Progress of Theoretical Physics, Vol. 106, No. 1, July 2001

BRS Symmetry, the Quantum Master Equation and the Wilsonian Renormalization Group

Yuji Igarashi, Katsumi Itoh and Hiroto So*

Faculty of Education, Niigata University, Niigata 950-2181, Japan
**Department of Physics, Niigata University, Niigata 950-2181, Japan*

(Received April 13, 2001)

Recently we made a proposal for realization of an effective BRS symmetry along the Wilsonian renormalization group flow. In this paper we show that the idea can be naturally extended to the most general gauge theories. Extensive use of the antifield formalism is made to reveal some remarkable structure of the effective BRS symmetry. The average action, defined with a continuum analog of the block spin transformation, obeys the quantum master equation (QME), provided that a UV action does so. We show that the RG flow described by the exact flow equations is generated by canonical transformations in the field-antifield space. Using the relation between the average action and the Legendre effective action, we establish the equivalence between the QME for the average action and the modified Ward-Takahashi identity for the Legendre action. The QME remains intact when the regularization is removed.

§1. Introduction

The Wilsonian renormalization group (RG) [1] is formulated in such a way that modes with frequencies higher than a reference scale k are integrated out to yield an effective action for lower momentum modes. The resulting action has been shown to obey the exact RG flow equations, [2)-5)] an invaluable tool in study of field theories. In realizing gauge symmetries, however, one needs to deal with all the momentum scales on an equal footing, and this conflicts with the introduction of such a cut-off. The reconciliation of regularizations and gauge symmetries is a long-standing problem in the RG approach.

In recent years, there has been considerable effort made in investigating this problem. [6)-17)] Becchi showed in his seminal paper [7] that symmetry breaking due to regularization can be compensated for by gauge non-invariant counter terms. The compensation called the "fine-tuning condition" has been analyzed in detail within a perturbative framework. [9] Further, Ellwanger made an important observation. [11] He showed that once the "modified Ward-Takahashi (WT) or Slavnov-Taylor (ST) identity" is satisfied at a fixed IR cutoff k, it holds everywhere along the RG flow. A perturbative formulation for solving the identities is given in Ref. 12).

The studies in the last decade suggest the possibility that there exists a cutoff-dependent effective gauge symmetry the deformation caused by regularization. This expectation has been made more plausible by a recent breakthrough for the realization of a chiral symmetry on the lattice: Lüscher constructed an exact chiral symmetry transformation for lattice fermions, [18] based on the Ginsparg-Wilson relation. [19] The transformation depends on the Dirac operator as well as the lattice spacing.

In our previous publications,[20),21)] we took a step further for the realization of effective symmetries. We showed that the WT identities for a Wilsonian effective action take the form of the quantum master equation (QME) in the Batalin-Vilkovisky (BV) antifield formalism,[22)] and thereby formulated *renormalized symmetries* realized along the RG flow. Our formalism is quite general and it applies to BRS as well as other global symmetries. We constructed our formalism based on the concepts introduced by Wetterich in a continuum analog of the block spin transformation:[5)] the average action of the macroscopic or IR fields, which are obtained through the coarse-graining of microscopic or UV fields.[*)] It should be noted in this connection that Ginsparg and Wilson also used the block spin transformation for lattice fermions in their pioneering work.[19)] Thus it is natural to ask if the QME and the renormalized transformation formulated for the chiral symmetry actually produce the continuum counterparts of the Ginsparg-Wilson relation and the Lüscher's chiral transformation. We showed in Ref. 20) that this is indeed the case. This result is certainly encouraging and considered as a nontrivial check of our formalism.

In the specific examples discussed in Ref. 20), the average actions have exact renormalized symmetries. However, this is not the case for more general interacting theories. In these, the non-vanishing variation of the average action should be canceled by some other contributions, which have been recognized as "symmetry breaking terms" in the literature. However, this does not necessarily imply the breakdown of the symmetry, because one should take account not only of the transformation of the action but also of the Jacobian factor associated with a change of the functional measure. Cancellation of these two effects makes it possible to define an exact symmetry for the quantum system under consideration. This is what the QME tells us.

Although the above summarized results may be considered as progress in the conceptual sense, there remain the following questions to be clarified. (1) How can we specify the symmetry of the UV action in the presence of a BRS non-invariant UV regularization? (2) Can the antifield formalism give new insights into the RG approach? (3) Does the QME reduce to the standard WT identity when the regularization is removed? (4) How is the QME related to other WT identities given to this time for the Wilsonian or the Legendre effective actions?

In this paper, we develop our formalism further for the purpose of answering these questions. To address question (1), we arrange a regularization in such a way that the integration of the UV fields is performed for those modes with momenta between k and a UV cutoff Λ. Our basic assumption regarding the UV action is that it is a solution of the QME. Justification of this assumption will be given in a forthcoming paper.[23)] With this assumption, we show that the resulting average action obeys the QME expressed with the IR fields. This demonstrates the presence of an exact BRS symmetry along the RG flow, which constitutes a generalization of our previous result, derived by assuming that the UV action satisfies the classical master equation.

*) A similar attempt, which introduces two kinds of the fields, is made in Ref. 10).

Concerning question (2), we will discuss the following two points in this paper. First, we treat the most general gauge theories with an open and/or reducible gauge algebra. It is straightforward to make this extension, once a local UV action for such a theory is given in the antifield formalism. Second, we show that a change of the average action along the RG flow can be described by a canonical transformation in the space of the IR fields and their antifields. It is known that a change in the Wilsonian effective action along the RG flow can be interpreted as a reparametrization of the fields. [13] The antifield formalism provides its natural extension. We note that the Jacobian factor of the canonical transformation should be added to the action. The action with this correction then satisfies the QME again.

In order to investigate questions (3) and (4), we define a subtracted average action, which is the generating functional of the connected cutoff Green functions for the UV fields. The subtracted average action is well-defined in the IR limit, while the average action has a regulator that diverges in the limit $k \to 0$. The Legendre effective action obtained from the subtracted action is also well-defined in the IR limit. There is a simple relation [24] between the average action for the IR fields and the Legendre effective action for the UV fields. Using this, we show that the QME for the average action is equivalent to the "modified WT identity" for the Legendre effective action. For the specific case of a pure Yang-Mills theory, it reduces to the "modified ST identity" given by Ellwanger. [11] The "symmetry breaking terms" in the "modified WT identity" are thus identified with those coming from the Jacobian factor in the path integral of the IR fields. [21] The boundary conditions on the cutoff functions imply the validity of the "modified WT identity" in the IR limit. As for the UV cutoff, one can take the UV limit $\Lambda \to \infty$ in renormalizable theories. We conclude, therefore, that the QME for the average action remains intact, and becomes equivalent to the Zinn-Justin equation for the Legendre effective action in the limit that the regulator is removed by taking $k \to 0,\ \Lambda \to \infty$.

This paper is organized as follows. The next section gives a brief summary of the antifield formalism and the construction of the average action. We show that the QME for the IR fields is obtained from the functional average of the QME for the UV fields. In §3, the exact RG flow equation is given for the average action. The evolution equation of the WT functional is obtained as well. We also construct the canonical transformation that generates the RG flow. Section 4 discusses the relation between the average action and the Legendre effective action. The equivalence between the QME and the "modified WT identity" is shown. The final section contains our conclusions and a short discussion of the outlook. Our notation and some notes concerning our computations are given in the appendices.

§2. The average action in the antifield formalism

2.1. *The antifield formalism*

To make this paper self-contained, we first briefly summarize the Batalin-Vilkovisky (BV) antifield formalism,[*] and then use it to construct the average

[*] See Ref. 25) for reviews of this subject.

action. Our formulation applies to the most general gauge theories. Their gauge algebra can be open and/or reducible.

