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Exact BRS Symmetry Realized along the Renormalization Group Flow

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Using the average action, defined with a continuum analog of the block spin transformation, we demonstrate the presence of a gauge symmetry along the Wilsonian renormalization group flow. As a reflection of this gauge symmetry, the average action satisfies the quantum master equation (QME). We show that the quantum part of the master equation can be naturally understood, once the measure contribution under the BRS transformation is taken into account. Furthermore, an effective BRS transformation acting on macroscopic fields may be defined from the QME. The average action is explicitly evaluated in terms of the saddle point approximation up to one-loop order. It is confirmed that the action satisfies the QME and the flow equation.

§1. Introduction

For the definition of the Wilsonian effective action,[1)-3)] one needs to introduce some regularization. Therefore, it is a nontrivial problem if symmetries such as chiral or gauge symmetry can survive along the renormalization group (RG) flow, and if so how they can be realized in the effective theory.

An important contribution to see a (modified or broken) gauge symmetry along the RG flow was made by Ellwanger.[4)] He showed that there exists the broken Ward-Takahashi (WT) or Slavnov-Taylor identity along the flow expressed as $\Sigma_k = 0$ in his notation,[*)] where k denotes an IR cutoff. Once we find a theory on the hypersurface defined by $\Sigma_k = 0$ in the coupling space, it remains on the surface along the RG flow, and in the limit of $k \to 0$ the identity reduces to the Zinn-Justin equation. The broken WT identity is, in this sense, connected to the usual WT identity. This viewpoint suggests that we could modify the gauge symmetry broken due to the regularization in such a way that it could be connected smoothly to the usual gauge symmetry.

It had been long believed that the realization of a chiral symmetry on the lattice was impossible.[5)] However Lüscher[6)] took an important step by finding an exact chiral symmetry on the lattice a decade after the Ginsparg-Wilson paper.[7)] His chiral symmetry has a different form than the continuum chiral symmetry.

The above example may suggest the possibility that a symmetry in a field theory could survive even after regularization and that its form after regularization could be generally different from its familiar form. In an earlier publication,[8)] we pursued this possibility in the context of Wilsonian RG. We defined a procedure to give an effective

[*)] We use the same notation, Σ_k, for the corresponding quantity in our formulation.

field theory with an IR cutoff. In this setting it was shown that we may define a quantity similar to Ellwanger's Σ_k: the equation $\Sigma_k = 0$ is found to be the quantum master equation (QME). We also explicitly constructed the symmetry transformation on the macroscopic fields, which was called the renormalized transformation. With this result we claimed that a symmetry survives the regularization and is maintained along the RG flow. We emphasize that the symmetry along the flow is exact and is not "modified" or "broken". The free Maxwell theory and the chiral symmetry were the two examples studied in Ref. 8). For the latter, we obtained continuum analogs of the Ginsparg-Wilson relation and the Lüscher symmetry. In these specific examples, the average actions are invariant under the renormalized transformations, and the QME reduces to the classical master equation. This is, however, not the case for more general interacting theories.

In the present paper we show that our procedure may be naturally extended to an interacting gauge theory, typically the non-Abelian gauge theory coupled to matter fields. A major difference from the earlier examples considered previously is the presence of the quantum part in the master equation. Although this had been regarded as a "breaking" term of the symmetry, we will see that its presence is necessary to maintain the symmetry. The renormalized BRS transformation is given as in our previous paper. To see more explicitly how our formulation works, we evaluate the average action with the saddle point approximation up to one-loop order. It is shown that this action satisfies both the master equation and the flow equation.

This paper is organized as follows. In §2, after a brief explanation of Batalin-Vilkovisky (BV) antifield formalism, [9),*)] the average action is introduced and shown to satisfy the QME and the RG flow equation. For the BRS invariance of the average action, the quantum part of the master equation naturally emerges. This is the subject of §3. The renormalized BRS transformation is also given there. In §4 we evaluate the average action with the saddle point approximation. The last section is devoted to summary and further discussion of the average action. An explanation of our notation is given in Appendix A. Some relations in §4 are proved in Appendices B and C.

Owing to the presence of Grassmann odd fields, we have to keep track of signs carefully. In order to make equations correct and, at the same time, as simple as possible, we introduce abbreviations whenever possible.

§2. The average action and its properties

The average action was introduced by Wetterich [11)] to realize a continuum analog of the block spin transformation. Before presenting it, let us describe the microscopic action and its properties in the antifield formalism.

*) For reviews, see Ref. 10)

2.1. *The antifield formalism*

In the following, ϕ_a denote all the fields in the system under consideration: e.g., gauge, ghosts, antighosts, B-fields and matters for the non-Abelian theory. Further, we introduce their antifields, denoted by ϕ^*_a. For the gauge fixing, we perform the canonical transformation $\phi_a \to \phi_a$, $\phi^*_a \to \phi^*_a + \partial\Psi/\partial\phi_a$, where Ψ is the gauge fermion, a function only of the fields. This gauge-fixed basis is convenient, since it retains the antifields. Let $S_0[\phi]$ be a BRS invariant gauge-fixed action in the new basis. We then consider an extended action, linear in the antifields,

$$S[\phi, \phi^*] \equiv S_0[\phi] + \phi^*\delta\phi. \tag{2·1}$$

Here $\delta\phi_a$ is the BRS transformation of ϕ_a. The full expression of the second term is given in Eq. (A·2).

