Progress of Theoretical Physics, Vol. 108, No. 6, December 2002

Gauge and Non-Gauge Tensor Multiplets in 5D Conformal Supergravity

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(Received September 23, 2002)

An off-shell formulation of two distinct tensor multiplets, a massive tensor multiplet and a tensor gauge multiplet, is presented in superconformal tensor calculus in five-dimensional space-time. Both contain a rank 2 antisymmetric tensor field, but there is no gauge symmetry in the former, while it is a gauge field in the latter. Both multiplets have 4 bosonic and 4 fermionic *on-shell* modes, but the former consists of 16 (boson)+16 (fermion) component fields, while the latter consists of 8 (boson)+8 (fermion) component fields.

§1. Introduction

Long ago, Townsend, Pilch and van Nieuwenhuizen [1] discussed supersymmetry multiplets containing rank $2k$ massive antisymmetric tensor fields $A_{\mu_1\cdots\mu_{2k}}$ in odd numbers of dimensions $d = 4k+1$ (and rank $2k-1$ fields in dimensions $d = 4k-1$), which satisfy a self-dual type equations of motion:

$$mA^{\mu_1\cdots\mu_{2k}} = \frac{i}{(2k)!}\epsilon^{\mu_1\cdots\mu_{2k}\nu_1\nu_2\cdots\nu_{2k+1}}\partial_{\nu_1}A_{\nu_2\cdots\nu_{2k+1}}. \quad (1\cdot1)$$

With the recent renewed interest in supergravity in five dimensions, [2] in connection with AdS/CFT dualities [3] and brane world scenarios, [4)−6] Günaydin and Zagermann [7] introduced tensor multiplets in 5D gauged supergravity system, generalizing the earlier work by Günayden, Sierra and Townsend [8] on 5D supergravity system coupled to vector multiplets. They found that the presence of a tensor multiplet gives a novel contribution to the scalar potential. Very recently, Bergshoeff et al. [9] have given a yet more general form of tensor-vector multiplet couplings in a 5D superconformal framework.

These works all give the so-called *on-shell* formulation for the *massive* tensor multiplets, in which auxiliary fields are missing and the supersymmetry algebra closes only on-shell in a particular system.

In a series of papers, [10)−14] we have given a general off-shell formulation for 5D supergravity — superconformal tensor calculus — and have discussed the general supergravity system coupled to Yang-Mills vector multiplets and hypermultiplets in the 5D bulk, [11] as well as its orbifold compactification on S^1/Z_2. [14]

The work presented in Refs. 10)−14) is, however, incomplete, because the tensor multiplets are missing, and therefore, in particular, the scalar potential is not quite

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general. This is because until now there had been no off-shell formulation of the tensor multiplet. The purpose of this paper is to present such an off-shell formulation. We have actually found an off-shell formulation not only of *massive (non-gauge)* tensor multiplets but also of massless tensor *gauge* multiplets, both containing rank 2 tensor fields. Although both multiplets have 4 bosonic and 4 fermionic *on-shell* modes, the former consists of 16 (boson)+16 (fermion) component fields, while the latter consists of 8 (boson)+8 (fermion) component fields. To make clear the distinction between the two tensor multiplets, we call the former the 'large (or, 'massive') tensor multiplet' and the latter the 'tensor gauge multiplet' (or, the 'small tensor multiplet').

In this paper we assume knowledge of 5D superconformal tensor calculus. We refer the reader for the details to Ref. 12) (see also Ref. 15)).

The rest of this paper is organized as follows. We start in §2 by performing a dimensional reduction of the known 6D superconformal tensor multiplet to 5D, and find the small (8+8) tensor multiplet. The main purpose of that section is, however, to motivate the form of the supersymmetry transformation rule of 5D (large or small) tensor multiplets. It is actually easier to compute directly in five dimensions for the purpose of determining the details of the multiplets. Therefore, in §3 we examine directly in 5D the form of the supersymmetry transformation rule suggested by the dimensional reduction in a slightly more general manner, and actually find a larger 16+16 multiplet, which we call the 'large tensor multiplet'. Here, we incorporate the point by Bergshoeff et al.[9] that the large tensor multiplets should be treated collectively with vector multiplets. In §4, we impose a stronger constraint, which reduces the large tensor multiplet to a smaller 8+8 multiplet, and show that it coincides with the small (8+8) tensor multiplet obtained from the above mentioned dimensional reduction. In this case, the original tensor field $\hat{B}_{ab}$ is made subject to the Bianchi-type constraint $\hat{\mathcal{D}}_\nu \hat{B}^{\mu\nu} + \cdots = 0$, and it is shown to be essentially expressed as a dual of the field strength $\partial_{[\lambda} A_{\mu\nu]}$ of a rank 2 tensor gauge field $A_{\mu\nu}$. In §5 we present the general form of the large tensor multiplet action. An explicit component form is given. It is shown that the mixing of the vector multiplets in the Yang-Mills gauge transformation of the tensor multiplets can generally be eliminated aside from the mixing of the central-charge vector multiplet which is effectively induced by the vector-tensor mixing mass terms. Finally, in §6 we briefly discuss the gauge invariant action for the small tensor multiplet. We also show there that the tensor gauge multiplet action is in fact dual to the vector multiplet action. An appendix is added to present the solution of an algebraic equation that appears when $\hat{B}_{ab}$ is rewritten in terms of the tensor gauge field $A_{\mu\nu}$.

§2. Dimensional reduction of a 6D tensor multiplet to 5D

A superconformal tensor multiplet is known in 6D,[16] but it is an on-shell multiplet; that is, the superconformal algebra closes only when the multiplet satisfies an equation of motion. We can, however, convert it into an off-shell 5D multiplet by making a dimensional reduction, in the same manner as we have done for the hypermultiplet.[10] When going down to 5D, we reinterpret the fifth spatial derivative

$\partial/\partial x^5 \equiv \partial/\partial z$ as a *central charge transformation* $\boldsymbol{Z}$, and then in 5D the constraint equations in 6D are no longer the equations of motion but become the defining equations of the central charge transformations of the relevant fields. The curved index as well as the coordinate for the fifth spatial dimension is denoted z, in distinction from the tangent index 5; $(x^{\mu=0,1,2,3,4}, x^z = z)$. The $U(1)$ gauge multiplet $V^0 = (M^0{\equiv}\alpha, W^0_\mu, \Omega^{0i}, Y^{0ij})$ which couples to this central charge is identified with the fifth spatial components of the vielbein, Rarita-Schwinger and conformal $SU(2)$ gauge fields, $\underline{e}_z{}^5$, $\underline{e}_\mu{}^5$, $\underline{\psi}^i_z$, $\underline{V}^{ij}_z$, as follows (in the gauge $\underline{e}_z{}^a = 0$): [12]

$$\alpha^{-1} = \underline{e}_z{}^5, \quad W^0_\mu = \alpha \underline{e}_\mu{}^5, \quad \Omega^i_0 = -\alpha^2 \underline{\psi}^i_z, \quad Y^{ij}_0 = \alpha^2 \underline{V}^{ij}_z - \frac{3i}{\alpha}\bar{\Omega}^i_0 \Omega^j_0. \tag{2·1}$$

The underline here denotes the fields in 6D. Note that $\underline{e}_a{}^z = -\underline{e}_a{}^\mu W^0_\mu$, so that the 6D derivative with flat index $\underline{\partial}_a = \underline{e}_a{}^\mu \partial_\mu + \underline{e}_a{}^z \partial_z$ indeed reduces to the $\boldsymbol{Z}$-covariant derivative $\underline{e}_a{}^\mu(\partial_\mu - W^0_\mu \boldsymbol{Z}) = \partial_a - W^0_a \boldsymbol{Z}$ in 5D.

The 6D tensor multiplet consists of a scalar $\underline{\sigma}$, $SU(2)$-Majorana-Weyl spinor $\underline{\tau}^{i}$ *) $(\gamma_7 \underline{\tau}^i = -\underline{\tau}^i)$, and selfdual rank 3 tensor F^+_{abc}, whose superconformal transformation rules and constraint equations were given by Bergshoeff, Sezgin and Van Proeyen. [16] If we perform the dimensional reduction explained in Ref. 10) and use the 5D supersymmetry transformation, which is identified with a certain linear combination of the 6D superconformal transformations, [12] then we obtain a 5D tensor multiplet $(\sigma, \tau^i, \hat{B}_{ab}, X^{ij}, \cdots)$. The supersymmetry transformation laws of the first two components are found to be

$$\begin{aligned}
&\delta_Q(\varepsilon)\sigma = 2i\bar{\varepsilon}\tau, \\
&\delta_Q(\varepsilon)\tau^i = -\tfrac{1}{4}\gamma^{ab}\hat{B}_{ab}\varepsilon^i - \tfrac{1}{2}\hat{\not{\mathcal{D}}}\sigma\varepsilon^i + \tfrac{1}{2}\alpha \boldsymbol{Z}\sigma\varepsilon^i + X^i{}_j\varepsilon^j,
\end{aligned} \tag{2·2}$$

and the fields appearing here are identified with the following combinations of 6D fields:

$$\begin{aligned}
\sigma &= \frac{1}{\alpha}\underline{\sigma}, \qquad \tau^i = \frac{1}{\alpha}(\underline{\tau}^i - \sigma\Omega_0), \\
\hat{B}_{ab} &= \frac{1}{\alpha}\left(F^+_{ab5} - 2\alpha\sigma v_{ab} - \tfrac{1}{2}\sigma\hat{F}_{ab}(W^0) - 2i\bar{\Omega}^0\gamma_{ab}\tau - \tfrac{i}{\alpha}\bar{\Omega}^0\gamma_{ab}\Omega^0\right), \\
X^{ij} &= -\frac{1}{\alpha}\left(\sigma Y^{ij}_0 - 2i\bar{\Omega}^{0(i}\tau^{j)}\right).
\end{aligned} \tag{2·3}$$

The powers of α have been multiplied such that σ, τ^i and $\hat{B}_{ab}$ carry Weyl weights $w = 1$, $3/2$ and 2, respectively.