Let us consider a gauge theory in d-dimensional Euclidean space. It consists of gauge and matter fields denoted collectively by ϕ_0^i, as well as ghosts, antighosts and B-fields. If the gauge algebra is reducible, we should further add ghosts for ghosts, their antighosts and B-fields. Let $\phi^A = \{\phi_0^i, \cdots\}$ be all the fields introduced and let ϕ^*_A be their antifields. The index A represents the Lorentz indices of tensor fields, the spinor indices of the fermions, and/or indices distinguishing different types of generic fields. The Grassmann parities for ϕ^A and ϕ^*_A are expressed as $\epsilon(\phi^A) = \epsilon_A$ and $\epsilon(\phi^*_A) = \epsilon_A + 1$. The antibracket in the space of $\{\phi,\ \phi^*\}$ is then defined as

$$\begin{aligned}(F,\ G)_\phi &\equiv \frac{\partial^r F}{\partial \phi^A}\frac{\partial^l G}{\partial \phi^*_A} - \frac{\partial^r F}{\partial \phi^*_A}\frac{\partial^l G}{\partial \phi^A} \\ &= \int \frac{d^d p}{(2\pi)^d}\left[\frac{\partial^r F}{\partial \phi^A(-p)}\frac{\partial^l G}{\partial \phi^*_A(p)} - \frac{\partial^r F}{\partial \phi^*_A(-p)}\frac{\partial^l G}{\partial \phi^A(p)}\right]. \end{aligned} \tag{2·1}$$

To make our equations simple, we use matrix notation in the momentum space, as given in Appendix A.

We begin with a gauge invariant action $S_0[\phi_0]$. The first step in the antifield formalism is to construct a classical extended action $\tilde{S}_{\rm cl}[\phi,\ \phi^*]$ as a power series expansion of the antifields:

$$\tilde{S}_{\rm cl}[\phi,\ \phi^*] = S_0[\phi_0] + \phi^*_A P^A[\phi] + \phi^*_A \phi^*_B Q^{AB}[\phi] + \cdots. \tag{2·2}$$

The coefficient functions, such as P^A and Q^{AB}, should be fixed by the requirement that the $\tilde{S}_{\rm cl}$ satisfy the (classical) master equation, [22]

$$\left(\tilde{S}_{\rm cl},\ \tilde{S}_{\rm cl}\right)_\phi = 0. \tag{2·3}$$

This master equation incorporates all the information of the underlying gauge algebra.

The next step is the gauge fixing. To accomplish this, one introduces the gauge fixing fermion $\Psi(\phi^A)$, a function which does not depend on the antifields. A possible way of gauge fixing is to eliminate the antifields by imposing the conditions $\phi^*_A = \partial\Psi/\partial\phi^A$. Alternatively, one may perform the canonical transformation in the space of fields and antifields, as $\phi^A \to \phi^A$, $\phi^*_A \to \phi^*_A + \partial\Psi/\partial\phi^A$, where Ψ acts as the generator of the canonical transformation. This choice of coordinates, called the gauge-fixed basis, allows us to retain antifields until the end of the calculations. In the following, we employ this basis and use the notation $S_{\rm cl}[\phi,\ \phi^*] \equiv \tilde{S}_{\rm cl}[\phi,\ \phi^* + \partial\Psi/\partial\phi]$. In the new basis, the classical master equation still holds, because an antibracket is invariant under a canonical transformation.

In the BV quantization, the classical action $S_{\rm cl}[\phi,\ \phi^*]$ should be replaced by the quantum action $S[\phi,\ \phi^*]$, on which one imposes the quantum master equation (QME) [22] in place of (2·3):

$$\Sigma[\phi,\ \phi^*] \equiv \hbar^2 \exp(S/\hbar)\Delta_\phi \exp(-S/\hbar) = \frac{1}{2}(S,\ S)_\phi - \hbar\Delta_\phi S = 0. \tag{2·4}$$

The functional Σ is called the WT functional in this paper. The Δ_ϕ derivative is defined as

$$\Delta_\phi \equiv (-)^{\epsilon_A+1}\frac{\partial^r}{\partial\phi^A}\frac{\partial^r}{\partial\phi^*_A} = (-)^{\epsilon_A+1}\int\frac{d^dp}{(2\pi)^d}\frac{\partial^r}{\partial\phi^A(-p)}\frac{\partial^r}{\partial\phi^*_A(p)}. \qquad (2\cdot5)$$

It is a nilpotent operator:

$$(\Delta_\phi)^2 = 0. \qquad (2\cdot6)$$

The QME ensures the BRS invariance of the quantum system.

2.2. *The IR fields and the average action*

In this subsection, we construct a Wilsonian effective action, the average action. Let us begin with the path integral representation of the generating functional for a local quantum action S in the presence of sources J_A:

$$Z[J] = \int\mathcal{D}\phi^*\prod_A\delta(\phi^*_A)Z[J,\ \phi^*],$$
$$Z[J,\ \phi^*] = \int\mathcal{D}\phi\exp\frac{1}{\hbar}\left(-S[\phi,\ \phi^*] + J_A\phi^A\right). \qquad (2\cdot7)$$

In this path integral, the antifields ϕ^*_A are integrated out for the gauge fixing. As seen below, this is important also for the study of the canonical structure in the space of fields and antifields. For the fields ϕ^A, the quantum modes with arbitrary momenta are to be integrated simultaneously. The main idea of the Wilsonian RG is to perform the integration successively: one integrates the high frequency modes of the fields ϕ^A to obtain an effective theory for the low frequency modes. For the division of momenta, one introduces an IR cutoff k. Furthermore, in order for the integration of the higher frequencies to be well-defined, the presence of a UV regulator is assumed. We consider here a regularization in which a UV cutoff Λ is introduced together with the IR cutoff k in a same regulator, regarding the frequencies between k and Λ as generating the "block spin action" for the frequencies lower than k. We construct this effective action, called the average action, slightly modifying Wetterich's method.[5] This formalism uses two kinds of fields, the microscopic or UV fields ϕ^A in (2·7), and the macroscopic or IR fields Φ^A identified roughly with the average fields obtained through the coarse-graining of the UV fields. In order to realize this formalism, we take the following steps. Consider a Gaussian integral

$$1 = N_{k\Lambda}\int\mathcal{D}\Phi\mathcal{D}\Phi^*\prod_A\delta\left(\Phi^*_A - f^{-1}_{k\Lambda}\phi^*_A\right)\ \exp -\frac{1}{\hbar}\left[\frac{1}{2}\left(\Phi^A - f_{k\Lambda}\phi^A - f^{-1}_{k\Lambda}J_C(R^{-1}_{k\Lambda})^{CA}\right)\right.$$
$$\left.\times R^{k\Lambda}_{AB}\left(\Phi^B - f_{k\Lambda}\phi^B - (R^{-1}_{k\Lambda})^{BD}f^{-1}_{k\Lambda}J_D\right)\right], \qquad (2\cdot8)$$

where we have used matrix notation and $N_{k\Lambda}$ is the normalization function. Shortly we describe the properties of the invertible matrix $(R^{k\Lambda})_{AB}$ and the function $f_{k\Lambda}$.

Let us insert (2·8) into (2·7) and rewrite it as

$$Z[J] = Z_{k\Lambda}[J]\ \exp -\frac{1}{\hbar}\left(\frac{1}{2} J_A f_{k\Lambda}^{-2} (R_{k\Lambda}^{-1})^{AB} J_B\right). \tag{2·9}$$

Here the cutoff-dependent partition function is given by a functional integral of the IR fields Φ^A:

$$\begin{aligned} Z_{k\Lambda}[J] &= \int \mathcal{D}\Phi^* \prod_A \delta\left(\Phi^*_A\right) Z_{k\Lambda}[J,\ \Phi^*], \\ Z_{k\Lambda}[J,\ \Phi^*] &= \int \mathcal{D}\Phi \exp \frac{1}{\hbar}\left(-W_{k\Lambda}[\Phi,\ \Phi^*] + J_A f_{k\Lambda}^{-1} \Phi^A\right). \end{aligned} \tag{2·10}$$

The Wilsonian effective action $W_{k\Lambda}$ has the path integral representation*)

$$\begin{aligned} \exp\left(-W_{k\Lambda}[\Phi,\ \Phi^*]/\hbar\right) = N_{k\Lambda} \int \mathcal{D}\phi \mathcal{D}\phi^* \prod_A \delta\left(f_{k\Lambda}\Phi^*_A - \phi^*_A\right) \\ \times \exp -S_{k\Lambda}[\phi, \Phi, \phi^*]/\hbar, \end{aligned} \tag{2·11}$$

where

$$S_{k\Lambda}[\phi, \Phi, \phi^*] = S[\phi, \phi^*] + \frac{1}{2}(\Phi - f_{k\Lambda}\phi)^A (R^{k\Lambda})_{AB} (\Phi - f_{k\Lambda}\phi)^B. \tag{2·12}$$

The action given in (2·11) is the average action, which was introduced by Wetterich [5]) to realize a continuum analog of the block spin transformation. The average action describes the dynamics below the IR cutoff. Obviously, the path integral (2·10) over the IR fields must be the same as the original partition function (2·7). The relation is given in (2·9): there is a factor depending on the source J, which produces a trivial IR cutoff dependence for $Z_{k\Lambda}$.