Under the set of BRS transformations

$$\begin{aligned}\delta\phi_a &= \frac{\overrightarrow{\partial}S}{\partial\phi^*_a} = (-1)^{\epsilon_a+1}\frac{\overleftarrow{\partial}S}{\partial\phi^*_a},\\ \delta\phi^*_a &= -\frac{\overrightarrow{\partial}S}{\partial\phi_a} = (-1)^{\epsilon_a+1}\frac{\overleftarrow{\partial}S}{\partial\phi_a},\end{aligned} \tag{2·2}$$

the extended action $S[\phi, \phi^*]$ is invariant:

$$\delta S[\phi, \phi^*] = \delta S_0[\phi] + \phi^*\delta^2\phi + (-1)^{\epsilon_a+1}\delta\phi^*_a\delta\phi_a = 0. \tag{2·3}$$

Here ϵ_a is the Grassmann parity of the field ϕ_a. The sign factor in the third term of Eq. (2·3) appears since we have chosen the BRS transformation to act from the right. Another important sign factor appears in changing a right derivative to a left derivative and vice versa, as in (2·2). (See (A·1) for a general formula.)

With the antibracket

$$(F, G)_\phi \equiv \frac{F\overleftarrow{\partial}}{\partial\phi}\frac{\overrightarrow{\partial}G}{\partial\phi^*} - \frac{F\overleftarrow{\partial}}{\partial\phi^*}\frac{\overrightarrow{\partial}G}{\partial\phi}, \tag{2·4}$$

the BRS transformation may be written as $\delta F \equiv (F, S)_\phi$. In terms of the antibracket, the gauge invariance of the action is nicely expressed as the classical master equation: $(S, S)_\phi = 0$. In Eq. (2·4), the summation over indices and the momentum integration are implicit.

For the following discussion, the action (2·1) is our starting point. Thus we assume that the action is linear in the antifield ϕ^*. This includes the Yang-Mills fields coupled to matter fields as a typical and important example. Actually, we can extend our consideration to the case of an action with nonlinear ϕ^* dependence. This will be discussed in Ref. 13).

2.2. *The average action*

The average action Γ_k, with IR cutoff k, can be written in terms of macroscopic fields Φ after integrating out the high frequency modes (see also (3·3)) as

$$e^{-\Gamma_k[\Phi,\phi^*]/\hbar} = \int \mathcal{D}\phi e^{-S_k[\phi,\Phi,\phi^*]/\hbar}, \tag{2·5}$$

$$S_k[\phi, \Phi, \phi^*] = S_0[\phi] + \phi^* \delta\phi + \frac{1}{2}(\Phi - f_k\phi)\ R_k\ (\Phi - f_k\phi). \qquad (2\cdot6)$$

The third term on the rhs of Eq. (2·6) represents our abbreviated notation for the full expression given in Eq. (A·3). The functions $f_k(p)$ and $R_k(p)$ should be chosen appropriately, so that the macroscopic fields carry momentum whose magnitude is less than k. Though we do not need their explicit forms in this paper, it would be instructive to see how the high-frequency modes are integrated out in the above path integral.

To realize a continuum analog of the block spin transformation, Wetterich specified some conditions on the functions. For example,

$$f_k(p) = \exp\left(-\alpha\left(\frac{p^2}{k^2}\right)^{\beta}\right),$$
$$[R_k(p)]_{ab} = (1 - f_k^2(p))^{-1} \times [\mathcal{R}_k(p)]_{ab},$$

with positive α and β are functions that satisfy these conditions. (See Ref. 11) for details.) The components of the matrix $[\mathcal{R}_k(p)]_{ab}$ are at most polynomials in p.

Note the following: 1) the function $f_k(p)$ is close to 1 for the momentum lower than k and decreases rapidly as the momentum increases, and 2) consequently the factor $(1-f_k^2(p))^{-1}$ in $R_k(p)$ is almost constant for high momenta and is very large for momenta lower than k. The p dependence of $[\mathcal{R}_k(p)]_{ab}$ adds only minor modulation to this behavior. This implies that $\Phi(p) \sim \phi(p)$ for $p < k$, while $\Phi(p)$ with $p > k$ does not carry any information of the microscopic dynamics and appears in a simple quadratic form in the average action. In the remainder of the paper, we do not need the explicit forms of the functions, and we only assume the following properties: $f_k(-p) = f_k(p)$ and $[R_k(p)]_{ab} = (-)^{\epsilon_a\epsilon_b}[R_k(-p)]_{ba}$, while the components of R_k vanish for mixed Grassmann parity indices.

2.3. *The quantum master equation*

An important question is how the gauge symmetry at the microscopic level is reflected in $\Gamma_k[\Phi, \phi^*]$. The answer was given in our earlier paper:[8)] the macroscopic action satisfies the QME.