We could continue this procedure of dimensional reduction to find the transformation rule of the $\hat{B}_{ab}$ field and to rewrite the constraint equations

$$G_{\underline{ab}} \equiv \hat{\underline{\mathcal{D}}}^c F^+_{\underline{abc}} + \cdots = 0, \tag{2·4a}$$

$$\Gamma^i \equiv \hat{\underline{\not{\mathcal{D}}}}\underline{\tau}^i + \cdots = 0, \tag{2·4b}$$

$$C \equiv \hat{\underline{\mathcal{D}}}^c \hat{\underline{\mathcal{D}}}_c \underline{\sigma} + \cdots = 0 \tag{2·4c}$$

*) Originally, in Ref. 16), this $SU(2)$-Majorana-Weyl spinor $\underline{\tau}^i$ was denoted by ψ^i. We prefer τ^i, since ψ^i is easily confused with the Rarita-Schwinger field ψ^i_μ.

in terms of these 5D fields. With this we would obtain an off-shell *tensor gauge multiplet* in 5D, since the $(\underline{ab}) = (a5)$ component of the constraints (2·4a) gives the Bianchi identity $\hat{\mathcal{D}}^b\hat{B}_{ab} + \cdots = 0$, implying that $\hat{B}_{\mu\nu}$ is the dual of the field strength $3\partial_{[\mu}A_{\nu\rho]}$ of a rank 2 tensor gauge field $A_{\mu\nu}$. The other (ab) component of Eq. (2·4a) defines the central charge transformation of B_{ab}, $\partial_z F^{+}_{ab5} \sim \boldsymbol{Z}\hat{B}_{ab}$, and the constraints (2·4b) and (2·4c) define the central charge transformations $\boldsymbol{Z}\tau^i$ and $\boldsymbol{Z}(Z\sigma)$, respectively, of the fermion τ^i and the auxiliary scalar $Z\sigma \equiv \boldsymbol{Z}\sigma$. Thus this tensor gauge multiplet consists of the 8 boson components $1(\sigma) + 6(A_{\mu\nu}) + 1(Z\sigma)$ and 8 fermion components τ^i. (Note that the rank r antisymmetric tensor field in d dimensions has ${}_{d-1}C_r = (d-1)!/r!(d-1-r)!$ off-shell degrees of freedom while the non-gauge tensor has ${}_dC_r = d!/r!(d-r)!$ components.)

We find a concrete form of these results for this tensor gauge multiplet in §4, working directly in 5D as a special case of the more general tensor multiplet, which we discuss in the next section.

§3. Large tensor multiplet $\boldsymbol{T}^\alpha$

Let $V^I = (M^I,\ W^I_\mu,\ \Omega^{Ii},\ Y^{Iij})$ $(I = 0, 1, 2, \cdots)$ be the vector multiplets [12] of the system in which the zero-th multiplet V^0 denotes the $U(1)$ vector multiplet coupling to the central charge $\boldsymbol{Z}$, and the other V^I $(I \geq 1)$ are the Yang-Mills multiplets of a gauge group G'. We write $U(1)_Z \times G' = G$. We consider a set of scalar fields $\{\sigma^\alpha\}_{\alpha=1,2,\cdots}$, which give a representation of the Yang-Mills group G' and carry the central charge $\boldsymbol{Z}$ as well. Let us start with the superconformal transformation law $\delta\sigma^\alpha \equiv (\delta_Q(\varepsilon) + \delta_S(\eta))\sigma^\alpha = 2i\bar{\varepsilon}^i\tau^\alpha_i \equiv 2i\bar{\varepsilon}\tau^\alpha$ of the scalar fields σ^α with Weyl weight w. This defines the $SU(2)$-Majorana fermion field τ^i. Then the 5D superconformal algebra presented in Ref. 12) generally determines the superconformal transformation of τ^i in the form

$$\delta\tau^i = -\ \tfrac{1}{4}\gamma^{ab}\hat{B}_{ab}\varepsilon^i -\ \tfrac{1}{2}\hat{\not{\mathcal{D}}}\sigma\varepsilon^i +\ \tfrac{1}{2}M_*\sigma\varepsilon^i - X^{ij}\varepsilon_j - Z^{ij}_a\gamma^a\varepsilon_j - w\,\sigma\,\eta^i, \qquad (3·1)$$

where $\hat{B}_{ab}$ is an anti-symmetric tensor, and X^{ij} and Z^{ij}_a are an $SU(2)$-triplet [i.e., (i,j)-symmetric] scalar and vector, respectively. We use $\Lambda_*\varphi$ to denote the $G = U(1)_Z \times G'$ gauge transformation of the field φ with parameters Λ^I:

$$\Lambda_*\sigma = \delta_G(\Lambda)\sigma = \Lambda^0\boldsymbol{Z}\sigma + \delta_{G'}(\Lambda)\sigma. \qquad (3·2)$$

We defer the presentation of the explicit form of the Yang-Mills G'-transformation $\delta_{G'}(\Lambda)\sigma$ of the tensor multiplet to §5, since the following discussion in this section is independent of it. Comparing this general form (3·1) with the previous $\delta\tau^i$ given in Eq. (2·2) for the tensor gauge multiplet, we see that the $SU(2)$-tensor vector term $Z^{ij}_a\gamma^a\varepsilon_j$ is missing in the latter. We are thus led to try the transformation rule (3·1) with the Z^{ij}_a term *omitted*. Then, using the 5D superconformal algebra, [12] we find the transformation rules

$$\begin{aligned}
&\delta\sigma^\alpha = 2i\bar{\varepsilon}\tau^\alpha, \\
&\delta\tau^{\alpha i} = -\ \tfrac{1}{4}\gamma^{ab}\hat{B}^\alpha_{ab}\varepsilon^i -\ \tfrac{1}{2}\hat{\not{\mathcal{D}}}\sigma^\alpha\varepsilon^i +\ \tfrac{1}{2}M_*\sigma^\alpha\varepsilon^i - X^{\alpha ij}\varepsilon_j - \sigma^\alpha\eta^i,
\end{aligned}$$

$$\begin{aligned}
\delta\hat{B}^\alpha_{ab} &= 4i\bar{\varepsilon}\gamma_{[a}\hat{\mathcal{D}}_{b]}\tau^\alpha - 2i\bar{\varepsilon}\gamma_{cd[a}\gamma_{b]}\tau^\alpha v^{cd} + 2i\bar{\varepsilon}\hat{R}_{ab}(Q)\sigma^\alpha \\
&\quad +2i\bar{\varepsilon}\gamma_{ab}\Omega_*\sigma^\alpha + 2i\bar{\varepsilon}\gamma_{ab}M_*\tau^\alpha - 4i\bar{\eta}\gamma_{ab}\tau^\alpha, \\
\delta X^{\alpha ij} &= 2i\bar{\varepsilon}^{(i}\hat{\not{\mathcal{D}}}\tau^{\alpha j)} - i\bar{\varepsilon}^{(i}\gamma\cdot v\tau^{\alpha j)} - \tfrac{i}{4}\bar{\varepsilon}^{(i}\chi^{j)}\sigma^\alpha \\
&\quad +4i\bar{\varepsilon}^{(i}\Omega_*^{j)}\sigma^\alpha + 2i\bar{\varepsilon}^{(i}M_*\tau^{\alpha j)} - 2i\bar{\eta}^{(i}\tau^{\alpha j)},
\end{aligned} \tag{3·3}$$

where we have fixed the Weyl weight value of σ to 1 for convenience. (We can adjust it by multiplying by $\alpha = M^0$ if necessary.) A dot between two tensors generally represents contraction; e.g., $\gamma\cdot v = \gamma^{ab}v_{ab}$. Closure of the algebra requires the constraints

$$\begin{aligned}
0 &= M_*\hat{B}^\alpha_{ab} + \hat{F}_{ab}(W)_*\sigma^\alpha + 4v_{ab}M_*\sigma^\alpha + 2i\bar{\Omega}_*\gamma_{ab}\tau^\alpha \\
&\qquad - \tfrac{1}{2}\epsilon_{abcde}\left(\hat{\mathcal{D}}^c\hat{B}^{\alpha de} + 2i\bar{\tau}^\alpha\gamma^c\hat{R}^{de}(Q)\right),
\end{aligned} \tag{3·4}$$

$$0 = M_*X^{\alpha ij} + Y_*^{ij}\sigma^\alpha - 2i\bar{\Omega}_*^{(i}\tau^{\alpha j)} \tag{3·5}$$

and the supersymmetry transformation descendants of (3·5), since it is not $\boldsymbol{Q}$-inert. Actually, we now see that the quantity on the RHS of Eq. (3·5) turns out to be the first component of a *linear multiplet.* [12)]