Let us discuss some properties of the functions appearing in the definition of the average action in (2·11) and (2·12). The function $f_{k\Lambda}(p^2)$ is for the coarse-graining of the UV fields. For the analysis given in this paper, we do not need its concrete form, but require it to behave as $f_{k\Lambda}(p^2) \approx 0$ for $k^2 < p^2 < \Lambda^2$ and $f_{k\Lambda}(p^2) \approx 1$ outside of this interval. The cutoff functions $(R^{k\Lambda})_{AB}$ are introduced to relate the IR fields with the UV fields. The IR fields are roughly equated with the average fields, $\Phi^A(p) \approx f_{k\Lambda}(p^2)\phi^A(p)$. Because of this relation for the fields, we impose the constraints $\Phi^*_A = f_{k\Lambda}^{-1}\phi^*_A$ for the antifields in (2·8) and (2·11) to keep the canonical structure in the space of fields and antifields. We may choose the cutoff functions as

$$\begin{aligned} (R^{k\Lambda})_{AB}(p, -q) &= (\mathcal{R}^{k\Lambda})_{AB}(p)(2\pi)^d \delta(p - q), \\ (\mathcal{R}^{k\Lambda})_{AB}(p) &= \frac{\bar{\mathcal{R}}_{AB}(p)}{f_{k\Lambda}(1 - f_{k\Lambda})}. \end{aligned} \tag{2·13}$$

The invertible matrix $(R^{k\Lambda})_{AB}$ has the signature $\epsilon((R^{k\Lambda})_{AB}) = \epsilon_A + \epsilon_B$. This matrix and its inverse satisfy

*) In our previous papers, Γ_k was used to represent the average action, but it is reserved here to denote the Legendre effective action.

$$(R^{k\Lambda})_{BA} = (-)^{\epsilon_A+\epsilon_B+\epsilon_A\epsilon_B}(R^{k\Lambda})_{AB},$$
$$\left(R^{-1}_{k\Lambda}\right)^{BA} = (-)^{\epsilon_A\epsilon_B}\left(R^{-1}_{k\Lambda}\right)^{AB}. \tag{2·14}$$

All non-vanishing components of $\bar{\mathcal{R}}_{AB}(p)$ are assumed to be polynomials in p. As a possible choice, it may be identified with D^{-1}_{AB}, the inverse (free) propagator for the fields ϕ^A and ϕ^B.

In (2·12), we find that the terms $\phi^A f^2_{k\Lambda}(R^{k\Lambda})_{AB}\phi^B$ can be regarded as a regulator and that the integration of the UV fields is performed for those modes with momenta between k and Λ. The terms $f_{k\Lambda}\Phi^B(R^{k\Lambda})_{BA}$ act as sources for the UV fields ϕ^A in place of J_A in (2·7). In order for this replacement to be realized, we included J dependent contributions in the Gaussian integral (2·8). The remaining terms $\Phi^A(R^{k\Lambda})_{AB}\Phi^B$ in the exponential do not affect the path integral of the UV fields. We may therefore define a subtracted average action by removing these terms from $W_{k\Lambda}$:

$$\hat{W}_{k\Lambda}[\Phi,\ \Phi^*] = W_{k\Lambda}[\Phi,\ \Phi^*] - \frac{1}{2}\Phi^A\left(R^{k\Lambda}\right)_{AB}\Phi^B. \tag{2·15}$$

It should be noted that $\hat{W}_{k\Lambda}$ is the generating functional of the connected cutoff Green functions of the UV fields.

We now discuss the behavior of the average action when the cutoff k reaches the IR and UV boundary values, 0 and Λ. At the UV scale, $k \to \Lambda$, we have

$$\lim_{k\to\Lambda} f_{k\Lambda}(p^2) = 1,$$
$$\lim_{k\to\Lambda}(\mathcal{R}^{k\Lambda})_{AB}(p) = \infty. \tag{2·16}$$

Then,

$$\lim_{k\to\Lambda} W_{k\Lambda}[\Phi,\ \Phi^*] = S[\phi,\ \phi^*]. \qquad (\Phi^A \to \phi^A) \tag{2·17}$$

This formally implies that the UV action $S[\phi,\ \phi^*]$ is defined at the UV scale Λ. For the IR limit $k \to 0$, we have

$$\lim_{k\to 0} f_{k\Lambda}(p^2) = 0,$$
$$\lim_{k\to 0}(\mathcal{R}^{k\Lambda})_{AB}(p) = \infty. \tag{2·18}$$

In this limit, we find in (2·12) that the sources $f_{k\Lambda}\Phi^B(\mathcal{R}^{k\Lambda})_{BA}$ become finite as $\lim_{k\to 0} f_{k\Lambda}(\mathcal{R}^{k\Lambda})_{AB} = \bar{\mathcal{R}}_{AB}$, and the regulator contributions $\phi^A f^2_{k\Lambda}(R^{k\Lambda})_{AB}\phi^B$ vanish. However, the remaining terms $\Phi^A(\mathcal{R}^{k\Lambda})_{AB}\Phi^B$ become divergent, and act as infinite "mass terms" for the average action $W_{k\Lambda}$. The subtracted average action given in (2·15) thus suits the study of the IR limit.

2.3. *The quantum master equation for the UV and IR fields*

Now we show that an exact gauge (BRS) symmetry is realized along the RG flow, though it is deformed due to the regularization. This symmetry is inherited from the original symmetry of the UV action, $S[\phi,\ \phi^*]$. Our basic assumption to specify the symmetry is that the UV action is a solution of the QME, $\Sigma[\phi,\ \phi^*]=0$. For a given classical action, such a solution is known to exist at least in perturbation theory,[25] when gauge anomaly is absent. It is a nontrivial problem to fix BRS non-invariant counter terms for a given regularization scheme. We will discuss this issue in a forthcoming paper[23] and assume here that the UV action solves the QME.

Let us consider the WT functional Σ for the IR fields,

$$\begin{aligned}\Sigma_{k\Lambda}[\Phi,\ \Phi^*] \equiv&\ \hbar^2 \exp(W_{k\Lambda}/\hbar)\Delta_\Phi \exp(-W_{k\Lambda}/\hbar)\\ =&\ \frac{1}{2}\left(W_{k\Lambda},\ W_{k\Lambda}\right)_\Phi - \hbar\Delta_\Phi W_{k\Lambda},\end{aligned} \tag{2·19}$$

where $(\ ,\ )_\Phi$ and Δ_Φ are the antibracket and Δ derivative for the IR fields. In order to relate the WT operator for the UV action (2·4) to that for the average action, we take the functional average of the former with respect to the regularized UV action. In the formulation given in this paper, the path integral of the UV fields includes integration over the antifields. This requires a slight modification of our previous treatment, given in Ref. 21). After some calculation, we find that

$$\begin{aligned}\langle\Sigma[\phi,\ \phi^*]\rangle_{\phi:f\Phi R} =&\ \hbar^2\exp(W_{k\Lambda}/\hbar)N_{k\Lambda}\int\mathcal{D}\phi\mathcal{D}\phi^*\prod_A\delta\left(f_{k\Lambda}\Phi^*_A-\phi^*_A\right)\\ &\times\exp\left[(S-S_{k\Lambda})/\hbar\right]\left[\Delta_\phi\exp(-S/\hbar)\right]\\ =&\ \hbar^2\exp(W_{k\Lambda}/\hbar)\Delta_\Phi\exp(-W_{k\Lambda}/\hbar)=\Sigma_{k\Lambda}[\Phi,\ \Phi^*].\end{aligned} \tag{2·20}$$

Here $\langle F\rangle_{\phi:J}$ denotes the functional average of F with respect to the fields ϕ^A in the presence of sources J_A.