Let us consider the identity

$$\int \mathcal{D}\phi e^{-S_k[\phi+\delta\phi\lambda,\Phi,\phi^*]/\hbar} - \int \mathcal{D}\phi e^{-S_k[\phi,\Phi,\phi^*]/\hbar} = 0, \qquad (2\cdot7)$$

with the Grassmann odd parameter λ. We have assumed here the BRS invariance of the measure $\mathcal{D}\phi$, and thus anomalies are not consider here. Because of the BRS invariance of the microscopic action, Eq. (2·7) is proportional to the path integral average of $(\Phi - f_k\phi)R_k f_k\delta\phi$. It is expressed as $\Delta_\Phi e^{-\Gamma_k/\hbar} = 0$, where

$$\Delta_\Phi \equiv \sum_a (-)^{\epsilon_a+1} \int \mathrm{d}p f_k(p) \frac{\partial^r}{\partial \Phi_a(-p)} \frac{\partial^r}{\partial \phi_a^*(p)}.$$

We thus obtain the QME,

$$\Sigma_k[\Phi, \phi^*] \equiv \hbar^2 e^{\Gamma_k/\hbar} \Delta_\Phi e^{-\Gamma_k/\hbar}$$

$$= \frac{1}{2}(\Gamma_k[\Phi,\phi^*],\Gamma_k[\Phi,\phi^*])_\Phi - \hbar\Delta_\Phi\Gamma_k[\Phi,\phi^*] = 0, \qquad (2\cdot 8)$$

where the bracket is defined in terms of Φ and ϕ^*:

$$(F,G)_\Phi \equiv \int \mathrm{d}^4 p f_k(p)[\frac{F\overleftarrow{\partial}}{\partial\Phi_a(-p)}\frac{\overrightarrow{\partial}G}{\partial\phi^*_a(p)} - \frac{F\overleftarrow{\partial}}{\partial\phi^*_a(-p)}\frac{\overrightarrow{\partial}G}{\partial\Phi_a(p)}]. \qquad (2\cdot 9)$$

A comparison of Eqs. (2·4) and (2·9) suggests that ϕ^*/f_k may be regarded as the antifield associated with Φ.

2.4. *The flow equation for the average action*

A straightforward calculation leads us to the flow equation:

$$\hbar\partial_k e^{-\Gamma_k[\Phi,\phi^*]/\hbar} = -\left[X + \frac{\hbar}{2}\mathrm{Str}(R_k^{-1}\partial_k R_k) + \hbar\ \mathrm{Str}(\partial_k(\ln f_k))\right] e^{-\Gamma_k[\Phi,\phi^*]/\hbar}, \quad (2\cdot 10)$$

$$X \equiv -\frac{\hbar^2}{2}\frac{\partial^l}{\partial\Phi}(\partial_k R_k^{-1})\frac{\partial^r}{\partial\Phi} + \partial_k(\ln f_k)\left[\hbar^2\frac{\partial^l}{\partial\Phi}R_k^{-1}\frac{\partial^r}{\partial\Phi} + \hbar\Phi\frac{\partial^l}{\partial\Phi}\right]. \quad (2\cdot 11)$$

Here we have used the fact that $(R_k)_{\mathrm{even\ odd}} = (R_k)_{\mathrm{odd\ even}} = 0$, in our choice for R_k.

An interesting property of the quantity $\Sigma_k[\Phi,\phi^*]$ was found by Ellwanger:[4)] using the flow equation (2·10), it can be shown that

$$\hbar\partial_k\Sigma_k = (e^{\Gamma_k/\hbar}Xe^{-\Gamma_k/\hbar})\Sigma_k - e^{\Gamma_k/\hbar}X\ (e^{-\Gamma_k/\hbar}\Sigma_k). \qquad (2\cdot 12)$$

Therefore, once we are on the hypersurface $\Sigma_k = 0$ in the space of couplings in Γ_k, the same condition will continue to hold, even if we change the IR cutoff k. Note that the rhs vanishes when $\Sigma_k = 0$, since the QME $\Sigma_k = 0$ for a given k is the identity for any Φ and ϕ^*, so that its functional derivatives should vanish.

§3. The QME and the renormalized BRS transformation

In earlier works it had been generally understood that the momentum cutoff breaks gauge invariance; we only have the condition[14)] so that the gauge invariance is recovered when the IR cutoff is removed. The condition was beautifully summarized in Ref. 4), and its connection to the QME was clarified in our earlier paper.[8)] The commonly shared view is that terms corresponding to $\Delta_\Phi\Gamma_k$ represent the breaking of the gauge invariance.[*)] Here we show that the BRS invariance is maintained, *including* $\Delta_\Phi\Gamma_k$ term.

In the following we first explain how a QME is related to the BRS invariance of a generic gauge-invariant system. We find the variation of the path integral measure is exactly the $\Delta_\Phi\Gamma_k$ term. Based on this understanding, we may define the renormalized BRS transformation for the macroscopic fields.

[*)] If one uses the average action, the condition is written in a very simple form as the QME. Of course, in other formalisms it looks completely different, and the “breaking terms” have very different forms.

3.1. *A generic gauge system*

Let us consider a generic gauge system with the action $\mathcal{A}[\eta,\eta^*]$, where (η,η^*) could be the microscopic fields (ϕ,ϕ^*) or the macroscopic fields (Φ,ϕ^*). In order to formulate the BRS invariance of this quantum system, we consider the identity

$$\int \mathcal{D}\eta e^{-\mathcal{A}[\eta,\eta^*]/\hbar} = \int \mathcal{D}\eta' e^{-\mathcal{A}[\eta',\eta^*]/\hbar},$$

with the new variables given by the transformation

$$\eta' = \eta + \delta\eta\lambda,$$
$$\delta\eta = (\eta,\mathcal{A})_\eta = \frac{\overrightarrow{\partial}\mathcal{A}}{\partial\eta^*},$$

where λ is the transformation parameter. Then, the BRS invariance of the path integral including the measure may be written as $\delta(\mathcal{A}[\eta,\eta^*]-\hbar\ln\mathcal{D}\eta)=0$. We will presently see that this is nothing but a QME and that its quantum part is due to the variation of the measure.