We should observe that this transformation rule (3·3) for the tensor multiplet T^α has exactly the same form as that [12)] for the vector multiplet V^I. The only difference is hidden in the G-transformation '$_*$' and its manifestation is that the vector multiplets carry no central charge, $\boldsymbol{Z}V^I = 0$. Then, it is easy to see that all the constraints in (3·4) and (3·5) are trivially satisfied when the component fields of the tensor multiplets $T^\alpha = (\sigma^\alpha, \tau^{\alpha i}, \hat{B}^\alpha_{ab}, X^{\alpha ij})$ are replaced by those of vector multiplets, $V^I = (M^I, \Omega^{Ii}, \hat{F}^I_{ab}, Y^{Iij})$. In any case, since the transformation rules take the same form, the embedding formula of the vector multiplets V^I into a linear multiplet $L(f(V))$, [12)] which applies to any homogeneous quadratic function $f(V) = \frac{1}{2}f_{IJ}V^IV^J$, is valid even when the tensor multiplets T^α are included, and the formula is generalized as follows. We write the vector and tensor multiplets collectively as $\mathcal{T}^A \equiv (V^I, T^\alpha)$, and their components as $\mathcal{T}^A = (\sigma^A, \tau^{Ai}, \hat{B}^A_{ab}, X^{Aij})$, following the notation for the tensor multiplet. For the vector multiplet index $A = I$, of course, the components should be understood as $(\sigma^I, \tau^{Ii}, \hat{B}^I_{ab}, X^{Iij}) = (M^I, \Omega^{Ii}, \hat{F}^I_{ab}(W), Y^{Iij})$. Then, for any quadratic function $f(\mathcal{T}) = \frac{1}{2}f_{AB}\mathcal{T}^A\mathcal{T}^B = \frac{1}{2}f_{IJ}V^IV^J + f_{I\alpha}V^IT^\alpha + \frac{1}{2}f_{\alpha\beta}T^\alpha T^\beta$, we have the following linear multiplet $L(f(\mathcal{T})) = (\,L^{ij}(f(\mathcal{T})),\, \varphi^i(f(\mathcal{T})),\, E_a(f(\mathcal{T})),\ N(f(\mathcal{T}))\,)$:

$$\begin{aligned}
L^{ij}(f(\mathcal{T})) &= f_AX^{Aij} - i\bar{\tau}^{Ai}\tau^{Bj}f_{AB}, \\
\varphi^i(f(\mathcal{T})) &= -\tfrac{1}{4}\chi^i f + 2\Omega_*^i f + \left(\hat{\not{\mathcal{D}}} - \tfrac{1}{2}\gamma\cdot v + M_*\right)\tau^{Ai}f_A \\
&\quad + \left(-\tfrac{1}{4}\gamma\cdot\hat{B}^A + \tfrac{1}{2}\hat{\not{\mathcal{D}}}\sigma^A - \tfrac{1}{2}M_*\sigma^A - X^A\right)\tau^{Bi}f_{AB}, \\
E_a(f(\mathcal{T})) &= \hat{\mathcal{D}}^b\left(4v_{ab}f + \hat{B}^A_{ab}f_A + i\bar{\tau}^A\gamma_{ab}\tau^Bf_{AB}\right) + \tfrac{1}{8}\epsilon_{abcde}\hat{B}^{Abc}\hat{B}^{Bde}f_{AB} \\
&\quad + \left(-\hat{\mathcal{D}}_aM_*\sigma^A + \hat{\mathcal{D}}_a\sigma^AM_*\right)f_A \\
&\quad + \left(-2i(\bar{\Omega}_*\gamma_a\tau)^A\sigma^B + 2i\bar{\tau}^A\gamma^a(M_*\tau)^B + 4i\bar{\tau}^A\gamma_a(\Omega_*\sigma)^B\right)f_{AB},
\end{aligned}$$

$$
\begin{aligned}
N(f(\mathcal{T})) = &-\hat{\mathcal{D}}^a\hat{\mathcal{D}}_a f + M_*M_*f - (\tfrac{1}{2}D + 3v\cdot v)f + (-2\hat{B}^A\cdot v + i\bar{\chi}\tau^A + 2i\bar{\Omega}_*\tau^A)f_A \\
&+ \Big(-\tfrac{1}{4}\hat{B}^A\cdot\hat{B}^B + \tfrac{1}{2}\hat{\mathcal{D}}^a\sigma^A\hat{\mathcal{D}}_a\sigma^B + 2i\bar{\tau}^A\hat{\not{\mathcal{D}}}\tau^B - i\bar{\tau}^A\gamma\cdot v\tau^B + X^A_{ij}X^{Bij} \\
&\qquad - \tfrac{3}{2}M_*\sigma^A M_*\sigma^B + 4i\bar{\tau}^A M_*\tau^B + 4i\bar{\tau}^A\Omega_*\sigma^B\Big) f_{AB}, \qquad (3\cdot6)
\end{aligned}
$$

where

$$
\begin{aligned}
&f \equiv f(\sigma) = \tfrac{1}{2}f_{AB}\sigma^A\sigma^B = \tfrac{1}{2}f_{IJ}M^IM^J + f_{I\alpha}M^I\sigma^\alpha + \tfrac{1}{2}f_{\alpha\beta}\sigma^\alpha\sigma^\beta, \\
&f_A = \frac{\partial f(\sigma)}{\partial\sigma^A}, \qquad f_{AB} = \frac{\partial^2 f(\sigma)}{\partial\sigma^A\partial\sigma^B} \qquad (3\cdot7)
\end{aligned}
$$

in this formula.

In view of the formula (3·6), we recognize that the RHS quantity of the constraint (3·5) is just the first component $L^{ij}(f(\mathcal{T}))$ of the linear multiplet $L(f(\mathcal{T}))$ for the choice $f(\mathcal{T}) = V_*T^\alpha$:

$$
f(\sigma) = M_*\sigma^\alpha = M^0\boldsymbol{Z}\sigma^\alpha + \delta_{G'}(M)\sigma^\alpha. \qquad (3\cdot8)
$$

Thus, the complete set of constraint equations for the tensor multiplets T^α are given by Eq. (3·4) and

$$
L(V_*T^\alpha) = \Big(L^{ij}(V_*T^\alpha),\, \varphi^i(V_*T^\alpha),\, E_a(V_*T^\alpha),\, N(V_*T^\alpha)\Big) = 0. \qquad (3\cdot9)
$$

It is, however, easy to see that the vector component constraint $E_a(V_*T) = 0$ here is automatically satisfied if the constraint (3·4) is satisfied. In confirming this, we note that the last two lines of $E_a(f(\mathcal{T}))$ in Eq. (3·6) vanish for $f(\mathcal{T}) = V\cdot T$, with the dot product '·' satisfying Eq. (4·3) in the footnote appearing subsequently, and thus in particular, for the simple product VT or $*$-product V_*T. We also need the equation

$$
\begin{aligned}
\hat{\mathcal{D}}_{[a}\hat{B}^\alpha_{bc]} + 2i\bar{\tau}^\alpha\gamma_{[a}\hat{R}_{bc]}(Q) = &\, e_{[a}{}^\mu e_b{}^\nu\mathcal{D}_{c]}B^\alpha_{\mu\nu} - 2i\bar{\psi}_{[a}\gamma_{bc]}\Omega_*\sigma^\alpha \\
&- 2i\bar{\psi}_{[a}\gamma_{bc]}M_*\tau^\alpha + 2i\bar{\psi}_{[a}\gamma_b\psi_{c]}M_*\sigma^\alpha, \qquad (3\cdot10)
\end{aligned}
$$

where we have introduced $B^A_{\mu\nu} = \{B^\alpha_{\mu\nu}, F^I_{\mu\nu}(W)\}$ without a hat, '$\hat{\ }$' , representing $\hat{B}^A_{\mu\nu}$ with supersymmetry covariantization terms subtracted:

$$
B^A_{ab} \equiv \hat{B}^A_{ab} - 4i\bar{\psi}_{[a}\gamma_{b]}\tau^A + 2i\bar{\psi}_a\psi_c\sigma^A. \qquad (3\cdot11)
$$

Equation (3·10) can be shown by using explicit expressions for $\hat{R}_{ab}(Q)$ and supercovariant derivatives as well as the identity $\gamma_d\psi^i_{[a}(\bar{\psi}_b\gamma^d\psi_{c]}) = \psi^i_{[a}(\bar{\psi}_b\psi_{c]})$ [See Eq. (A·11) in Ref. 10).]. Thus the independent constraints are given by Eq. (3·4), $L^{ij}(V_*T) = 0, \varphi^i(V_*T) = 0$ and $N(V_*T) = 0$, which are interpreted as defining equations of the central charge transformation of $\hat{B}_{ab}$, X^{ij}, $Z\tau^i$ and $Z^2\sigma$, respectively. ($Z^n\phi$ represents the field $\boldsymbol{Z}^n\phi$ obtained by performing central charge transformation n times.) We thus finally see that this tensor multiplet consists of

$$
T^\alpha = \Big(\sigma^\alpha,\ \tau^{\alpha i},\ \hat{B}^\alpha_{ab},\ X^{\alpha ij},\ Z\sigma^\alpha,\ Z\tau^{\alpha i},\ Z^2\sigma^\alpha\Big) \qquad (3\cdot12)
$$

Table I. Field content of the multiplets.

field	type	restrictions	$SU(2)$	Weyl-weight
		large tensor multiplet $\boldsymbol{T}$		
σ	boson	real	**1**	1
τ^i	fermion	$SU(2)$-Majorana	**2**	$\frac{3}{2}$
$\hat{B}_{ab}$	boson	real, antisymmetric	**1**	2
X^{ij}	boson	$X^{ij}=X^{ji}=(X_{ij})^*$	**3**	2
$Z\sigma$	boson	real	**1**	1
$Z\tau^i$	fermion	$SU(2)$-Majorana	**2**	$\frac{3}{2}$
$Z^2\sigma$	boson	real	**1**	1
		tensor gauge multiplet $\boldsymbol{A}$		
$A_{\mu\nu}$	boson	real, $\delta A_{\mu\nu}=2\partial_{[\mu}\Lambda_{\nu]}$	**1**	0
σ	boson	real	**1**	1
τ^i	fermion	$SU(2)$-Majorana	**2**	$\frac{3}{2}$
$Z\sigma$	boson	real	**1**	1

for each α and contains $1(\sigma)+10(\hat{B}_{ab})+3(X^{ij})+1(Z\sigma)+1(Z^2\sigma)=16$ bose field components and $8(\tau^i)+8(Z\tau^i)=16$ fermi field components. Note that $\hat{B}_{ab}$ is a mere tensor field, neither a field strength nor a gauge potential field. We call this 16+16 multiplet the 'large tensor multiplet', contrasting it with the smaller 'tensor gauge multiplet'. The components of this multiplet and their properties are listed in Table I.