Because $\Sigma[\phi,\ \phi^*]=0$, the average action automatically obeys the QME, $\Sigma_{k\Lambda}[\Phi,\ \Phi^*]=0$, for any k. This clearly demonstrates the presence of an exact BRS symmetry along the RG flow. We call it the renormalized BRS (rBRS) symmetry. The result given here generalizes our previous results,[20),21)] where we assumed that the UV action is linear in the antifields and satisfies the classical master equation.

The QME is understood as follows. Let us consider a set of rBRS transformations:

$$\Phi^A \to \Phi^A + \delta_r\Phi^A\lambda, \qquad \delta_r\Phi^A = \left(\Phi^A,\ W_{k\Lambda}\right)_\Phi. \tag{2·21}$$

Here λ is an anti-commuting constant. In general, the average action cannot remain invariant under (2·21). It transforms as

$$W_{k\Lambda} \to W_{k\Lambda} + \frac{1}{2}\left(W_{k\Lambda},\ W_{k\Lambda}\right)_\Phi\lambda. \tag{2·22}$$

At the same time, the functional measure*) transforms as

$$\mathcal{D}\Phi \to \mathcal{D}\Phi\left(1+\Delta_\Phi W_{k\Lambda}\lambda\right). \tag{2·23}$$

*) Note that the functional measure for the IR fields is flat in (2·8).

If the QME is satisfied, these two contributions cancel, leaving the functional integral $\mathcal{D}\Phi \exp(-W_{k\Lambda}/\hbar)$ invariant. The importance of the contribution from the Jacobian factor in the QME was noted already in Refs. 26) and 27).

Because of the presence of the $\Delta_\Phi W_{k\Lambda}$ term, one may introduce another effective transformation, called the quantum BRS transformation [26] (see also Ref. 25)). For any operator $F[\Phi,\ \Phi^*]$, it is defined by

$$\delta_Q F \equiv (F,\ W_{k\Lambda})_\Phi - \hbar \Delta_\Phi F. \tag{2·24}$$

The quantum BRS transformation δ_Q is nilpotent,

$$(\delta_Q)^2 F = (F,\ \Sigma_{k\Lambda})_\Phi = 0, \tag{2·25}$$

if the QME is satisfied. It is, however, no longer a graded derivation:

$$\delta_Q(FH) = F(\delta_Q H) + (-)^{\epsilon_H}(\delta_Q F)H - \hbar(-)^{\epsilon_H}(F,\ H)_\Phi. \tag{2·26}$$

One may define the cohomology using the quantum BRS transformation δ_Q. Observables can be specified as elements of the cohomology. A violation of the QME may induce a violation of the nilpotency condition for δ_Q, and it corresponds to a gauge anomaly.

§3. The exact RG flow equations and canonical transformation

3.1. *The exact RG flow equations*

The change in the average action resulting from lowering k is described by the exact RG flow equations. [2)-5)] It is obtained by differentiating (2·12) with respect to k. We find

$$\begin{aligned}\partial_k \exp\left(-W_{k\Lambda}[\Phi,\Phi^*]/\hbar\right) = \int \mathcal{D}\phi\mathcal{D}\phi^*\partial_k \Bigg[& N_{k\Lambda}\prod_A \delta\Big(f_{k\Lambda}\Phi^*_A - \phi^*_A\Big) \\ & \times \exp - \Big(S_{k\Lambda}[\phi,\Phi,\phi^*]/\hbar\Big)\Bigg],\end{aligned} \tag{3·1}$$

where the normalization function is given by $N_{k\Lambda} = \exp[\mathrm{Str}(\ln R^{k\Lambda})/2]$. This yields

$$\partial_k \exp\left(-W_{k\Lambda}[\Phi,\Phi^*]/\hbar\right) = -\left(X + \mathrm{Str}\left[\partial_k(\ln f_{k\Lambda})\right]\right)\exp\left(-W_{k\Lambda}[\Phi,\Phi^*]/\hbar\right), \tag{3·2}$$

where

$$\begin{aligned} X &= (-)^{\epsilon_A+\epsilon_B+1}\,\frac{\hbar}{2}\frac{\partial^r}{\partial\Phi^A}\mathcal{M}^{AB}\frac{\partial^r}{\partial\Phi^B} + \partial_k(\ln f_k)\left[\Phi^A\frac{\partial^l}{\partial\Phi^A} - \Phi^*_A\frac{\partial^l}{\partial\Phi^*_A}\right], \\ \mathcal{M}^{AB} &\equiv f^2_{k\Lambda}\partial_k\left[f^{-2}_{k\Lambda}(R^{-1}_{k\Lambda})^{AB}\right]. \end{aligned} \tag{3·3}$$

The operator X is the fundamental operator that characterizes the RG flow. The first term in X originates from the k dependence of $f^2_{k\Lambda}R^{k\Lambda}_{AB}$. The second corresponds to the effects of a scale transformation on Φ^A and Φ^*_A. The scale transformation on the antifields appears in (3·3) because of the constraints $\Phi^*_A = f^{-1}_{k\Lambda}\phi^*_A$.

We showed in the previous section that the QME, $\Sigma_{k\Lambda}[\Phi,\ \Phi^*] = 0$, at an arbitrary scale k $(< \Lambda)$ results from the QME $\Sigma[\phi,\ \phi^*] = 0$. The same conclusion may be obtained from the flow equation for $\Sigma_{k\Lambda}$.[11)] The exact RG flow equation (3·2) and the functional $\Sigma_{k\Lambda}$ in (2·19) are characterized by differential operators, X and Δ_Φ, respectively. Since X contains a term related to the scale transformation on Φ^*,[*)] it commutes with Δ_Φ:

$$[\Delta_\Phi,\ X] = 0. \tag{3·4}$$

It follows from (3·2), (2·19) and (3·4) that

$$\partial_k \Sigma_{k\Lambda} = \exp(W_{k\Lambda}/\hbar)\left[X \exp(-W_{k\Lambda}/\hbar)\right] \Sigma_{k\Lambda} - \exp(W_{k\Lambda}/\hbar) X \left[\exp(-W_{k\Lambda}/\hbar)\Sigma_{k\Lambda}\right]. \tag{3·5}$$

The rhs of this equation consists of terms proportional to functional derivatives of $\Sigma_{k\Lambda}$. Suppose that the QME, $\Sigma_{k\Lambda}[\Phi,\ \Phi^*] = 0$, holds for some k. This is an identity for *any* Φ and Φ^*, so that all functional derivatives of $\Sigma_{k\Lambda}$ should also vanish. Therefore, if $\Sigma_{k\Lambda} = 0$ at some k, then $\Sigma_{(k+dk)\Lambda} = 0$ $(dk < 0)$. Thus, the flow equation for the WT functional Σ ensures that the rBRS invariance of the quantum system persists along the RG flow.

3.2. *The canonical transformation generating the RG flow*

Let us discuss the BRS invariance realized along the RG flow from a new perspective. In the antifield formalism, we may consider the canonical transformations, which leave the classical master equation invariant, since it is written in the form of the antibracket. However, the transformations do change the QME or the operator Δ, since a canonical transformation in the field-antifield space induces a nontrivial Jacobian factor (see Appendix B). In order for a canonical transformation to make the QME invariant, the associated Jacobian factor must be taken into account: the action should be transformed suitably to cancel the Jacobian factor as shown in Appendix B.

Now, take two actions on a RG flow, $W_{k\Lambda}[\Phi,\ \Phi^*]$ and $W_{(k+dk)\Lambda}[\Phi,\ \Phi^*]$ $(dk < 0)$, which satisfy the QME. An interesting question is whether there is a canonical transformation which relates these actions. Below we show that such a transformation does exist.