Let us look at the first term in the above-mentioned equation,

$$\delta\mathcal{A}[\eta,\eta^*] = \frac{\mathcal{A}\overleftarrow{\partial}}{\partial\eta}\delta\eta = \frac{\mathcal{A}\overleftarrow{\partial}}{\partial\eta}\frac{\overrightarrow{\partial}\mathcal{A}}{\partial\eta^*} = \frac{1}{2}(\mathcal{A},\mathcal{A})_\eta.$$

If we assume that the path integral measure is flat, $\mathcal{D}\eta=\prod_a \mathrm{d}\eta_a$, the logarithm of the measure transforms as $\ln\mathcal{D}\eta' = \ln\mathcal{D}\eta + (\delta\ln\mathcal{D}\eta)\lambda$:*)

$$(\delta\ln\mathcal{D}\eta)\lambda = \ln\,\mathrm{Sdet}\frac{\partial^r}{\partial\eta_a}(\eta+\frac{\partial^l\mathcal{A}}{\partial\eta^*}\lambda)_b \sim \frac{\overrightarrow{\partial}}{\partial\eta_a^*}\mathcal{A}[\eta,\eta^*]\frac{\overleftarrow{\partial}}{\partial\eta_a}\lambda. \tag{3·1}$$

Therefore, including the contribution from the measure, we obtain the QME,

$$\frac{1}{2}(\mathcal{A}[\eta,\eta^*],\mathcal{A}[\eta,\eta^*])_\eta - \hbar\frac{\overrightarrow{\partial}}{\partial\eta_a^*}\mathcal{A}[\eta,\eta^*]\frac{\overleftarrow{\partial}}{\partial\eta_a} = 0. \tag{3·2}$$

3.2. *The average action*

Consider the path integral

$$\int \mathcal{D}\phi e^{-S[\phi,\phi^*]/\hbar}$$
$$= \int \mathcal{D}\Phi\mathcal{D}\phi\; e^{-S_k[\phi,\Phi,\phi^*]/\hbar} = \int \mathcal{D}\Phi e^{-\Gamma_k[\Phi,\phi^*]/\hbar}. \tag{3·3}$$

To the original path integral, we insert the gaussian integration with respect to Φ and reverse the order of the integrations. Then we find the path integral over the average action with a flat measure for the Φ-integration. The gauge symmetry of the original system is expressed as the classical master equation. The path integral of the average action carries the same information. As evident from our general argument, the symmetry can be expressed as the QME, with its quantum part $\Delta_\Phi\Gamma_k$ coming from the transformation of the path integral measure.

*) The argument leading to Eq. (3·1) is taken from Ref. 12).

3.3. *The renormalized BRS transformation*

From the above argument, we see that the renormalized BRS transformation may be read off from the classical part of the QME: [8)]

$$\delta_r \Phi \equiv f_k \frac{\overrightarrow{\partial} \Gamma_k}{\partial \phi^*} = f_k \langle \delta\phi \rangle_\phi, \tag{3·4}$$

$$\delta_r \phi^* \equiv -f_k \frac{\overrightarrow{\partial} \Gamma_k}{\partial \Phi} = -\langle f_k R_k (\Phi - f_k \phi) \rangle_\phi. \tag{3·5}$$

Here we have used the definition

$$\langle \mathcal{O} \rangle_\phi \equiv \int \mathcal{D}\phi \; \mathcal{O} \; e^{-S_k/\hbar} \Big/ \int \mathcal{D}\phi \; e^{-S_k/\hbar}. \tag{3·6}$$

In Ref. 14), the cutoff-dependent BRS transformation was considered in a different approach.

Some comments are in order. First, let us emphasize that the quantum part had long been understood to suggest the breaking of the gauge symmetry, which is not the correct understanding from our viewpoint. Second, as far as we know, this is the second example in which the quantum part of a QME plays an important role. (The first example is the string field theory (SFT). [12)]) It is probably very important to keep in mind that the QME is deeply related to the unitarity of the SFT.

§4. The average action in the saddle point approximation

It is usually unfeasible to fully evaluate the path integral (2·5) to construct an average action. In order to understand the formalism in more concrete terms, a systematic evaluation of the average action in (2·5) is quite instructive. The loop expansion with the saddle point method suits our purpose: it provides a way to integrate out high frequency modes systematically. In this section we calculate the average action up to one-loop order.

The saddle point, $\phi(p) = \phi_0(p)$, is determined by the equation

$$-f_k R_k (\Phi - f_k \phi_0) + \frac{\overrightarrow{\partial} (\phi_a^* P_a[\phi_0] + S_0[\phi_0])}{\partial \phi_0} = 0, \tag{4·1}$$

where $P_a[\phi]$ denotes the BRS transformation of ϕ_a: $P_a[\phi] \equiv \delta\phi_a$. Note that in Eq. (4·1) we have omitted the indices and the momentum dependence for simplicity. The left derivative, $\overrightarrow{\partial}/\partial\phi_0$, in the second term is taken with ϕ^* fixed. The saddle point equation gives an implicit function, $\phi_0 = \phi_0[\Phi, \phi^*]$.

