbol{x})) = d\bar{\mu}(\Lambda(\boldsymbol{x}))\,dU$ is the Haar measure of $\Omega \in SU(N)$.

We introduce ω as $\Omega(\boldsymbol{x}) = e^{i\omega(\boldsymbol{x})}$, and represent the $SU(N)$ generator by the fields as

$$G(\omega) = -\frac{1}{g}\int d^3x[E_i^a(D_i\omega)^a + g\omega^a\bar{\psi}\gamma_0 T^a\psi]. \tag{A·8}$$

Then using

$$|A_i^\Omega,-\psi^\Omega,-\bar{\psi}^\Omega\rangle = e^{iG(\omega)}|A_i,-\psi,-\bar{\psi}\rangle,$$

we find

$$\begin{aligned} \int_{\Omega(\infty)=\Omega_\infty} D\mu\left(\Omega(\boldsymbol{x})\right)|\Psi^\Omega\rangle &= e^{iG(\tilde{\lambda})}\int_{\Omega(\infty)=\Omega_\infty} D\mu\left(\Omega(\boldsymbol{x})\right)e^{-iG(\tilde{\lambda})}e^{iG(\omega)}|\Psi\rangle \\ &= e^{iG(\tilde{\lambda})}\int_{\Omega(\infty)=\Omega_\infty} D\mu\left(\Omega(\boldsymbol{x})\right)|\Psi^{\Omega_\infty^{-1}\Omega}\rangle \\ &= e^{iG(\tilde{\lambda})}\int_{\Omega'(\infty)=1} D\mu\left(\Omega'(\boldsymbol{x})\right)|\Psi^{\Omega'}\rangle, \end{aligned}$$

where $\Omega' = \Omega_\infty^{-1}\Omega$ satisfies $\lim_{|\boldsymbol{x}|\to\infty}\Omega'(\boldsymbol{x}) = 1$, and the left invariance of the Haar measure has been used. Thus we obtain

$$\int_{\Omega(\infty)=\Omega_\infty} D\mu\left(\Omega(\boldsymbol{x})\right)|\Psi^\Omega\rangle = e^{iG(\tilde{\lambda})}P_0|\Psi\rangle, \tag{A·9}$$

where, by neglecting the prime for Ω',

$$P_0|\Psi\rangle = \int_{\Omega(\infty)=1} D\mu\left(\Omega(\boldsymbol{x})\right)|\Psi^\Omega\rangle \tag{A·10}$$

$$= \int_{\Omega(\infty)=1} D\mu\left(\Omega(\boldsymbol{x})\right)e^{iG(\omega)}|\Psi\rangle. \tag{A·11}$$

Substituting (A·9) into (A·7), we find

$$Z(\tilde{\lambda}) = \int DA_i(\boldsymbol{x})D\psi(\boldsymbol{x})D\bar{\psi}(\boldsymbol{x})\langle A_i, \psi, \bar{\psi}|e^{-\beta H}e^{iG(\tilde{\lambda})}P_0|A_i, -\psi, -\bar{\psi}\rangle. \qquad \text{(A·12)}$$

Using the trace formula (2·4), (A·12) becomes $\mathrm{Tr}(e^{-\beta H+iG(\tilde{\lambda})}P_0)$. We thus see that $Z(\tilde{\lambda})$ in (A·2) is tantamount to $\hat{Z}(\tilde{\lambda})$ in (A·1).

Finally, we define gauge invariant states and rewrite $Z(\tilde{\lambda})$. Since the left and right invariant measures are equivalent for $SU(N)$, $d\mu(\Omega) = d\mu(\Omega^{\tilde{\lambda}})$ with $\Omega^{\tilde{\lambda}} = \Omega_\infty \Omega \Omega_\infty^{-1}$ holds. Integrating the identity

$$e^{iG(\tilde{\lambda})}e^{iG(\omega)} = e^{iG(\omega^{\tilde{\lambda}})}e^{iG(\tilde{\lambda})}, \quad e^{iG(\omega^{\tilde{\lambda}})} = e^{iG(\tilde{\lambda})}e^{iG(\omega)}e^{-iG(\tilde{\lambda})}$$

over Ω, we find

$$\begin{aligned} e^{iG(\tilde{\lambda})}\int_{\Omega(\infty)=1} D\mu\left(\Omega(\boldsymbol{x})\right)e^{iG(\omega)} &= \int_{\Omega(\infty)=1} D\mu\left(\Omega(\boldsymbol{x})\right)e^{iG(\omega^{\tilde{\lambda}})}e^{iG(\tilde{\lambda})} \\ &= \int_{\Omega^{\tilde{\lambda}}(\infty)=1} D\mu\left(\Omega^{\tilde{\lambda}}(\boldsymbol{x})\right)e^{iG(\omega^{\tilde{\lambda}})}e^{iG(\tilde{\lambda})}. \end{aligned}$$

Therefore

$$e^{iG(\tilde{\lambda})}P_0 = P_0 e^{iG(\tilde{\lambda})}. \qquad \text{(A·13)}$$

Using (A·13) and $P_0^2 = P_0$, which is a property of the projection operator, (A·12) becomes

$$Z(\tilde{\lambda}) = \int DA_i(\boldsymbol{x})D\psi(\boldsymbol{x})D\bar{\psi}(\boldsymbol{x})\langle(A_i, \psi, \bar{\psi})_{\mathrm{inv}}|e^{-\beta H+iG(\tilde{\lambda})}|(A_i, -\psi, -\bar{\psi})_{\mathrm{inv}}\rangle, \qquad \text{(A·14)}$$

where $|(\Psi)_{\mathrm{inv}}\rangle = P_0|\Psi\rangle$ is an invariant state under the gauge transformation $e^{iG(\alpha)}$ with $\lim_{|\boldsymbol{x}|\to\infty}\alpha(\boldsymbol{x}) = 0$. When fermions are absent, (A·14) is (2·5).