As shown in Appendix B, under the canonical transformation with the generator G,

$$\begin{aligned}\Phi^A \to \bar{\Phi}^A &= \Phi^A + (\Phi^A,\ G)_\Phi,\\ \Phi^*_A \to \bar{\Phi}^*_A &= \Phi^*_A + (\Phi^*_A,\ G)_\Phi,\end{aligned} \tag{3·6}$$

the action changes by an amount $-\delta_Q G$. For the generator

$$G[\Phi,\ \Phi^*] = (-)^{\epsilon_B+1}\,\frac{1}{2}\Phi^*_A \mathcal{M}^{AB}\frac{\partial^r W_{k\Lambda}}{\partial \Phi^B}dk - \Phi^*_A \partial_k\left(\ln f_k\right)\Phi^A dk, \tag{3·7}$$

[*)] In Ref. 21), such a term was not included in X, so that there, $[\Delta_\Phi,\ X] \neq 0$.

we obtain, up to $O((dk)^2)$,

$$-\delta_Q G = \partial_k W_{k\Lambda}[\Phi,\ \Phi^*]dk + \frac{1}{2}\Phi^*_B \mathcal{M}^{BC}\frac{\partial^r}{\partial \Phi^C}\Sigma_{k\Lambda}[\Phi, \Phi^*]dk. \tag{3·8}$$

The second term on the rhs of (3·8) vanishes for an average action $W_{k\Lambda}$ satisfying the QME. However, Eq. (3·8) itself holds for *any* average action defined as (2·11), which does not necessarily satisfy the QME. A few more comments are in order. 1) The second term on the rhs of (3·8) is proportional to the antifields other than the WT operator Σ. This term is zero for the BRS invariant system due to the gauge fixing condition, $\Phi^*_A = 0$. 2) Note that the canonical transformation (3·6) with (3·7) does not affect the gauge fixing conditions $\Phi^*_A = 0$ in (2·10), as the new antifields $\bar{\Phi}^*_A$ are proportional to Φ^*_A.

Equation (3·8) may be rewritten as

$$W_{(k+dk)\Lambda}[\Phi,\ \Phi^*] = W_{k\Lambda}[\Phi,\ \Phi^*] - \delta_Q G. \tag{3·9}$$

As discussed in Appendix B, this is exactly the change of the action which makes the QME invariant.

Thus there exists a canonical transformation that generates the infinitesimal change of the action along the RG flow, keeping the QME intact. The entire RG flow is generated by a successive series of canonical transformations in the space of fields and antifields. Reaching the physical limit $k \to 0$ is equivalent to finding the corresponding finite canonical transformation.

It has been pointed out that the RG flow for the effective action can be regarded as reparametrizations of the fields.[13)] The antifield formalism provides us with its extension in the form of canonical transformations. It is certainly intriguing to realize this new perspective of the RG flow for gauge theories.

§4. The average action and the Legendre effective action

We have given in previous sections a general formulation of the renormalized symmetry realized on the RG flow. The concept of the average action is of crucial importance to reveal the structure of the symmetry. In the literature, however, the RG flow has been discussed often by using the Legendre effective action rather than the average action. These two kinds of effective actions may play complementary roles. Construction of the Legendre effective action for the classical UV fields is the subject of this section. It allows us to make clear the relation between our approach and others.

We begin with the effective action

$$\exp -\hat{W}_{k\Lambda}[\Phi,\ \Phi^*]/\hbar = N_{k\Lambda}\int \mathcal{D}\phi\mathcal{D}\phi^* \prod_A \delta\left(f_{k\Lambda}\Phi^*_A - \phi^*_A\right)$$
$$\times \exp -\frac{1}{\hbar}\left(S[\phi,\ \phi^*] + \frac{1}{2}\phi^A f^2_{k\Lambda} R^{k\Lambda}_{AB}\phi^B - \Phi^A f_{k\Lambda} R^{k\Lambda}_{AB}\phi^B\right). \tag{4·1}$$

This action is the generating functional of the connected cutoff Green functions of the UV fields, and it is related to the average action by (2·15). In (4·1), the background

IR fields act as the sources for the UV fields in the combinations

$$j_A = \Phi^B f_{k\Lambda} R^{k\Lambda}_{BA}. \tag{4·2}$$

We may perform the Legendre transformation

$$\hat{\Gamma}_{k\Lambda}[\varphi,\ \varphi^*] = \hat{W}_{k\Lambda}[\Phi,\ \Phi^*] + j_A\varphi^A, \tag{4·3}$$

where the classical UV fields φ^A are defined as the expectation values of the UV fields ϕ^A in the presence of the sources j_A:

$$\varphi^A = -\frac{\partial^l \hat{W}_{k\Lambda}}{\partial j_A},$$
$$j_A = \frac{\partial^r \hat{\Gamma}_{k\Lambda}}{\partial \varphi^A}. \tag{4·4}$$

The antifields are related as

$$\varphi^*_A = \phi^*_A = f_{k\Lambda}\Phi^*_A. \tag{4·5}$$

The above equations (4·2), (4·4) and (4·5) give the functional relations $\varphi^A = \varphi^A[\Phi, \Phi^*]$. Using

$$\frac{\partial^r \hat{W}_{k\Lambda}}{\partial \Phi^A} = -\varphi^B f_{k\Lambda} R^{k\Lambda}_{BA}, \tag{4·6}$$

Eqs. (2·15), (4·2) and (4·4) we further obtain

$$\frac{\partial^r W_{k\Lambda}}{\partial \Phi^A} = f^{-1}_{k\Lambda}\frac{\partial^r \Gamma_{k\Lambda}}{\partial \varphi^A}, \tag{4·7}$$

where the Legendre effective action is defined by

$$\begin{aligned}\Gamma_{k\Lambda}[\varphi,\ \varphi^*] &\equiv \hat{\Gamma}_{k\Lambda}[\varphi,\ \varphi^*] - \frac{1}{2}\varphi^A f^2_{k\Lambda} R^{k\Lambda}_{AB}\varphi^B \\ &= W_{k\Lambda}[\Phi,\ \Phi^*] - \frac{1}{2}\left(\Phi^A - f_{k\Lambda}\varphi^A\right) R^{k\Lambda}_{AB}\left(\Phi^B - f_{k\Lambda}\varphi^B\right).\end{aligned} \tag{4·8}$$

From (4·8), we find the relations

$$\left.\frac{\partial^l \hat{W}_{k\Lambda}}{\partial \Phi^*_A}\right|_{\Phi\ \text{fixed}} = \left.\frac{\partial^l W_{k\Lambda}}{\partial \Phi^*_A}\right|_{\Phi\ \text{fixed}} = f_{k\Lambda}\left.\frac{\partial^l \hat{\Gamma}_{k\Lambda}}{\partial \varphi^*_A}\right|_{\varphi\ \text{fixed}} = f_{k\Lambda}\left.\frac{\partial^l \Gamma_{k\Lambda}}{\partial \varphi^*_A}\right|_{\varphi\ \text{fixed}}. \tag{4·9}$$

The identity $\partial\varphi^A/\partial\varphi^B = \delta^A_B$ leads to

$$\left(\frac{\partial^l\partial^r \hat{\Gamma}_{k\Lambda}}{\partial\varphi\partial\varphi}\right)^{-1}_{AC} \equiv (-)^{\epsilon_C+1}\left(\frac{\partial^l\partial^r \hat{W}_{k\Lambda}}{\partial j_A\partial j_C}\right). \tag{4·10}$$

Using (4·1), (4·3) and (4·10), we derive the flow equation for the Legendre effective action,

$$\partial_k\Gamma_{k\Lambda} = \frac{\hbar}{2}(-)^{\epsilon_A}\left[\partial_k\left(f^2_{k\Lambda}R^{k\Lambda}_{AB}\right)\right]\left(\frac{\partial^l\partial^r \hat{\Gamma}_{k\Lambda}}{\partial\varphi\partial\varphi}\right)^{-1}_{BA} - \frac{\hbar}{2}\ \partial_k \mathrm{Str}(\ln R^{k\Lambda}). \tag{4·11}$$