Now the average action at the tree level is given as

$$\Gamma_k^{(0)}[\Phi, \phi^*] \equiv S_k[\phi_0[\Phi, \phi^*], \Phi, \phi^*], \tag{4·2}$$

and the one-loop correction is the superdeterminant,

$$\Gamma_k^{(1)}[\Phi, \phi^*] = \frac{\hbar}{2} \ln \mathrm{Sdet}(A[\phi_0, \phi^*]), \tag{4·3}$$

of the matrix A:

$$A_{ab}[\phi_0,\phi^*] = f_k^2[R_k]_{ab} + \frac{\overrightarrow{\partial}}{\partial\phi_0^a}(\phi_c^* P_c[\phi_0] + S_0[\phi_0])\frac{\overleftarrow{\partial}}{\partial\phi_0^b}. \qquad (4\cdot4)$$

We next show that the one-loop average action, $\mathit{\Gamma}_k^{(0)} + \mathit{\Gamma}_k^{(1)}$, satisfies both the QME and the flow equation.

4.1. *The one-loop QME*

The QME to be proved may be rewritten as

$$(\mathit{\Gamma}_k^{(0)}, \mathit{\Gamma}_k^{(0)})_\Phi = 0, \qquad (4\cdot5)$$

$$(\mathit{\Gamma}_k^{(0)}, \mathit{\Gamma}_k^{(1)})_\Phi - \hbar\Delta_\Phi \mathit{\Gamma}_k^{(0)} = 0, \qquad (4\cdot6)$$

where we have used the fact that $\frac{1}{2}(\mathit{\Gamma}_k^{(0)}, \mathit{\Gamma}_k^{(1)})_\Phi = \frac{1}{2}(\mathit{\Gamma}_k^{(1)}, \mathit{\Gamma}_k^{(0)})_\Phi$, which is easily seen by using $\overrightarrow{\partial}\mathit{\Gamma}_k^{(0)}/\partial\Phi_a = (-1)^{\epsilon_a}\mathit{\Gamma}_k^{(0)}\overleftarrow{\partial}/\partial\Phi_a$, etc.

The validity of the tree level master equation (4·5) may be confirmed by using the tree level renormalized BRS transformations for Φ and ϕ^*:

$$\delta_r^{(0)}\Phi = (\Phi, \mathit{\Gamma}_k^{(0)})_\Phi = f_k P[\phi_0], \qquad (4\cdot7)$$

$$\begin{aligned}\delta_r^{(0)}\phi^* &= (\phi^*, \mathit{\Gamma}_k^{(0)})_\Phi = -f_k R_k(\Phi - f_k\phi_0)\\ &= -\frac{\overrightarrow{\partial}(\phi_a^* P_a[\phi_0] + S_0[\phi_0])}{\partial\phi_0}.\end{aligned} \qquad (4\cdot8)$$

The final expression in Eq. (4·8) follows from the saddle point equation (4·1). Further, using Eqs. (4·7) and (4·8), we may obtain the transformation of the implicit function $\phi_0^a[\Phi,\phi^*]$:

$$\begin{aligned}\delta_r^{(0)}\phi_0^a[\Phi,\phi^*] &= (\phi_0\overleftarrow{\partial}/\partial\Phi)_{ab}\delta_r^{(0)}\Phi_b + (\phi_0\overleftarrow{\partial}/\partial\phi^*)_{ab}\delta_r^{(0)}\phi_b^*\\ &= P_a[\phi_0].\end{aligned} \qquad (4\cdot9)$$

This is shown in Appendix B. From (4·2) we see that $\mathit{\Gamma}_k^{(0)}$ may be written as the rhs of (2·6), with ϕ replaced by ϕ_0. It is easy to see that the first and second terms of that expression are invariant under Eqs. (4·8) and (4·9). Also, its third term is invariant under Eqs. (4·7) and (4·9). Therefore $\mathit{\Gamma}_k^{(0)}$ is invariant under (4·7), (4·8) and (4·9). This proves the tree level master equation (4·5).

In Appendix C, we show that the lhs of (4·6) reduces to

$$\begin{aligned}&(\mathit{\Gamma}_k^{(0)}\,, \mathit{\Gamma}_k^{(1)})_\Phi - \hbar\Delta_\Phi\mathit{\Gamma}_k^{(0)}\\ &\quad= \frac{\hbar}{2}(A^{-1})_{ab}\frac{\overrightarrow{\partial}}{\partial\phi_0^b}[(\phi_d^* P_d + S_0)\frac{\overleftarrow{\partial}}{\partial\phi_0^c}P_c]\frac{\overleftarrow{\partial}}{\partial\phi_0^a} - \hbar\frac{\overrightarrow{\partial}}{\partial\phi_a^*}S[\phi_0,\phi^*]\frac{\overleftarrow{\partial}}{\partial\phi_0^a},\end{aligned} \qquad (4\cdot10)$$

where $S[\phi_0,\phi^*]$ is the extended action (2·1) evaluated at the saddle point. The first term of Eq. (4·10) vanishes, owing to the relations

$$\frac{S_0[\phi_0]\overleftarrow{\partial}}{\partial\phi_0^a}P_a[\phi_0] = 0, \qquad (4\cdot11)$$

$$\frac{P_a[\phi_0]\overleftarrow{\partial}}{\partial\phi_0^b}P_b[\phi_0]=0. \qquad (4\cdot12)$$

These respectively come from the BRS invariance of the action S_0 and the nilpotency of the BRS transformation at the microscopic level. Similarly, it is easy to observe that the second term of Eq. (4·10) is nothing but the quantum part of the QME for $S[\phi,\phi^*]$. It vanishes, since we assumed that the measure $\mathcal{D}\phi$ is BRS invariant.