The superconformal transform action rules for the auxiliary fields $Z\sigma^\alpha$, $Z\tau^{\alpha i}$ and $Z^2\sigma^\alpha$ are omitted, since they trivially follow from the stipulation that the central charge transformation $\boldsymbol{Z}$ be *central*, i.e., that they commute with all transformations.

§4. Tensor gauge multiplet $\boldsymbol{A}$

If the tensor multiplet carries no gauge group G' charges other than the central charge, that is, $V_*T=V^0\boldsymbol{Z}T$, then we can derive a smaller tensor multiplet by requiring

$$L(V^0T)=\Big(L^{ij}(V^0T),\,\varphi^i(V^0T),\,E_a(V^0T),\,N(V^0T)\Big)=0, \tag{4·1}$$

which is a stronger constraint than the previous one, $L(V_*T)=0$, in Eq. (3·9). Indeed, the latter follows from the former; $L(V_*T)=\boldsymbol{Z}L(V^0T)=0$, so that the constraint (4·1) plus Eq. (3·4) gives a sufficient set of conditions for the multiplet to exist. Note that the first component of this linear multiplet constraint (4·1) reads

$$L^{ij}(V^0T)=\alpha X^{ij}+\sigma Y_0^{ij}-2i\bar{\Omega}_0^{(i}\tau^{j)}=0, \tag{4·2}$$

which can be solved with respect to X^{ij} so that X^{ij} is no longer an independent field! This contrasts with the case of the large tensor multiplet, in which the constraint $L^{ij}(V_*T)=0$ merely defines the central charge transformation of X^{ij}, $\boldsymbol{Z}X^{ij}$. We here notice that Eq. (4·2) is the same relation as the last equation in Eq. (2·3) that we obtained by dimensional reduction from the 6D tensor multiplet. We have therefore found that the relation (2·3) is the first component of the linear multiplet constraint (4·1), and hence that the multiplet obtained by imposing the stronger constraints $L(V^0T)=0$, (4·1), is nothing but the *tensor gauge multiplet* possessing 8 boson + 8 fermion components that we would get by dimensional reduction from the 6D tensor multiplet.

Note that the constraints $\varphi^i(V^0T)=0$ and $N(V^0T)=0$ determine the central charge transformations $\boldsymbol{Z}\tau^i$ and $\boldsymbol{Z}(Z\sigma)$, respectively. The additional constraint (3·4) determines the the central charge transformation of $\hat{B}_{ab}$. As we now show in detail, the final constraint $E_a(V^0T)=0$ corresponds to the (a5) components of the 6D constraints (2·4a) and is the Bianchi identity $\hat{\mathcal{D}}^b\hat{B}_{ab}+\cdots=0$.*)

Generally, the linear multiplet (L^{ij},φ^i,E_a,N) satisfies the constraint

$$\hat{\mathcal{D}}_aE^a+i\bar{\varphi}\gamma\cdot\hat{R}(Q)+M_*N+4i\bar{\Omega}_*\varphi+2Y_*^{ij}L_{ij}=0, \tag{4·5}$$

which can be rewritten in the form [12]

$$\begin{aligned}&e^{-1}\partial_\lambda(e\mathcal{E}^\lambda)+2\mathcal{H}_{VL}=0, &(4·6)\\ &\text{with}\quad \mathcal{E}^\lambda\equiv E^\lambda-2i\bar{\psi}_\rho\gamma^{\rho\lambda}\varphi+2i\bar{\psi}_\rho\gamma^{\lambda\rho\sigma}L\psi_\sigma,\\ &\qquad\mathcal{H}_{VL}\equiv Y_*^{ij}L_{ij}+2i\bar{\Omega}_*\varphi+2i\bar{\psi}_i^a\gamma_a\Omega_{j*}L^{ij}-\tfrac{1}{2}W_{a*}\mathcal{E}^a\\ &\qquad\qquad+\tfrac{1}{2}M_*\left(N-2i\bar{\psi}_b\gamma^b\varphi-2i\bar{\psi}_a^{(i}\gamma^{ab}\psi_b^{j)}L_{ij}\right). &(4·7)\end{aligned}$$

In the present case of the linear multiplet $L(V^0T)$, it is G'-neutral and the $*$-operation is only the $\boldsymbol{Z}$ transformation. If we here use the constraints $L^{ij}(V^0T)=\varphi^i(V^0T)=N(V^0T)=0$ other than $E^\lambda(V^0T)=0$, this equation (4·6) is reduced to

$$e^{-1}\partial_\lambda(eE^\lambda(V^0T))-W_\lambda^0\boldsymbol{Z}E^\lambda(V^0T)=0. \tag{4·8}$$

But $\boldsymbol{Z}E^\lambda(V^0T)=E^\lambda(V_*T)$, which vanishes automatically because of the constraint (3·4), as remarked before. Thus we have $\partial_\lambda(eE^\lambda(V^0T))=0$, just as in the case when $L(V^0T)$ is completely neutral. This implies that $eE^\lambda(V^0T)$ can be written as the divergence of a rank 2 antisymmetric tensor density $E^{\lambda\rho}$. Indeed, inspecting the formula (3·6) for $f(M)=\alpha\sigma$, we can show that $E_a(V^0T)$ can be written in the form

$$\begin{aligned}E^\lambda(V^0T)&=-e^{-1}\partial_\rho(E^{\lambda\rho}(V^0T)), &(4·9)\\ -E^{\lambda\rho}(V^0T)&=e\left(4v^{\lambda\rho}\alpha\sigma+\alpha\hat{B}^{\lambda\rho}+\hat{F}^{\lambda\rho}(W^0)\sigma+2i\bar{\Omega}^0\gamma^{\lambda\rho}\tau\right)\\ &\quad+\tfrac{1}{2}\epsilon^{\lambda\rho\sigma\mu\nu}\left\{W_\sigma^0B_{\mu\nu}+2i\bar{\psi}_\mu\gamma_\sigma\psi_\nu\alpha\sigma+2i\bar{\psi}_\sigma\gamma_{\mu\nu}(\alpha\tau+\Omega^0\sigma)\right\}. &(4·10)\end{aligned}$$

*) Actually, this tensor gauge multiplet can be defined even if we replace the linear multiplet constraint (4·2), $L(V^0T)=0$, defined with a simple product of V^0 and T, by a more general one, $L(V\cdot T)=0$, defined with a *dot product* '$\cdot$' that satisfies the Jacobi-like identity

$$[V_1,\,V_2;\,T]\equiv V_1\cdot(V_{2*}T)-V_2\cdot(V_{1*}T)-(V_{1*}V_2)\cdot T=0 \tag{4·3}$$

and invertibility. Then $L(V\cdot T)=0$ implies $L(V_*T)=0$, since

$$\delta_G(\Lambda)L(V\cdot T)=\Lambda_*L(V\cdot T)=L((\Lambda_*V)\cdot T+V\cdot(\Lambda_*T))=\Lambda\cdot L((V_*T))-L([\Lambda,V;T]). \tag{4·4}$$

The Bianchi identity $E_a(V\cdot T)=0$ can also be solved with a rank 2 tensor gauge field $A_{\mu\nu}$ in the same way as for the simple product case $E_a(V^0T)=0$. In the main text, however, we deal with only the latter case for simplicity of notation.

This expression actually has the same form as $E^{\lambda\rho}(V_1V_2)$ [12] for the case of the completely neutral vectors V_1 and V_2.

Now, because of the form (4·9), the constraint $E^\lambda(V^0T)=0$ implies that there exists an antisymmetric tensor gauge field $A_{\mu\nu}$ with which $-E^{\lambda\rho}$ can be written as $\frac{1}{3!}\epsilon^{\lambda\rho\sigma\mu\nu}\partial_{[\sigma}A_{\mu\nu]}$; thus the constraint $E^\lambda(V^0T)=0$ is rewritten as

$$\hat{F}_{\lambda\mu\nu}(A)-\ \tfrac{1}{2}\epsilon_{\lambda\mu\nu\rho\sigma}\Big(4v^{\rho\sigma}\alpha\sigma+\alpha\hat{B}^{\rho\sigma}+\hat{F}^{\rho\sigma}(W^0)\sigma+2i\bar{\Omega}^0\gamma^{\rho\sigma}\tau\Big)=0 \tag{4·11}$$

in terms of the covariant field strength of $A_{\mu\nu}$:

$$\begin{aligned}\hat{F}_{\lambda\mu\nu}(A)\equiv{}&3\partial_{[\lambda}A_{\mu\nu]}-3W^0_{[\lambda}B_{\mu\nu]}\\&-6i\bar{\psi}_{[\lambda}\gamma_{\mu\nu]}(\alpha\tau+\Omega^0\sigma)+6i\bar{\psi}_{[\lambda}\gamma_\mu\psi_{\nu]}\alpha\sigma\,.\end{aligned} \tag{4·12}$$