The WT identity for the Legendre effective action can be obtained from the QME for the average action. We find from (4·7) and (4·9) that

$$\frac{1}{2}\left(W_{k\Lambda},\ W_{k\Lambda}\right)_{\Phi} = \frac{\partial^r W_{k\Lambda}}{\partial \Phi^A}\ \frac{\partial^l W_{k\Lambda}}{\partial \Phi^*_A} = \frac{\partial^r \Gamma_{k\Lambda}}{\partial \varphi^A}\ \frac{\partial^l \Gamma_{k\Lambda}}{\partial \varphi^*_A} = \frac{1}{2}\left(\Gamma_{k\Lambda},\ \Gamma_{k\Lambda}\right)_{\varphi}, \qquad (4\cdot12)$$

and from (4·4) and (4·10) that

$$\begin{aligned}\Delta_{\Phi} W_{k\Lambda} &= \frac{\partial^r \partial^l W_{k\Lambda}}{\partial \Phi^A \partial \Phi^*_A} = \left(f_{k\Lambda}\frac{\partial^r \partial^l \Gamma_{k\Lambda}}{\partial \varphi^B \partial \varphi^*_A}\right)\frac{\partial^r \varphi^B}{\partial \Phi^A} \\ &= (-)^{1+\epsilon_A(1+\epsilon_C)}\left(\frac{\partial^l \partial^r \Gamma_{k\Lambda}}{\partial \varphi^*_A \partial \varphi^B}\right)\left(\frac{\partial^l \partial^r \hat{W}_{k\Lambda}}{\partial j_B \partial j_C}\right) f^2_{k\Lambda} R^{k\Lambda}_{AC} \\ &= \left(\frac{\partial^l \partial^r \Gamma_{k\Lambda}}{\partial \varphi^*_A \partial \varphi^B}\right)\left(\frac{\partial^l \partial^r \hat{\Gamma}_{k\Lambda}}{\partial \varphi \partial \varphi}\right)^{-1}_{BC} f^2_{k\Lambda} R^{k\Lambda}_{CA}. \qquad (4\cdot13)\end{aligned}$$

Therefore, the QME takes the form

$$\Sigma_{k\Lambda}[\Phi,\ \Phi^*] = \frac{\partial^r \Gamma_{k\Lambda}}{\partial \varphi^A}\ \frac{\partial^l \Gamma_{k\Lambda}}{\partial \varphi^*_A} - \hbar\left(\frac{\partial^l \partial^r \Gamma_{k\Lambda}}{\partial \varphi^*_A \partial \varphi^B}\right)\left(\frac{\partial^l \partial^r \hat{\Gamma}_{k\Lambda}}{\partial \varphi \partial \varphi}\right)^{-1}_{BC} f^2_{k\Lambda} R^{k\Lambda}_{CA} = 0. \qquad (4\cdot14)$$

When applied to the pure Yang-Mills theory, (4·14) reduces to the "modified ST identity" obtained by Ellwanger.[11] This result can be understood as follows. The first term of (4·14) is equal to the antibrackets $(\Gamma_{k\Lambda},\ \Gamma_{k\Lambda})_{\varphi}/2 = (W_{k\Lambda},\ W_{k\Lambda})_{\Phi}/2$, which cannot vanish, because the symmetry is violated by the regularization. It should be compensated for by the remaining "symmetry breaking terms." This is the reason that (4·14) has been called as the broken or modified WT identity. From our point of view, the origin of this symmetry breaking terms becomes more transparent. They are nothing but the $\Delta_{\Phi} W_{k\Lambda}$ term arising from the nontrivial Jacobian factor associated with the non-invariance of the functional measure for the IR fields Φ^A under the rBRS transformation. It is *necessary*, therefore, for the quantum system under consideration to be BRS invariant. Note that this interpretation becomes possible only when we consider the average action $W_{k\Lambda}[\Phi, \Phi^*]$. On the other hand, the RG flow equations are shown to take a simpler form when expressed in terms of the $\Gamma_{k\Lambda}[\varphi, \varphi^*]$. This is because the Legendre effective action consists only of the one-particle irreducible (1PI) cutoff vertex functions. In this sense, the average action and the Legendre effective action play complementary roles.

We close this section with the following remarks.

(1) Let us consider the IR limit, $k \to 0$. In the path integral (4·1), the regulator terms proportional to $f^2_{k\Lambda}\mathcal{R}^{k\Lambda}_{AB}$ are removed in this limit. Thus $\lim_{k\to 0}\hat{W}_{k\Lambda}$ and $\lim_{k\to 0}\hat{\Gamma}_{k\Lambda} = \lim_{k\to 0}\Gamma_{k\Lambda}$ are found to be free of any singularities and well-defined. They are the generating functionals of the connected Green function and 1PI vertex function, respectively. In the WT identity (4·14), "the symmetry breaking terms" are again proportional to $f^2_{k\Lambda}\mathcal{R}^{k\Lambda}_{AB}$ and vanish in the IR limit. Thus, we conclude that $\lim_{k\to 0}\Sigma_{k\Lambda}[\Phi,\ \Phi^*] = \lim_{k\to 0}(W_{k\Lambda},\ W_{k\Lambda})_{\Phi}/2 = \lim_{k\to 0}(\Gamma_{k\Lambda},\ \Gamma_{k\Lambda})_{\varphi}/2 = 0$: all

the quantum fluctuations of the UV fields are integrated out to yield the Zinn-Justin equation.

(2) There is another way to obtain the Zinn-Justin equation. We may consider the standard Legendre effective action based on the path integral (2·10) for the IR fields,

$$\boldsymbol{\Gamma}_{k\Lambda}[\Phi_{\rm cl},\ \Phi^*] = -\hbar \ln Z_{k\Lambda}[J,\ \Phi^*] + f_{k\Lambda}^{-1} J_A {\Phi_{\rm cl}}^A, \tag{4·15}$$

where

$$f_{k\Lambda}^{-1}\Phi_{\rm cl}^A \equiv \hbar \frac{\partial^l \ln Z_{k\Lambda}}{\partial J_A}. \tag{4·16}$$

In the construction of this effective action, all the quantum fluctuations of the UV fields are integrated out. Therefore, the action should obey the Zinn-Justin equation. Actually, one obtains

$$\frac{1}{2}\left(\boldsymbol{\Gamma}_{k\Lambda},\ \boldsymbol{\Gamma}_{k\Lambda}\right)_{\Phi_{\rm cl}} = \frac{\partial^r \boldsymbol{\Gamma}_{k\Lambda}}{\partial \Phi_{\rm cl}^A}\frac{\partial^l \boldsymbol{\Gamma}_{k\Lambda}}{\partial \Phi_A^*} = \langle \Sigma_{k\Lambda}[\Phi,\ \Phi^*]\rangle_{\Phi:f_{k\Lambda}^{-1}J}\,. \tag{4·17}$$

Thus, the QME $\Sigma_{k\Lambda} = 0$ yields the Zinn-Justin equation for the Legendre effective action, $(\boldsymbol{\Gamma}_{k\Lambda},\ \boldsymbol{\Gamma}_{k\Lambda})_{\Phi_{\rm cl}} = 0$.

§5. Conclusions

We have shown here that symmetries are not violated but only deformed by regularizations. This conclusion emerges from a careful study of the WT identities for the effective theory. It is actually a nontrivial problem to derive them in the RG approach. Our observation is that, when applied to a Wilsonian effective action called the average action, they take the form of the QME in the antifield formalism. It makes it conceptually clear that there exist exact renormalized symmetries realized along the RG flow.

Because of the generic interactions among the UV fields, neither the IR action nor the functional measure of the IR fields can remain BRS invariant. The QME ensures the cancellation of these contributions. We have used the relation between the average and the Legendre effective action to show that the QME for the former is equivalent to the "modified WT or ST identity" for the latter. This leads to the identification of the "symmetry breaking terms" with the Jacobian factor mentioned above.

The use of the antifield formalism allows us not only to deal with most general gauge theories with open and/or reducible gauge algebras, but also to reveal the interesting structure of the RG flow and associated renormalized BRS symmetry. First, we may define the quantum BRS transformation for the symmetry. This transformation is nilpotent, and does not obey the Leibniz rule. Second any two average actions on the RG flow are shown to be connected via a canonical transformation.