4.2. *The flow equation for the one-loop average action*

Let us now show that the one-loop average action satisfies the flow equation as well. This is a consistency check of our calculation. We have

$$\begin{aligned}&-\partial_k\Gamma_k+e^{\Gamma_k/\hbar}[X+\frac{\hbar}{2}\mathrm{Str}(R_k^{-1}\partial_kR_k)+\hbar\mathrm{Str}(\partial_k(\ln f_k))]e^{-\Gamma_k/\hbar}\\&\quad\sim-\frac{\Gamma_k^{(1)}\overleftarrow{\partial}}{\partial\phi_0}\partial_k\phi_0-\frac{\Gamma_k^{(1)}\overleftarrow{\partial}}{\partial\Phi}[(\partial_kR_k^{-1})R_k(\Phi-f_k\phi_0)-\partial_k(\ln f_k)(\Phi-2f_k\phi_0)].\end{aligned} \qquad (4\cdot13)$$

The cancellation of $O(\hbar^0)$ terms follows trivially. Thus here on the rhs we have included only $O(\hbar)$ terms. Remember that $\Gamma_k^{(1)}$ depends on Φ only through its ϕ_0 dependence. Therefore we can rewrite the Φ derivative of (4·13) as a ϕ_0 derivative. Then, using (B·1) and the relation

$$-\partial_kf_kR_k(\Phi-2f_k\phi_0)-f_k\partial_kR_k(\Phi-f_k\phi_0)+A\partial_k\phi_0=0, \qquad (4\cdot14)$$

the vanishing of the rhs of (4·13) follows. The relation (4·14) is obtained by differentiating the saddle point equation.

§5. Summary and discussion

By using the average action formalism, we have shown that the claim we made in an earlier publication [8] may be justified even for an interacting gauge theory: i.e., a gauge symmetry survives even in the presence of a cutoff, and the corresponding renormalized BRS transformation may be constructed from the QME.

The average action satisfies the QME if the original classical action is gauge invariant. At this point we have found that the antifield formalism is very convenient to describe the symmetry properties of the average action. The flow equation also follows. This also implies that once the system satisfies the WT identity with some IR cutoff, it will continue to satisfy this identity along the RG flow.

The saddle point evaluation was performed for the average action up to the one-loop order. The QME and the flow equation were confirmed explicitly. As we have seen above, there is no essential difficulty to extend our analysis to higher orders. It is worth pointing out that the construction of an action satisfying both equations had not been done until this time. However, there is a related calculation due to Ellwanger. [4] In that calculation the gauge mass term was obtained from the master and flow equations independently, and these results were found to coincide.

The quantum part of a QME had been regarded as an obstacle for the gauge symmetry. We have shown, contrastingly, that it is necessary for the symmetry, since the measure is not invariant under the renormalized BRS transformation: the jacobian under the transformation is exactly the quantum part of the QME. This argument implies also that we may read off the renormalized BRS transformation as we did earlier for free field theories. The transformation for the averaged field is particularly simple: $\delta_r \Phi = f_k \langle \delta\phi \rangle_\phi$. Similarly, the quantity Σ_k defined referring to the cutoff scale k can also be expressed as a path integral average. Let us briefly explain this in the following.

As seen below, our argument is applicable even for a microscopic action with symmetry breaking terms or anomalies. For this reason, let us consider for the moment the average action $\Gamma_k[\Phi,\ \Phi^*]$ defined with Eq. (2·5), but with an action $S[\phi,\ \phi^*]$ which is not necessarily BRS invariant. For the microscopic fields, we define the quantity Σ as

$$\Sigma[\phi,\ \phi^*] \equiv \frac{1}{2}\left(S,\ S\right)_\phi - \hbar \Delta_\phi S = \hbar^2 \exp(S/\hbar) \Delta_\phi \exp(-S/\hbar).$$

The functional average of this quantity can be rewritten as

$$\begin{aligned} \langle \Sigma[\phi,\ \phi^*] \rangle_\phi &= \hbar^2 e^{\Gamma_k/\hbar} \int \mathcal{D}\phi\ e^{(S-S_k)/\hbar} \left(\Delta_\phi e^{-S/\hbar} \right) \\ &= \hbar^2 e^{\Gamma_k/\hbar} \Delta_\Phi e^{-\Gamma_k/\hbar} \equiv \Sigma_k[\Phi, \phi^*]. \end{aligned} \qquad (5·1)$$

For $S[\phi,\ \phi^*]$, which does satisfy the (classical) master equation, Eq. (5·1) tells us that the average action satisfies the QME, $\Sigma_k[\Phi, \phi^*] = 0$. This is an important result: the QME for the average action is obtained from the master equation for the microscopic action. Note that the relation (5·1) holds even in the case that Σ does not vanish. This fact must have further implications. For example, it tells us how a symmetry-breaking term changes along the RG flow.