With Eq. (4·11), the original tensor $\hat{B}_{ab}$ is now rewritten in terms of the *tensor gauge field* $A_{\mu\nu}$, which has fewer components, ${}_{5-1}C_2=6$, than $\hat{B}_{ab}$, due to gauge invariance under $\delta A_{\mu\nu}=2\partial_{[\mu}\Lambda^B_{\nu]}$. Solving $\hat{B}_{ab}$ in terms of $A_{\mu\nu}$ is, however, not quite trivial, since Eq. (4·11) contains $\hat{B}_{ab}$ in two places; that is, it has the form

$$3W^0_{[\lambda}\hat{B}_{\mu\nu]}+\ \tfrac{1}{2}\alpha\epsilon_{\lambda\mu\nu\rho\sigma}\hat{B}^{\rho\sigma}=\mathcal{H}_{\lambda\mu\nu}, \tag{4·13}$$

$$\begin{aligned}\mathcal{H}_{\lambda\mu\nu}\equiv{}&3\partial_{[\lambda}A_{\mu\nu]}+6iW^0_{[\lambda}(2\bar{\psi}_\mu\gamma_{\nu]}\tau-\bar{\psi}_\mu\psi_{\nu]}\sigma)-6i\bar{\psi}_{[\lambda}\gamma_{\mu\nu]}(\alpha\tau+\Omega^0\sigma)\\&+6i\bar{\psi}_{[\lambda}\gamma_\mu\psi_{\nu]}\alpha\sigma-\ \tfrac{1}{2}\epsilon_{\lambda\mu\nu\rho\sigma}\Big(4v^{\rho\sigma}\alpha\sigma+\hat{F}^{\rho\sigma}(W^0)\sigma+2i\bar{\Omega}^0\gamma^{\rho\sigma}\tau\Big)\,.\end{aligned} \tag{4·14}$$

This is solved in the Appendix to yield

$$\hat{B}_{ab}=\frac{\alpha}{\alpha^2-(W^0)^2}\left(\frac{1}{3!}\epsilon_{abcde}\mathcal{H}^{cde}-\frac{1}{\alpha}W^{0c}\mathcal{H}_{abc}+\frac{2}{3!\alpha^2}W^0_{[a}\epsilon_{b]cdef}W^{0c}\mathcal{H}^{def}\right). \tag{4·15}$$

The transformation law of $A_{\mu\nu}$ can be read from this covariant field strength (4·12) as

$$\begin{aligned}\delta A_{\mu\nu}={}&2i\bar{\varepsilon}\gamma_{\mu\nu}(\alpha\tau+\Omega^0\sigma)-4i\bar{\varepsilon}\gamma_{[\mu}\psi_{\nu]}\alpha\sigma+W^0_{[\mu}(4i\bar{\varepsilon}\gamma_{\nu]}\tau-4i\bar{\varepsilon}\psi_{\nu]}\sigma)\\&+2\partial_{[\mu}\Lambda^B_{\nu]}+\Lambda^0B_{\mu\nu},\end{aligned} \tag{4·16}$$

where Λ^0 is the parameter of the central charge transformation $\boldsymbol{Z}$. Thus, the independent components of the tensor gauge multiplet are

$$A=\Big(\sigma,\ \tau^i,\ A_{\mu\nu},\ Z\sigma\Big), \tag{4·17}$$

and their properties are listed in Table I.

We add here the transformation law of the field strength $\hat{F}_{\mu\nu\rho}(A)$ and its Bianchi identity (equivalent to $E_a(V^0T)=0$):

$$\begin{aligned}\delta\hat{F}_{abc}(A)={}&6i\bar{\varepsilon}\gamma_{[ab}\hat{\mathcal{D}}_{c]}(\alpha\tau+\Omega^0\sigma)+6i\bar{\varepsilon}\gamma_{[a}\hat{R}_{bc]}(Q)\alpha\sigma\\&+3i\bar{\varepsilon}\gamma_{de[a}\gamma_{bc]}(\alpha\tau+\Omega^0\sigma)v^{de}+6i\bar{\varepsilon}\gamma_{[a}(\hat{F}_{bc]}(W^0)\tau+\Omega^0\hat{B}_{bc]})\\&+6i\bar{\eta}\gamma_{abc}(\alpha\tau+\Omega^0\sigma)+3\Lambda^0(\hat{\mathcal{D}}_{[a}\hat{B}_{bc]}+2i\bar{\tau}\gamma_{[a}\hat{R}_{bc]}(Q))\,,\\0={}&\hat{\mathcal{D}}_{[a}\hat{F}_{bcd]}(A)+\ \tfrac{3}{4}\hat{B}^A_{[ab}\hat{B}^B_{cd]}f_{AB}\ .\end{aligned} \tag{4·18}$$

§5. An invariant action for large tensor multiplets

We first discuss the explicit form of the Yang-Mills G'-gauge transformation of the large tensor multiplets T^α. We can interpret the vector multiplets as special tensor multiplets that are $\boldsymbol{Z}$-inert. This fact enabled us to treat the tensor and vector multiplets collectively as $\mathcal{T}^A \equiv (V^I, T^\alpha)$ in the embedding formula $L(f(\mathcal{T}))$ given in Eq. (3·6). This suggests that the tensor and vector multiplets transform collectively also under the G'-gauge transformation; that is, $\delta_{G'}(\Lambda)\mathcal{T}^A = \Lambda^J(t_J)^A{}_B\mathcal{T}^B$, or more explicitly,

$$\delta_{G'}(\Lambda)\begin{pmatrix} V^I \\ T^\alpha \end{pmatrix} = \sum_{J\geq 1}\Lambda^J\begin{pmatrix} (t_J)^I{}_K & 0 \\ (t_J)^\alpha{}_K & (t_J)^\alpha{}_\beta \end{pmatrix}\begin{pmatrix} V^K \\ T^\beta \end{pmatrix}. \tag{5·1}$$

Here, we have used 0 for the top-right entry of the generator matrix $(t_J)^A{}_B$, because the tensor multiplets are $\boldsymbol{Z}$-variant and therefore cannot appear in the G'-gauge transformation of $\boldsymbol{Z}$-inert vector multiplets V^I. The other off-diagonal entry, $(t_J)^\alpha{}_K$, can be non-vanishing. Specifically, the G'-gauge transformation of the scalar components σ^α of the tensor multiplets, $\delta_{G'}(\Lambda)\sigma^\alpha$, can generally contain the scalar fields M^I of a vector multiplet V^I as well, as pointed out very recently by Bergshoeff et al.[9] Nevertheless, as we see shortly, this mixing of vector multiplets in the G'-transformation of the tensor multiplets is only apparent, and it must vanish in the field basis in which the kinetic terms of tensor and vector multiplets do not mix with each other: $(t_I)^\alpha{}_K = 0$ for $I \geq 1$. We see below, however, that the introduction of a mass term effectively induces an off-diagonal entry $(t_0)^\alpha{}_K$ only for the central charge transformation, $I = 0$.

A general invariant action for the tensor multiplets is obtained by using the VL action formula[12] as follows:

$$\mathcal{L}_T = \mathcal{L}_{\rm VL}\left(V^0 L(h(\mathcal{T}))\right), \qquad h(T) = -\mathcal{T}^A(\boldsymbol{Z}\mathcal{T}^B)d_{AB} + \mathcal{T}^A\mathcal{T}^B\eta_{AB}. \tag{5·2}$$

Here we have put a negative sign in front of the kinetic term for later convenience. Because the linear multiplet $L(h(\mathcal{T}))$ has a non-zero central charge, V^0 in Eq. (5·2) must be the central charge vector multiplet in order for the action to be $\boldsymbol{Z}$-invariant. Because $\boldsymbol{Z}V^I = 0$, we can take $d_{AJ} = (d_{\alpha J}, d_{IJ}) = 0$. We can also assume that the submatrix $d_{\alpha\beta}$ is invertible, because $T^\alpha(\boldsymbol{Z}T^\beta)d_{\alpha\beta}$ gives the kinetic term of the tensor multiplets, as we see below. Then, we can redefine the tensor multiplets as $T^\alpha \to T^\alpha - V^I d_{I\beta}(d^{-1})^{\beta\alpha}$ without changing the superconformal transformation rule (3·3) to cancel the off-diagonal part of the kinetic term, $V^I(ZT)^\alpha d_{I\alpha}$. Further, the contribution of $V^IV^J\eta_{IJ}$ in $\mathcal{T}^A\mathcal{T}^B\eta_{AB}$ can be absorbed into the kinetic term of the vector multiplets, $\mathcal{L}_{\rm VL}\left(V^IL(V^JV^K)\right)c_{IJK}$, which we also discuss later. The function $h(\mathcal{T})$ can, therefore, generally be assumed to be

$$h(\mathcal{T}) = -T^\alpha(\boldsymbol{Z}T)^\beta d_{\alpha\beta} + T^\alpha T^\beta\eta_{\alpha\beta} + 2T^\alpha V^I\eta_{\alpha I}. \tag{5·3}$$

We can also assume without loss of generality that the metric tensor $d_{\alpha\beta}$ is anti-symmetric, $d_{\alpha\beta} = -d_{\beta\alpha}$ (and $\eta_{\alpha\beta}$ is symmetric, $\eta_{\alpha\beta} = \eta_{\beta\alpha}$). This is because the symmetric part $d^S_{\alpha\beta}$ of $d_{\alpha\beta}$, if it exists, yields a total central-charge transformed linear multiplet $\boldsymbol{Z}L(T^\alpha T^\beta)d^S_{\alpha\beta}$, but the action of the form $\mathcal{L}_{\rm VL}(V_0\boldsymbol{Z}L)$ is seen to vanish up to total derivative terms.