Our arguments on the renormalized BRS symmetry given here are based on the assumption that the UV action or the average action at some IR cutoff $k = k_0$ obeys the QME. In perturbation theory, imposing the QME or the WT identities at some value of k is called the "fine-tuning." This is discussed extensively in Refs. 7), 9), 10) and 12). There, a regularization is used in which the IR and UV cutoffs are incorporated in the same regulator, and the boundary conditions are imposed on the relevant and irrelevant operators separately. This procedure makes solving the QME rather complicated. In a forthcoming paper,[23] we discuss an alternative method using the Pauli-Villars UV regularization. It allows us to directly confirm that the QME holds at one-loop level for a given anomaly-free UV action.

It has been recognized that the QME plays an important role in the investigation of unitarity in string field theory.[27] The formalism for the renormalized symmetry given here provides another example for which the QME plays a crucial role. It is difficult but highly desirable to solve the QME using a non-perturbative truncation of the average action.

Acknowledgements

K. I. would like to thank M. Bonini for discussions on the subject and the theory group at Parma University for their kind hospitality. H. S. (K. I.) is supported in part by the Grants-in-Aid for Scientific Research No. 12640259 (12640258) from the Japan Society for the Promotion of Science.

Appendix A
—— *A Matrix Notation* ——

In this work we use matrix notation which corresponds to DeWitt's condensed notation in the d-dimensional Euclidean momentum space. In this notation, the discrete indices $A,\ B, \cdots$, indicates the momentum variables as well. We also use a generalized summation convention, in which a repeated index implies not only a summation over various quantum numbers but also a momentum integration. For example,

$$f^A_{\ B} = g^A_{\ C}\ h^C_{\ B} \tag{A·1}$$

is a shorthand expression of

$$f^A_{\ B}(p,-q) = \sum_C \int \frac{d^dk}{(2\pi)^d} g^A_{\ C}(p,-k)\ h^C_{\ B}(k,-q). \tag{A·2}$$

The functional derivative is normalized as

$$\frac{\partial \phi^A}{\partial \phi^B} = \delta^A_{\ B} \equiv \delta_{AB}(2\pi)^d \delta(p-q). \tag{A·3}$$

We often use

$$\begin{aligned}\phi^A M^{(1)}_{AB} \cdots M^{(n)}_{CD} \phi^D &= \int \frac{d^dp_1}{(2\pi)^d} \cdots \int \frac{d^dp_{n+1}}{(2\pi)^d} \phi^A(-p_1) \\ &\quad \times M^{(1)}_{AB}(p_1,-p_2) \cdots M^{(n)}_{CD}(p_n,-p_{n+1}) \phi^D(p_{n+1}), \\ \frac{\partial^r}{\partial \phi^A} M^{(1)AB} \cdots M^{(n)CD} \frac{\partial^l}{\partial \phi^*_D} &= \int \frac{d^dp_1}{(2\pi)^d} \cdots \int \frac{d^dp_{n+1}}{(2\pi)^d} \frac{\partial^r}{\partial \phi^A(p_1)} M^{(1)AB}(p_1,-p_2) \cdots\end{aligned}$$

$$\times M^{(n)CD}(p_n, -p_{n+1})\frac{\partial^l}{\partial\phi^*_D(-p_{n+1})}, \tag{A·4}$$

where $\partial^r/\partial\phi^A$ $(\partial^l/\partial\phi^A)$ denotes a right (left) derivative with respect to ϕ^A. These derivatives are related as

$$\frac{\partial^r F}{\partial\phi^A} = (-)^{\epsilon_A(\epsilon_F+1)}\frac{\partial^l F}{\partial\phi^A}. \tag{A·5}$$

Appendix B
—— The Canonical Transformation for the RG Flow ——

Consider a generic action $W[\Phi,\Phi^*]$ and an infinitesimal canonical transformation with generator $G[\Phi,\ \Phi^*]$:

$$\begin{aligned}\bar\Phi^A &= \Phi^A + (\Phi^A,\ G)_\Phi,\\ \bar\Phi^*_A &= \Phi^*_A + (\Phi^*_A,\ G)_\Phi.\end{aligned} \tag{B·1}$$

The path integral identity

$$\begin{aligned}&\int \mathcal{D}\bar\Phi^*\delta(\bar\Phi^*)\mathcal{D}\bar\Phi\exp(-W[\bar\Phi,\ \bar\Phi^*]/\hbar)\\ &\quad = \int \mathcal{D}\Phi^*\delta(\Phi^*)\mathcal{D}\Phi\exp\left[-W[\bar\Phi,\ \bar\Phi^*]/\hbar + \ln\mathrm{Sdet}\left(\frac{\mathcal{D}\bar\Phi}{\mathcal{D}\Phi}\right)\right]\end{aligned} \tag{B·2}$$

implies that the infinitesimal transformation of the action is $-\delta_Q G \equiv -(G,\ W)_\Phi + \hbar\Delta_\Phi G$:

$$\begin{aligned}&W[\bar\Phi,\ \bar\Phi^*] - \hbar\ln\mathrm{Sdet}\frac{\mathcal{D}\bar\Phi}{\mathcal{D}\Phi}\\ &\quad = W[\Phi+(\Phi,G),\ \Phi^*+(\Phi^*,G)] - \hbar\ln\left(1+(-)^{\epsilon_A}\frac{\partial^r}{\partial\Phi^A}\frac{\partial^l}{\partial\Phi^*_A}G\right)\\ &\quad = W[\Phi,\ \Phi^*] - (G,\ W)_\Phi + \hbar\Delta_\Phi G = W[\Phi,\ \Phi^*] - \delta_Q G.\end{aligned} \tag{B·3}$$

A comment is in order here. In writing (B·2), we assumed that the generator G itself is proportional to Φ^*, so that the canonical transformation does not change the gauge fixing condition. The generator in (3·7) is of this type. However, the change of the action obtained above is correct for a more generic situation (see, for example, Ref. 25)).

Now choose the generator $G[\Phi,\Phi^*]$ in the form of (3·7) with $W_{k\Lambda}$ replaced by W. Let us consider the contribution from the first term of (3·7),

$$G_1[\Phi,\Phi^*] = (-)^{1+\epsilon_B}\frac{1}{2}\Phi^*_A\mathcal{M}^{AB}\frac{\partial^r W}{\partial\Phi^B}dk. \tag{B·4}$$

The IR fields transform as

$$\bar\Phi^A = \Phi^A + \left(\Phi^A, G_1\right)_\Phi$$

$$= \Phi^A - \frac{1}{2}(-)^{\epsilon_B}\mathcal{M}^{AB}\frac{\partial^r W}{\partial \Phi^B}dk - \frac{1}{2}(-)^{\epsilon_C+(\epsilon_A+1)(\epsilon_C+1)}\Phi^*_B\mathcal{M}^{BC}\frac{\partial^l\partial^r W}{\partial \Phi^*_A \Phi^C}dk,$$

$$\bar{\Phi}^*_A = \Phi^*_A + (\Phi^*_A, G_1)_\Phi$$
$$= \Phi^*_A + \frac{1}{2}(-)^{\epsilon_C+\epsilon_A(\epsilon_C+1)}\Phi^*_B\mathcal{M}^{BC}\frac{\partial^l\partial^r W}{\partial \Phi^A \Phi^C}dk, \tag{B·5}$$

which yield

$$W[\Phi + (\Phi, G_1)_\Phi, \Phi^* + (\Phi^*, G_1)_\Phi] - W[\Phi, \Phi^*]$$
$$= -\frac{1}{2}\frac{\partial^r W}{\partial \Phi^A}\left[(-)^{\epsilon_B}\mathcal{M}^{AB}\frac{\partial^r W}{\partial \Phi^B} + (-)^{\epsilon_C+(\epsilon_A+1)(\epsilon_C+1)}\Phi^*_B\mathcal{M}^{BC}\frac{\partial^l\partial^r W}{\partial \Phi^*_A \Phi^C}\right]dk$$
$$+\frac{1}{2}(-)^{\epsilon_C+\epsilon_A(\epsilon_C+1)}\frac{\partial^r W}{\partial \Phi^*_A}\Phi^*_B\mathcal{M}^{BC}\frac{\partial^l\partial^r W}{\partial \Phi^A \partial \Phi^C}dk$$
$$= -\frac{1}{2}(-)^{\epsilon_A+\epsilon_B}\mathcal{M}^{BA}\frac{\partial^r W}{\partial \Phi^A}\frac{\partial^r W}{\partial \Phi^B}dk + \frac{1}{4}\Phi^*_B\mathcal{M}^{BC}\frac{\partial^r}{\partial \Phi^C}\Big[(W, W)_\Phi\Big]dk. \tag{B·6}$$