We make some comments here. First we note the relation between $\tilde{\lambda}$ and the constant part a_q in (1·6). From (A·3) with (A·4), we find

$$\tilde{\lambda} = g\beta a_q. \qquad \text{(A·15)}$$

Equation (A·15) holds to any loop order in the above gauge, which is equivalent to the static background gauge in Ref. 10). In other gauges, (A·15) holds at leading order. Thus we can apply (A·15) to the one-loop free energy density in the text.

The second comment is in regard to the projection operator. In (A·14), P_0 operates on the external states to make them gauge invariant. As $\Omega(\boldsymbol{x})$ in P_0 satisfies $\Omega(\infty) = 1$, (A·13) holds. Contrastingly, let us consider the operator P defined by

$$P|\Psi\rangle = \int D\mu\left(\Omega(\boldsymbol{x})\right)|\Psi^{\Omega}\rangle, \qquad \text{(A·16)}$$

where the restriction $\Omega(\infty) = 1$ is not imposed. This operator also satisfies $P^2 = P$, and makes external states gauge invariant. However, as

$$\begin{aligned} e^{iG(\tilde{\lambda})}P|\Psi\rangle &= \int D\mu\left(\Omega(\boldsymbol{x})\right)|\Psi^{\Omega_\infty\Omega}\rangle \\ &= \int D\mu\left(\Omega_\infty\Omega(\boldsymbol{x})\right)|\Psi^{\Omega_\infty\Omega}\rangle, \end{aligned}$$

P satisfies

$$e^{iG(\tilde{\lambda})}P = P \tag{A·17}$$

instead of (A·13). The partition function with P is discussed in the Summary.

Appendix B
—— *Proof of the Inequality* ——

Let us consider the amplitude

$$\langle \Psi | \Psi^U \rangle,$$

where Ψ^U is obtained by performing a unitary transformation U on Ψ. From the Schwarz inequality, we have

$$\left| \langle \Psi | \Psi^U \rangle \right| \leq \| \Psi \| \| \Psi^U \|. \tag{B·1}$$

Since $\|\Psi^U\|^2 := \langle \Psi^U | \Psi^U \rangle = \langle \Psi | \Psi \rangle = \|\Psi\|^2$, (B·1) becomes

$$\left| \langle \Psi | \Psi^U \rangle \right| \leq \langle \Psi | \Psi \rangle.$$

Using

$$\langle \Psi | \Psi^U \rangle \leq \left| \langle \Psi | \Psi^U \rangle \right|, \tag{B·2}$$

we find

$$\langle \Psi | \Psi^U \rangle \leq \langle \Psi | \Psi \rangle. \tag{B·3}$$

From (B·1) and (B·2), the equality holds if and only if

$$|\Psi^U \rangle = |\Psi \rangle. \tag{B·4}$$

Now we consider examples used in the text. For the internal symmetry $SU(N)$, let us put $U = e^{i\tilde{\lambda}}$, where $\tilde{\lambda} = \sum_{a=1}^{N-1} \tilde{\lambda}^a T^a$ with diagonal and traceless $N \times N$ matrices T^a. Since $A_i^U = U A_i U^{-1} = A_i$ for $U = z_n \in Z_N$ and $\psi^U = U\psi = \psi$ for $U = 1$, we obtain

$$|A_i^U \rangle = e^{iG(\tilde{\lambda})} |A_i \rangle = |A_i \rangle \quad \text{if} \quad e^{i\tilde{\lambda}} = z_n \in Z_N, \tag{B·5}$$

$$\begin{aligned} |A_i^U, -\psi^U, -\bar{\psi}^U \rangle &= e^{iG(\tilde{\lambda})} |A_i, -\psi, -\bar{\psi} \rangle \\ &= |A_i, -\psi, -\bar{\psi} \rangle \quad \text{if} \quad e^{i\tilde{\lambda}} = 1. \end{aligned} \tag{B·6}$$

As another example, let us choose $U = e^{i\tilde{\lambda}} e^{-i\theta}$. Since the $U(1)$ phase $e^{-i\theta}$ does not change A_i, we have $A_i^U = U A_i U^{-1} = A_i$ for $e^{i\tilde{\lambda}} = z_n$. With this value, $\psi^U = e^{i\tilde{\lambda}} e^{-i\theta} \psi = \psi$ for $e^{-i\theta} = z_n^*$. Therefore

$$\begin{aligned} |A_i^U, -\psi^U, -\bar{\psi}^U \rangle &= e^{iG(\tilde{\lambda})} e^{i\theta Q_f} |A_i, -\psi, -\bar{\psi} \rangle \\ &= |A_i, -\psi, -\bar{\psi} \rangle \quad \text{if} \quad e^{i\tilde{\lambda}} = z_n, e^{-i\theta} = z_n^*. \end{aligned} \tag{B·7}$$

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