We examine the invariance of the action (5·2) with $h(\mathcal{T})$ in Eq. (5·3) under the G'-transformation (5·1). It is easily seen that $h(\mathcal{T})$ itself must be G'-invariant and that the kinetic term part $T^\alpha(\boldsymbol{Z}T)^\beta d_{\alpha\beta}$ and the mass term part $T^\alpha T^\beta\eta_{\alpha\beta}+2T^\alpha V^I\eta_{\alpha I}$ must be separately invariant. The G'-transformation of the former gives

$$\delta_{G'}(\Lambda)\Big(T^\alpha(\boldsymbol{Z}T)^\beta d_{\alpha\beta}\Big) = \Lambda^J(t_J)^\alpha{}_K V^K(\boldsymbol{Z}T)^\beta d_{\alpha\beta} + \Lambda^J\Big((t_J)^\gamma{}_\alpha d_{\gamma\beta}+d_{\alpha\gamma}(t_J)^\gamma{}_\beta\Big)\Big(T^\alpha(\boldsymbol{Z}T)^\beta\Big), \qquad (5·4)$$

and therefore it is necessary that the antisymmetric tensor $d_{\alpha\beta}$ is G' invariant,

$$(t_J)^\gamma{}_\alpha d_{\gamma\beta}+d_{\alpha\gamma}(t_J)^\gamma{}_\beta = 0, \qquad (J=1,2,\cdots) \qquad (5·5)$$

and the off-diagonal entry $(t_J)^\alpha{}_K$ vanishes,

$$(t_J)^\alpha{}_K = 0, \qquad (J=1,2,\cdots) \qquad (5·6)$$

as stated above. Similarly, the invariance of the mass term $T^\alpha T^\beta\eta_{\alpha\beta}+2T^\alpha V^I\eta_{\alpha I}$ requires the G'-invariance of the symmetric tensors $\eta_{\alpha\beta}$ and $\eta_{\alpha I}$:

$$(t_J)^\gamma{}_\alpha\eta_{\gamma\beta}+\eta_{\alpha\gamma}(t_J)^\gamma{}_\beta = 0, \qquad (t_J)^\gamma{}_\alpha\eta_{\gamma I}+\eta_{\alpha K}(t_J)^K{}_I = 0. \qquad (J=1,2,\cdots) \quad (5·7)$$

We now wish to obtain an explicit component expression of the action (5·2) with $h(\mathcal{T})$ in Eq. (5·3). For that purpose, we first compute the expression for the following simpler action without mass terms:

$$\mathcal{L}_T = -\mathcal{L}_{\rm VL}\Big(V^0L(T^\alpha(\boldsymbol{Z}T)^\beta)\Big)\,d_{\alpha\beta}. \qquad (5·8)$$

As we show below, the expression for the general action with mass terms can easily be obtained from the result in this case.

The component expression of the action (5·8) is computed in the following way. The constraints (3·9) give complicated expressions for the central-charge transformed quantities $\boldsymbol{Z}\hat{B}^\alpha_{ab}$, $\boldsymbol{Z}X^{\alpha ij}$, $\boldsymbol{Z}(Z\tau^{\alpha i})$, $\boldsymbol{Z}(Z^2\sigma^\alpha)$, which should be expressed in terms of the independent fields σ^α, $\tau^{\alpha i}$, $\hat{B}^\alpha_{ab}$, $X^{\alpha ij}$, $Z\tau^{\alpha i}$, $Z\sigma^\alpha$ and $Z^2\sigma^\alpha$. However, fortunately, this can be done relatively easily as follows. Because

$$V_*T^\alpha = V^0\boldsymbol{Z}T^\alpha + gV^\alpha{}_\beta T^\beta \qquad \text{with} \quad V^\alpha{}_\beta = \sum_{I\geq 1}V^I(t_I)^\alpha{}_\beta, \qquad (5·9)$$

the constraints (3·9), $L(V_*T^\alpha)=0$, can be rewritten in the form

$$L(V^0\boldsymbol{Z}T^\alpha) = -L(gV^\alpha{}_\beta T^\beta). \qquad (5·10)$$

If we can use this relation, the unwanted central-charge transformed quantities $\boldsymbol{Z}\hat{B}^\alpha_{ab}$, $\boldsymbol{Z}X^{\alpha ij}$, $\boldsymbol{Z}(Z\tau^{\alpha i})$, $\boldsymbol{Z}(Z^2\sigma^\alpha)$, which are contained in the LHS, can immediately be rewritten in terms of the independent variables. In order to utilize this relation, we first recall the following facts. First, $\mathcal{L}_{\rm VL}(V^1\,L(V^2\,V^3))$ is trilinear in the three vector multiplets V^1, V^2 and V^3, and it is completely symmetric under their interchange *if* they are all G-neutral (i.e., Abelian); that is, we have the identity

$$\mathcal{L}_{\rm VL}\left(V^1\,L(V^2\,V^3)\right) = \mathcal{L}_{\rm VL}\left(V^2\,L(V^1\,V^3)\right) + \left[\mathcal{L}_{\rm VL}\left(V^1\,L(V^2\,V^3)\right) - \mathcal{L}_{\rm VL}\left(V^2\,L(V^1\,V^3)\right)\right]_{*\text{-terms}}, \tag{5·11}$$

where '$*$-terms' indicates all the G-transformation terms containing the $*$-symbol that are absent when the vector multiplets V are Abelian. When we wish to generalize this identity to cases including tensor multiplets, we must *define* the quantity $\mathcal{L}_{\rm VL}(T\,L)$ for the tensor multiplet T, because the VL action formula $\mathcal{L}_{\rm VL}(V\,L)$ explicitly contains the vector component W_μ of V, to which the tensor multiplet T has no counterpart. However, we recall that the VL action formula can be rewritten into a form in which the vector component W_μ appears only in the field strength $F_{\mu\nu}(W)$ *if* V and L are both G-neutral. Thus, as a definition of the quantity '$\mathcal{L}_{\rm VL}(T\,L)$', we introduce the following function $\mathcal{L}_{\rm TL}(T^\alpha L(V^I T^\beta))$, which reduces to the VL invariant action when the multiplets T and V^I are all Abelian vector multiplets. Writing $f(\mathcal{T}) = V^I T^\beta$, we have

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm TL}\left(T^\alpha L(f(\mathcal{T}))\right) &\equiv X^{\alpha ij}L_{ij}(f(\mathcal{T})) + 2i\bar\tau^\alpha\Big(\varphi(f(\mathcal{T})) + L(f(\mathcal{T}))\gamma\cdot\psi\Big) \\
&\quad + \tfrac{1}{2}\sigma^\alpha\left(N(f(\mathcal{T})) - 2i\bar\psi\cdot\gamma\varphi(f(\mathcal{T})) - 2i\bar\psi^i_a\gamma^{ab}\psi^j_b L_{ij}(f(\mathcal{T}))\right) + \tfrac{1}{4}B^\alpha_{\mu\nu}E^{\mu\nu}(f(\mathcal{T})), \\
E^{\mu\nu}(V^I T^\beta) &\equiv -\Big((4v^{\mu\nu} + i\bar\psi_\rho\gamma^{\mu\nu\rho\sigma}\psi_\sigma)M^I\sigma^\beta + (F^{\mu\nu I}(W)\sigma^\beta + M^I B^{\beta\mu\nu}) \\
&\quad + 2i\bar\Omega^I\gamma^{\mu\nu}\tau^\beta - 2i\bar\psi_\lambda\gamma^{\mu\nu\lambda}(\Omega^I\sigma^\beta + M^I\tau^\beta) + \tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho\sigma}W^I_\lambda B^\beta_{\rho\sigma}\Big),
\end{aligned} \tag{5·12}$$

where $B^\alpha_{\mu\nu}$ (without a hat) was introduced in Eq. (3·10). We should, however, note a misleading point of our notation in Eq. (5·12): The function $\mathcal{L}_{\rm TL}(T^\alpha L(V^I T^\beta))$ depends directly on V^I and T^β through $E^{\mu\nu}$, $(V^I T^\beta)$ which is not actually a component of the linear multiplet $L(V^I T^\beta)$ unless V^I and T^β are both Abelian vector multiplets. Therefore, for instance, although we have the equation $L(V^0\boldsymbol{Z}T^\alpha) + L(gV^\alpha{}_\beta T^\beta) = 0$ in Eq. (5·10), we have a non-vanishing difference,

$$\begin{aligned}
&\mathcal{L}_{\rm TL}\left(T^\alpha L(V^0\boldsymbol{Z}T^\beta)\right) + \mathcal{L}_{\rm TL}\left(T^\alpha L(gV^\beta{}_\gamma T^\gamma)\right) \\
&\quad = \tfrac{1}{4}B^\alpha_{\mu\nu}\left(E^{\mu\nu}(V^0\boldsymbol{Z}T^\beta)) + E^{\mu\nu}(gV^\beta{}_\gamma T^\gamma)\right) = -\tfrac{1}{8}\epsilon^{\mu\nu\lambda\rho\sigma}B^\alpha_{\mu\nu}\partial_\lambda B^\beta_{\rho\sigma}.
\end{aligned} \tag{5·13}$$

In deriving this, we need to use the previous identity (3·10). Applying the equations (5·13), (5·11) and (5·12) and the constraint relation (5·10), we can rewrite the action (5·8) as

$$\begin{aligned}
\mathcal{L}_T &= -\mathcal{L}_{\rm VL}\left(V^0 L(T^\alpha\boldsymbol{Z}T^\beta)\right)d_{\alpha\beta} = -\mathcal{L}_{\rm TL}\left(T^\alpha L(V^0\boldsymbol{Z}T^\beta)\right)d_{\alpha\beta} + \Delta\mathcal{L}'(*\text{-terms}) \\
&= \mathcal{L}_{\rm TL}\left(T^\alpha L(gV^\beta{}_\gamma T^\gamma)\right)d_{\alpha\beta}\Big|_{*\text{-free}} + \tfrac{1}{8}\epsilon^{\mu\nu\lambda\rho\sigma}B^\alpha_{\mu\nu}\partial_\lambda B^\beta_{\rho\sigma}d_{\alpha\beta} + \Delta\mathcal{L}(*\text{-terms}), \\
\Delta\mathcal{L}'(*\text{-terms}) &\equiv d_{\alpha\beta}\Big\{-\mathcal{L}_{\rm VL}\left(V^0 L(T^\alpha\boldsymbol{Z}T^\beta)\right)\Big|_{*\text{-terms}}
\end{aligned}$$

$$+\mathcal{L}_{\rm TL}\left(T^\alpha L(V^0 \boldsymbol{Z} T^\beta)\right)\Big|_{*\text{-terms}}\Big\},$$

$$\begin{aligned}\varDelta\mathcal{L}(*\text{-terms}) &\equiv \varDelta\mathcal{L}'(*\text{-terms}) + \mathcal{L}_{\rm TL}\left(T^\alpha L(gV^\beta{}_\gamma T^\gamma)\right) d_{\alpha\beta}\Big|_{*\text{-terms}} \\ &= d_{\alpha\beta}\Big\{-\mathcal{L}_{\rm VL}\left(V^0 L(T^\alpha \boldsymbol{Z} T^\beta)\right)\Big|_{*\text{-terms}} + \mathcal{L}_{\rm TL}\left(T^\alpha L(V_* T^\beta)\right)\Big|_{**\text{-terms}}\Big\}, \quad (5{\cdot}14)\end{aligned}$$

where the subscript '$*$-free' indicates that the $*$-terms are all discarded.