The Δ derivative of the generator reads

$$\Delta_\Phi G_1 = -\frac{1}{2}(-)^{\epsilon_A+\epsilon_B+1}\frac{\partial^r}{\partial \Phi^A}\frac{\partial^r}{\partial \Phi^*_A}\left(\Phi^*_C\mathcal{M}^{CB}\frac{\partial^r W}{\partial \Phi^B}\right)dk$$
$$= \frac{1}{2}(-)^{\epsilon_A+\epsilon_B}\frac{\partial^r}{\partial \Phi^A}\mathcal{M}^{AB}\frac{\partial^r W}{\partial \Phi^B}dk - \frac{1}{2}\Phi^*_B\mathcal{M}^{BC}\frac{\partial^r}{\partial \Phi^C}\Delta_\Phi W dk. \tag{B·7}$$

From (B·6) and (B·7) we obtain the change of the action $-\delta_Q G_1$ as

$$-\delta_Q G_1 = W[\Phi + (\Phi, G_1)_\Phi, \Phi^* + (\Phi^*, G_1)_\Phi] - W[\Phi, \Phi^*] + \hbar\Delta_\Phi G_1$$
$$= \frac{1}{2}(-)^{\epsilon_A+\epsilon_B+1}\left(\mathcal{M}^{AB}\frac{\partial^r W}{\partial \Phi^B}\frac{\partial^r W}{\partial \Phi^A} - \hbar\frac{\partial^r}{\partial \Phi^A}\mathcal{M}^{AB}\frac{\partial^r W}{\partial \Phi^B}\right)dk$$
$$+\frac{1}{2}\Phi^*_B\mathcal{M}^{BC}\frac{\partial^r}{\partial \Phi^C}\left[\frac{1}{2}(W, W)_\Phi - \hbar\Delta_\Phi W\right]dk. \tag{B·8}$$

Similarly, for the second term of the generator $G[\Phi, \Phi^*]$,

$$G_2[\Phi, \Phi^*] = -\Phi^*_A\left[\partial_k(\ln f_{k\Lambda})\right]\Phi^A dk, \tag{B·9}$$

we obtain

$$-\delta_Q G_2 = -\partial_k(\ln f_{k\Lambda})\left(\Phi^A\frac{\partial^l W}{\partial \Phi^A} - \Phi^*_A\frac{\partial^l W}{\partial \Phi^*_A}\right)dk + \hbar\ \mathrm{Str}[\partial_k(\ln f_{k\Lambda})]dk. \tag{B·10}$$

Equations (B·8) and (B·10) are combined to give

$$-\delta_Q G = (-)^{\epsilon_A+\epsilon_B+1}\ \frac{1}{2}\left(\mathcal{M}^{AB}\frac{\partial^r W}{\partial \Phi^B}\frac{\partial^r W}{\partial \Phi^A} - \hbar\frac{\partial^r}{\partial \Phi^A}\mathcal{M}^{AB}\frac{\partial^r W}{\partial \Phi^B}\right)dk$$
$$+\hbar\ \mathrm{Str}\left[\partial_k(\ln f_{k\Lambda})\right]dk - \partial_k\left(\ln f_{k\Lambda}\right)\left(\Phi^A\frac{\partial^l W}{\partial \Phi^A} - \Phi^*_A\partial_k\frac{\partial^l W}{\partial \Phi^*_A}\right)dk$$
$$+\frac{1}{2}\Phi^*_B\mathcal{M}^{BC}\frac{\partial^r}{\partial \Phi^C}\Sigma[\Phi, \Phi^*]dk. \tag{B·11}$$

When the action $W[\Phi, \Phi^*]$ is the average action defined in (2·11), Eq. (B·11) may be written

$$-\delta_Q G = \partial_k W[\Phi,\ \Phi^*]dk + \frac{1}{2}\Phi^*_B \mathcal{M}^{BC} \frac{\partial^r}{\partial \Phi^C} \Sigma[\Phi, \Phi^*]dk. \qquad \text{(B·12)}$$

Here we have used the flow equation for the average action which may be obtained from (3·2).

References

1) K. G. Wilson and J. Kogut, Phys. Rep. **C12** (1974), 75.
2) F. J. Wegner and A. Houghton, Phys. Rev. **A8** (1973), 401.
3) J. F. Nicoll and T. S. Chang, Phys. Lett. **62A** (1977), 287.
4) J. Polchinski, Nucl. Phys. **B231** (1984), 269.
5) C. Wetterich, Nucl. Phys. **B352** (1991), 529; Phys. Lett. **B301** (1993), 90; Z. Phys. **C60** (1993), 461.
6) B. J. Warr, Ann. of Phys. **183** (1988), 1.
7) C. Becchi, "On the construction of renormalized quantum field theory using renormalization group techniques," in *Elementary particles, Field theory and Statistical mechanics*, ed. M. Bonini, G. Marchesini and E. Onofri (Parma University, 1993).
8) M. Reuter and C. Wetterich, Nucl. Phys. **B417** (1994), 181; **B427** (1994), 291.
F. Freire and C. Wetterich, Phys. Lett. **B380** (1996), 337.
9) M. Bonini, M. D'Attanasio and G. Marchesini, Nucl. Phys. **B418** (1994), 81; **B421** (1994), 429; **B437** (1995), 163; Phys. Lett. **B346** (1995), 87.
M. Bonini and E. Tricarico, Nucl. Phys. **B585** (2000), 253.
10) M. Bonini and G. Marchesini, Phys. Lett. **B389** (1996), 566.
11) U. Ellwanger, Phys. Lett. **B335** (1994), 364.
U. Ellwanger, M. Hirsch and A. Weber, Z. Phys. **C69** (1996), 687.
12) M. D'Attanasio and T. R. Morris, Phys. Lett. **B378** (1996), 213.
13) T. R. Morris, "A Manifestly Gauge Invariant Exact Renormalization Group," *Lectures given at Workshop on the Exact Renormalization Group, Faro, Portugal, 10-12 Sep 1998*, hep-th/9810104
14) T. R. Morris, Nucl. Phys. **B573** (2000), 97; J. High Energy Phys. **12** (2000), 012.
15) M. Simionato, Int. J. Mod. Phys. **A15** (2000), 2121, 2153, 4811.
16) M. Pernici, M. Raciti and F. Riva, Nucl. Phys. **B520** (1998), 469.
17) K.-I. Aoki, K. Morikawa, J.-I. Sumi, H. Terao and M. Tomoyose, Prog. Theor. Phys. **97** (1997), 479
K.-I. Aoki, K. Takagi, H. Terao and M. Tomoyose, Prog. Theor. Phys. **103** (2000), 815.
18) M. Lüscher, Phys. Lett. **B428** (1998), 342; Nucl. Phys. **B549** (1999), 295.
19) P. Ginsparg and K. Wilson, Phys. Rev. **D25** (1982), 2649.
20) Y. Igarashi, K. Itoh and H. So, Phys. Lett **B479** (2000), 336.
21) Y. Igarashi, K. Itoh and H. So, Prog. Theor. Phys. **104** (2000), 1053, hep-th/0006180.
22) I. A. Batalin and G. A. Vilkovisky, Phys. Lett. **102B** (1981), 27.
23) Y. Igarashi, K. Itoh and H. So, in preparation.
24) T. R. Morris, Int. J. Mod. Phys. **A9** (1994), 2411.
25) M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, 1992).
J. Gomis, J. Paris and S. Samuel, Phys. Rept. **259** (1995), 1.
W. Troost and A. Van Proeyen, "An introduction to Batalin-Vilkovisky Lagrangian Quantisation," unpublished notes.
26) P. M. Lavrov and I. V. Tyutin, Sov. J. Nucl. Phys. **41** (1985), 1049.
27) H. Hata, Phys. Lett. **B217** (1989), 438, 445; Nucl. Phys. **B329** (1990), 698.