In our formulation, there remain a couple of questions to be clarified. Among others, the following two are particularly important: 1) whether our QME reduces to the usual Zinn-Justin equation in the limit of $k \to 0$; 2) how we prepare the UV theory. In a forthcoming paper [13] we will show that the approach presented here can be extended to most general gauge theories. [9), 10)] The relations to other approaches [14) - 17)] will be given as well. At the same time, it will be explained how the Zinn-Justin equation is realized in the limit of $k \to 0$. [18)] The second question will be discussed by introducing an UV cutoff Λ and imposing appropriate boundary conditions on the average action.

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Appendix A
—— *On Notation* ——

The left and right derivatives are written as

$$\frac{\overrightarrow{\partial} F}{\partial \phi} \equiv \frac{\partial^l F}{\partial \phi}, \quad \frac{F\overleftarrow{\partial}}{\partial \phi} \equiv \frac{\partial^r F}{\partial \phi}.$$

We find that the notation on the lhs of these expressions provide us with simpler expressions for many equations. However, for the purpose of avoiding possible confusion, we use those on the rhs also.

The sign associated with the change from a right derivative to a left derivative or vice versa is very important:

$$\frac{F\overleftarrow{\partial}}{\partial \chi} = (-1)^{\epsilon_\chi(\epsilon_F+1)}\frac{\overrightarrow{\partial} F}{\partial \chi}. \tag{A·1}$$

Here we explain our abbreviated notation for some examples. The second term of $S[\phi, \phi^*] \equiv S_0[\phi] + \phi^* \delta\phi$ is defined as

$$\phi^* \delta\phi \equiv \sum_a \int \mathrm{d}^4 p \phi_a^*(-p)\delta\phi_a(p). \tag{A·2}$$

In the multiplication on the lhs, the summation over the index a and the momentum integration are implicit. Similarly, in the block spin transformation we use the following:

$$\begin{aligned}&(\Phi - f_k\phi)R_k(\Phi - f_k\phi)\\ &\quad\equiv \int d^4 p(\Phi(-p) - f_k(-p)\phi(-p))^a [R_k(p)]_{ab}(\Phi(p) - f_k(p)\phi(p))^b.\end{aligned} \tag{A·3}$$

Appendix B
—— *A Proof of Eq. (4·9):* $\delta_r^{(0)}\phi_0 = P[\phi_0]$ ——

Here we prove the equation

$$P[\phi_0] = (\partial^r \phi_0/\partial\Phi)\delta_r^{(0)}\Phi + (\partial^r\phi_0/\partial\phi^*)\delta_r^{(0)}\phi^*.$$

By differentiating the saddle point equation (4·1) with respect to Φ and ϕ^*, we obtain the relations

$$f_k R_k = A(\partial^r\phi_0/\partial\Phi), \tag{B·1}$$

$$\frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_a^* P_a)\frac{\overleftarrow{\partial}}{\partial\phi^*}\bigg|_{\phi_0} + A(\partial^r\phi_0/\partial\phi^*) = 0, \tag{B·2}$$

where $A_{ab}[\phi_0, \phi^*]$ is defined in Eq. (4·4). In Eq. (B·2), the ϕ^* derivative in the first term is taken with ϕ_0 fixed, which is denoted by the subscript ϕ_0.

Using Eqs. (B·1) and (B·2), and the tree level renormalized BRS transformation, the equation to be proved may be rewritten as

$$P = A^{-1} f_k^2 R_k P + A^{-1} \left[\frac{\overrightarrow{\partial}}{\partial \phi_0} (\phi_a^* P_a) \frac{\overleftarrow{\partial}}{\partial \phi^*}\bigg|_{\phi_0} \right] \frac{\overrightarrow{\partial}}{\partial \phi_0} (\phi_b^* P_b + S_0).$$

We now demonstrate the vanishing of the difference of lhs and rhs multiplied by A:

$$\begin{aligned} &(A - f_k^2 R_k)_{ab} P_b - \left[\frac{\overrightarrow{\partial}}{\partial \phi_0} (\phi_c^* P_c) \frac{\overleftarrow{\partial}}{\partial \phi^*}\bigg|_{\phi_0} \right]_{ab} \frac{\overrightarrow{\partial}}{\partial \phi_0^b} (\phi_d^* P_d + S_0) \\ &= \left[\frac{\overrightarrow{\partial}}{\partial \phi_0} (\phi_c^* P_c + S_0) \frac{\overleftarrow{\partial}}{\partial \phi_0} \right]_{ab} P_b - \left[\frac{\overrightarrow{\partial}}{\partial \phi_0} (\phi_c^* P_c) \frac{\overleftarrow{\partial}}{\partial \phi^*}\bigg|_{\phi_0} \right]_{ab} \frac{\overrightarrow{\partial}}{\partial \phi_0^b} (\phi_d^* P_d + S_0). \quad \text{(B·3)} \end{aligned}$$