Now, this equation is evaluated as follows. Because the $*$-free terms are the same as those in the Abelian vector multiplet case, the first term is

$$\mathcal{L}_{\rm TL}\left(T^\alpha L(gV^\beta{}_\gamma T^\gamma)\right) d_{\alpha\beta}\Big|_{*\text{-free}} = \mathcal{L}_{\rm VL}\left(gV^\beta{}_\gamma L(T^\alpha T^\gamma)\right) d_{\alpha\beta}\Big|_{*\text{-free}}. \qquad (5{\cdot}15)$$

The explicit expression of this term can be directly read from our previous result [11)] for the vector multiplet action $\mathcal{L}_{\rm VL}(V^I L(V^J V^K))c_{IJK}$, which is solely characterized by the 'norm function' $\mathcal{N}(M) = c_{IJK}M^I M^J M^K$. The $\varDelta\mathcal{L}(*\text{-terms})$ can be directly computed by picking out only the terms containing the $*$-symbol in $\mathcal{L}_{\rm VL}(V^0 L(T^\alpha \boldsymbol{Z} T^\beta))$ and $\mathcal{L}_{\rm TL}(T^\alpha L(V_* T^\beta))$ (in fact, appearing twice in the latter, since one $*$ is already contained in $V_* T^\beta$).[*)]

In this way, we find the following component expression for the tensor action (5·8):

$$\begin{aligned}
e^{-1}\mathcal{L}_T = {}& \tfrac{1}{2}\mathcal{N}\Big(\tfrac{1}{2}D - \tfrac{1}{2}i\bar\psi\cdot\gamma\chi - \tfrac{1}{4}R(M) - \tfrac{i}{2}\bar\psi_a\gamma^{abc}R_{bc}(Q) + 3v^2 \\
&\qquad + i\bar\psi_a\gamma^{abcd}\psi_b v_{cd} - 11i\bar\psi_a\psi_b v^{ab} - 6\bar\psi_a\psi_b\bar\psi^a\psi^b + 2\bar\psi_a\psi_b\bar\psi_c\gamma^{abcd}\psi_d\Big) \\
& + \tfrac{1}{2}\mathcal{N}_A\Big(i\bar\tau^A\chi + i\bar\tau^A\gamma\cdot R(Q) + 2v^{ab}B^A_{ab} \\
&\qquad - 2i\bar\tau^A\gamma^{abc}\psi_a v_{bc} + 6i\bar\tau^A\gamma_a\psi_b v^{ab} \\
&\qquad + \tfrac{i}{2}\bar\psi_a\gamma^{abcd}\psi_b B^A_{cd} - 2i\bar\psi^a\psi^b B^A_{ab} - 4\bar\psi_a\psi_b\bar\psi_c\gamma^{ab}\gamma^c\tau^A\Big) \\
& - \tfrac{1}{2}\mathcal{N}_{AB}\Big(-\tfrac{1}{4}B^A_{ab}B^{abB} + \tfrac{1}{2}\mathcal{D}'^a\sigma^A\mathcal{D}'_a\sigma^B + 2i\bar\tau^A\not{\mathcal{D}}'\tau^B + X^A_{ij}X^{Bij} \\
&\qquad - i\bar\tau^A\gamma\cdot v\tau^B + i\bar\psi_a(\gamma\cdot B^A - 2\not{\mathcal{D}}'\sigma^A)\gamma^a\tau^B \\
&\qquad - \bar\psi_a\psi_b\tau^A\gamma^{ab}\tau^B - 2\bar\psi_a\gamma_b\tau^A\bar\psi_c\gamma^{ab}\gamma^c\tau^B - 2\bar\psi_a\tau^A\bar\psi_b\gamma^a\gamma^b\tau^B\Big) \\
& + \mathcal{N}_{ABC}\Big(i\bar\tau^A X^B\tau^C + \tfrac{i}{4}\bar\tau^A\gamma\cdot B^B\tau^C \\
&\qquad - \tfrac{2}{3}\bar\psi_a\gamma_b\tau^A\bar\tau^B\gamma^{ab}\tau^C - \tfrac{2}{3}\bar\psi^i\cdot\gamma\tau^{Aj}\bar\tau^B_i\tau^C_j\Big) \\
& + \tfrac{1}{8}e^{-1}\epsilon^{\mu\nu\lambda\rho\sigma}B^\alpha_{\mu\nu}\mathcal{D}'_\lambda B^\beta_{\rho\sigma}d_{\alpha\beta} + \tfrac{1}{4}\mathcal{N}_{AB}(gM\sigma)^A(gM\sigma)^B - 2\mathcal{N}_A ig\bar\Omega\tau^A \\
& - \mathcal{N}_{AB}i\bar\tau^A gM\tau^B + \mathcal{N}_A i\bar\psi\cdot\gamma gM\tau^A + \alpha\left(\alpha^2 - (W_0)^2\right)(Z\sigma)^\alpha(Z^2\sigma)^\beta d_{\alpha\beta} \\
& + \tfrac{1}{2}\left(3\alpha^2 - (W_0)^2\right)(Z\sigma)^\alpha(gMZ\sigma)^\beta d_{\alpha\beta} \\
& + (Z\sigma)^\alpha\Big(\alpha W^a_0(\mathcal{D}'_a Z\sigma)^\beta + 2i\bar\Omega^0(3\alpha - \not{W}^0)(Z\tau)^\beta \\
&\qquad - 2\alpha i\bar\psi_a(\alpha + \not{W}_0)\gamma^a(Z\tau)^\beta\Big)d_{\alpha\beta}
\end{aligned}$$

[*)] To avoid possible confusion, we should note that these $**$-terms are those in the expression of $\mathcal{L}_{\rm TL}(T^\alpha L(V_* T^\beta))$ *before* the constraint $L(V_* T^\beta) = 0$ is applied.

$$+ 2\alpha i(Z\bar{\tau})^\alpha(\alpha - \not{W}_0)(Z\tau)^\beta d_{\alpha\beta}, \tag{5·16}$$

where the 'norm function' here is given by

$$\mathcal{N} \equiv -\sigma^\alpha d_{\alpha\beta}(gM)^\beta{}_\gamma\sigma^\gamma = -\sigma^\alpha d_{\alpha\beta}\sum_{I\geq 1} M^I(gt_I)^\beta{}_\gamma\sigma^\gamma, \tag{5·17}$$

and $\mathcal{N}_A = \partial\mathcal{N}(\sigma)/\partial\sigma^A$, $\mathcal{N}_{AB} = \partial^2\mathcal{N}(\sigma)/\partial\sigma^A\partial\sigma^B$, etc. The operator $\mathcal{D}_a$ is a 'homogeneous covariant derivative', which is covariant under all the homogeneous transformations, $\boldsymbol{M}_{ab}$, $\boldsymbol{D}$ and $\boldsymbol{U}_{ij}$, and the gauge transformations $G = G' \times U(1)_Z$, and $\mathcal{D}'_a$ is equivalent to $\mathcal{D}_a$ with $U(1)_Z$ covariantization omitted:

$$\begin{aligned}\mathcal{D}_\mu &= \partial_\mu - \tfrac{1}{2}\omega_\mu^{ab}\boldsymbol{M}_{ab} - b_\mu\boldsymbol{D} - \tfrac{1}{2}V_\mu^{ij}\boldsymbol{U}_{ij} - W_\mu^0\boldsymbol{Z} - \sum_{I\geq 1} W_\mu^I\boldsymbol{G}_I, \\ &= \mathcal{D}'_\mu - W_\mu^0\boldsymbol{Z}.\end{aligned} \tag{5·18}$$

The contributions from $\varDelta\mathcal{L}$($*$-terms) are the terms in the last four lines, starting with $\frac{1}{4}\mathcal{N}_{AB}(gM\sigma)^A(gM\sigma)^B$, aside from the G'-covariantization terms contained in $\mathcal{D}'_a$.