Here we have used Eq. (4·4) to make a substitution on the rhs. Applying the ϕ^*-differentiation, we rewrite the second term as

$$\begin{aligned} &- \left[\frac{\overrightarrow{\partial}}{\partial \phi_0} (\phi_c^* P_c) \frac{\overleftarrow{\partial}}{\partial \phi^*}\bigg|_{\phi_0} \right]_{ab} \frac{\overrightarrow{\partial}}{\partial \phi_0^b} (\phi_d^* P_d + S_0) \\ &= - \left(\frac{\overrightarrow{\partial}}{\partial \phi_0^a} P_b (-)^{\epsilon_b + 1} \right) \frac{\overrightarrow{\partial}}{\partial \phi_0^b} (\phi_d^* P_d + S_0) \\ &= \left(\frac{\overrightarrow{\partial} P_b}{\partial \phi_0^a} \right) \left[(\phi_d^* P_d + S_0) \frac{\overleftarrow{\partial}}{\partial \phi_0^b} \right] = (-)^{\epsilon_a \epsilon_b} \left[(\phi_d^* P_d + S_0) \frac{\overleftarrow{\partial}}{\partial \phi_0^b} \right] \left(\frac{\overrightarrow{\partial} P_b}{\partial \phi_0^a} \right). \end{aligned}$$

Thus the rhs of Eq. (B·3) may be rewritten as

$$\frac{\overrightarrow{\partial}}{\partial \phi_0^a} \left([\phi_c^* P_c + S_0] \frac{\overleftarrow{\partial}}{\partial \phi_0^b} P_b \right),$$

which vanishes, owing to Eqs. (4·11) and (4·12).

Appendix C
—— *A Proof of Eq. (4·10): the QME to One-Loop Order* ——

In (4·10), the first term is the variation of $\Gamma_k^{(1)}$ under the tree level BRS transformation given in (4·7) and (4·8):

$$\begin{aligned} &(\Gamma_k^{(0)}, \Gamma_k^{(1)})_\Phi - \hbar \Delta_\Phi \Gamma_k^{(0)} \\ &= \frac{\hbar}{2} \mathrm{Str} A^{-1} \left(\left(\frac{A \overleftarrow{\partial}}{\partial \phi_a^*}\bigg|_{\phi_0} \right) \delta_r^{(0)} \phi_a^* + \left(\frac{A \overleftarrow{\partial}}{\partial \phi_0^a} \right) \delta_r^{(0)} \phi_0^a \right) - \hbar \mathrm{tr}(f_k \partial^r P / \partial \Phi). \quad \text{(C·1)} \end{aligned}$$

Since the matrix A is a function of ϕ_0 and ϕ^*, the variation under the tree level BRS transformation is taken with respect to these variables. The derivatives in the first term of (C·1) should be understood accordingly. The second term is the trace (not the supertrace) of the matrix $\partial^r P_a / \partial \Phi_b$, which may be rewritten by using Eqs. (4·4) and (B·1) as

$$\begin{aligned}\varDelta_\Phi \varGamma_k^{(0)} &= \mathrm{tr}(f_k \partial^r P/\partial\phi_0 \cdot \partial^r \phi_0/\partial\Phi)) \\ &= \mathrm{tr}(\partial^r P/\partial\phi_0 A^{-1} f_k^2 R_k) \\ &= \mathrm{tr}\left(\partial^r P/\partial\phi_0 A^{-1}[A - \frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_c^* P_c + S_0)\frac{\overleftarrow{\partial}}{\partial\phi_0}]\right).\end{aligned}$$

Therefore Eq. (C·1) becomes

$$\begin{aligned}&(\varGamma_k^{(0)}, \varGamma_k^{(1)})_\Phi - \hbar\varDelta_\Phi \varGamma_k^{(0)} \\ &= -\hbar\mathrm{tr}(\partial^r P/\partial\phi_0) + \frac{\hbar}{2}\mathrm{Str}A^{-1}\left(-\left(\frac{A\overleftarrow{\partial}}{\partial\phi_a^*}\bigg|_{\phi_0}\right)\frac{\overrightarrow{\partial}}{\partial\phi_0^a}(\phi_c^* P_c + S_0) + \left(\frac{A\overleftarrow{\partial}}{\partial\phi_0^a}\right)P_a\right) \\ &\quad + \hbar\ \mathrm{tr}\left[A^{-1}\left(\frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_c^* P_c + S_0)\frac{\overleftarrow{\partial}}{\partial\phi_0}\right)\partial^r P/\partial\phi_0\right].\end{aligned}$$

We may write the rhs more explicitly. After the ϕ^*-differentiation, the second and third terms can be written as

$$\begin{aligned}&\frac{\hbar}{2}(-)^{\epsilon_a} A_{ab}^{-1}\ \left(-(-)^{(\epsilon_c+1)(\epsilon_a+1)}\left(\frac{\overrightarrow{\partial}}{\partial\phi_0^b}P_c\frac{\overleftarrow{\partial}}{\partial\phi_0^a}\right)\frac{\overrightarrow{\partial}}{\partial\phi_0^c}(\phi_d^* P_d + S_0) + \left(\frac{A_{ba}\overleftarrow{\partial}}{\partial\phi_0^c}\right)P_c\right) \\ &\qquad + \hbar A_{ab}^{-1}\left(\frac{\overrightarrow{\partial}}{\partial\phi_0^b}(\phi_d^* P_d + S_0)\frac{\overleftarrow{\partial}}{\partial\phi_0^c}\right)\left(\frac{P_c\overleftarrow{\partial}}{\partial\phi_0^a}\right).\end{aligned}$$

An easy calculation then leads us to Eq. (4·10). In this calculation, one must treat signs carefully, in particular those coming from Eq. (A·1).

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