Now, let us include the mass terms and consider the general action,

$$\mathcal{L}_T = -\mathcal{L}_{\rm VL}\left(V^0L(T^\alpha d_{\alpha\beta}(\boldsymbol{Z}T)^\beta)\right) + \mathcal{L}_{\rm VL}\left(V^0L(T^\alpha\eta_{\alpha\beta}T^\beta + 2T^\alpha\eta_{\alpha I}V^I)\right). \tag{5·19}$$

Interestingly, the component expression for this case turns out to take the same form as the above Eq. (5·16), provided that the following replacements are made. First, the central charge transformation is replaced by the new one, $\tilde{\boldsymbol{Z}}$:*)

$$\begin{aligned}\boldsymbol{Z}T^\alpha \ \rightarrow\ \tilde{\boldsymbol{Z}}T^\alpha &\equiv \boldsymbol{Z}T^\alpha - (gt_0)^\alpha{}_\beta T^\beta - (gt_0)^\alpha{}_I V^I, \\ (gt_0)^\alpha{}_\beta &\equiv (d^{-1})^{\alpha\gamma}\eta_{\gamma\beta}, \qquad (gt_0)^\alpha{}_I \equiv (d^{-1})^{\alpha\beta}\eta_{\beta I}.\end{aligned} \tag{5·20}$$

Second, the norm function now reads

$$\begin{aligned}\mathcal{N} &\equiv -\sigma^\alpha d_{\alpha\beta}(gM)^\beta{}_\gamma\sigma^\gamma - 2\alpha\sigma^\alpha d_{\alpha\beta}(gt_0)^\beta{}_I M^I = \sigma^\alpha\sigma^\beta M^I c_{I\alpha\beta} - 2\alpha\sigma^\alpha M^I\eta_{\alpha I}\,, \\ c_{I\alpha\beta} &\equiv -d_{\alpha\gamma}(gt_I)^\gamma{}_\beta\,. \qquad (I = 0, 1, 2, \cdots)\end{aligned} \tag{5·21}$$

Note that, in contrast to the previous $\mathcal{N}$ in (5·17), summations over repeated I here and henceforth always include $I = 0$ with $M^0 = \alpha$ and $(gt_0)^\alpha{}_A$ defined in Eq. (5·20). Third, the primed homogeneous covariant derivative $\mathcal{D}'_\mu$ should now be understood to also contain covariantization with respect to the 'homogeneous part' $\boldsymbol{G}_0$ of the central charge transformation $\boldsymbol{Z}$:

$$\begin{aligned}\mathcal{D}'_\mu \equiv \mathcal{D}_\mu + W_\mu^0\tilde{\boldsymbol{Z}} &= \partial_\mu - \tfrac{1}{2}\omega_\mu^{ab}\boldsymbol{M}_{ab} - b_\mu\boldsymbol{D} - \tfrac{1}{2}V_\mu^{ij}\boldsymbol{U}_{ij} - W_\mu^0\boldsymbol{G}_0 - \sum_{I\geq 1} W_\mu^I\boldsymbol{G}_I, \\ \boldsymbol{G}_0 T^\alpha &\equiv (gt_0)^\alpha{}_\beta T^\beta + (gt_0)^\alpha{}_I V^I.\end{aligned} \tag{5·22}$$

*) Thanks to the G'-invariance of the tensors $\eta_{\alpha\beta}$ and $\eta_{\alpha I}$ in Eq. (5·7), this $\tilde{\boldsymbol{Z}}$ transformation commutes with the G' transformation, and therefore it is still natural to call it the 'central transformation'.

The group action implied in terms like $gM\sigma^\alpha$, $gM\tau^\alpha$, etc., is also understood to contain the 'off-diagonal' $(gt_0)^\alpha{}_I$ part:

$$gM\sigma^\alpha = (gM)^\alpha{}_\beta \sigma^\beta + \alpha (gt_0)^\alpha{}_I M^I. \tag{5·23}$$

Finally, the 'Chern-Simons' like term $(e^{-1}/8)\epsilon^{\mu\nu\lambda\rho\sigma} B^\alpha_{\mu\nu} \mathcal{D}'_\lambda B^\beta_{\rho\sigma} d_{\alpha\beta}$ is replaced by

$$\begin{aligned}&\tfrac{1}{8}e^{-1}\epsilon^{\mu\nu\lambda\rho\sigma} B^\alpha_{\mu\nu} \mathcal{D}''_\lambda B^\beta_{\rho\sigma} d_{\alpha\beta} \\ &\quad = \tfrac{1}{8}e^{-1}\epsilon^{\mu\nu\lambda\rho\sigma} B^\alpha_{\mu\nu} \left(\partial_\lambda B^\alpha_{\rho\sigma} - W^I_\lambda (gt_I)^\alpha{}_\beta B^\beta_{\rho\sigma} - 2W^0_\lambda (gt_0)^\alpha{}_I F_{\rho\sigma}{}^I(W)\right).\end{aligned} \tag{5·24}$$

Note the factor 2 multiplying the last covariantization term in $\mathcal{D}''_\lambda B^\beta_{\rho\sigma}$, which differs from the factor 1 in the above definitions of $\mathcal{D}'_\mu$ and the group action gM.

We now explain why these replacement rules appear. First, consider the case in which there exist no V-T mixing mass terms $2T^\alpha \eta_{\alpha I} V^I$. Then, the purely tensor mass terms can easily be absorbed into the tensor kinetic terms,

$$\mathcal{L}^{\rm pure}_T = -\mathcal{L}_{\rm VL}\left(V_0 L(T^\alpha d_{\alpha\beta}(\boldsymbol{Z}T)^\beta - T^\alpha \eta_{\alpha\beta} T^\beta)\right) = -\mathcal{L}_{\rm VL}\left(V_0 L(T^\alpha d_{\alpha\beta}(\tilde{\boldsymbol{Z}}T)^\beta)\right), \tag{5·25}$$

by using the redefined central charge transformation $\tilde{\boldsymbol{Z}}$ given above in Eq. (5·20) (but with $\eta_{\alpha I}$ set equal to 0 in this case). This is the same form as the previous tensor kinetic term, $\mathcal{L}_T$, in Eq. (5·8) with $\boldsymbol{Z}$ replaced by $\tilde{\boldsymbol{Z}}$. Note, however, that the central charge transformations $\boldsymbol{Z}$ contained in the $*$-operations of all the other equations, e.g., the constraint equation (3·9) and the embedding formula (3·6), remain the same as the original $\boldsymbol{Z}$. We need to rewrite them in terms of the new $\tilde{\boldsymbol{Z}}$. We have

$$\begin{aligned}&V_* T^\alpha = V^0(\boldsymbol{Z}T)^\alpha + \sum_{I\geq 1} V^I (gt_I)^\alpha{}_\beta T^\beta = V^0(\tilde{\boldsymbol{Z}}T)^\alpha + V^I (gt_I)^\alpha{}_\beta T^\beta \\ &\text{with} \quad V^I (gt_I)^\alpha{}_\beta T^\beta = \left(V^0 (gt_0)^\alpha{}_\beta + \sum_{I\geq 1} V^I (gt_I)^\alpha{}_\beta\right) T^\beta.\end{aligned} \tag{5·26}$$

It is thus seen that, after rewriting the central charge, the tensor multiplets T^α become $U(1)_Z$-charged in the usual sense, with generators $(t_0)^\alpha{}_\beta$. Therefore, the G'-transformation should now include $I=0$ everywhere, with $(gt_0)^\alpha{}_\beta = (d^{-1})^{\alpha\gamma}\eta_{\gamma\beta}$, and so we obtain the above replacement rules, Eqs. (5·20)–(5·24), in the case $(gt_0)^\alpha{}_I = 0$. Note that the central charge term $\tilde{\boldsymbol{Z}}T$ vanishes if equations of motion are used. We know that a very similar situation exists also for the mass term of the hypermultiplets. [11)]

Next, we consider the general action (5·19), also containing the V-T transition mass term $\mathcal{L}_{\rm VL}\left(V_0 L(2T^\alpha \eta_{\alpha I} V^I)\right)$. This case can in fact be reduced to the pure tensor mass case considered above as follows. Let us consider the pure tensor multiplet system, as in Eq. (5·25), in which the tensor multiplets consist of three categories: $T^A \equiv (T^I, T^{\bar{I}}, T^\alpha)$. Here, T^I is an adjoint representation, $T^{\bar{I}}$ is another adjoint tensor multiplet, and T^α is the rest, which can be of arbitrary representation. We chose invariant d and η tensors in the forms

$$d_{AB} = \begin{pmatrix} 0 & d_{I\bar{J}} & 0 \\ d_{\bar{I}J} = -d_{J\bar{I}} & 0 & 0 \\ 0 & 0 & d_{\alpha\beta} \end{pmatrix} = -d_{BA}, \qquad \eta_{AB} = \begin{pmatrix} 0 & 0 & \eta_{I\beta} \\ 0 & 0 & 0 \\ \eta_{\alpha J} & 0 & \eta_{\alpha\beta} \end{pmatrix} = \eta_{BA}. \tag{5·27}$$

Then the action (5·25) becomes

$$\begin{aligned} \mathcal{L}'_T = &-\mathcal{L}_{\rm VL}\left(V_0 L(T^\alpha d_{\alpha\beta}(\boldsymbol{Z}T)^\beta + T^I d_{I\bar{J}}(\boldsymbol{Z}T)^{\bar{J}} + T^{\bar{I}} d_{\bar{I}J}(\boldsymbol{Z}T)^J)\right) \\ &+ \mathcal{L}_{\rm VL}\left(V_0 L(T^\alpha \eta_{\alpha\beta} T^\beta + 2T^\alpha \eta_{\alpha I} T^I)\right). \end{aligned} \tag{5·28}$$

Note that we have included no mass terms containing $T^{\bar{I}}$. Then, this system is a pure tensor multiplet system in any case, and therefore the previous result for the component expression (5·16) of the action applies with replacement rules given in Eqs. (5·20)–(5·24), with $(\eta_{\alpha\beta}, \eta_{\alpha I}) \to (\eta_{AB}, 0)$ understood in this case. Now, we stipulate that the first adjoint tensor multiplet T^I be a vector multiplet V^I; i.e., We set $T^I = V^I$. This is allowed, because the vector multiplet is a special tensor multiplet (which is $\boldsymbol{Z}$-invariant), and all the manipulations in computing the above component expression remain valid also for vector multiplets. Then, the action (5·28) of this system clearly reduces to the desired general action (5·19), because $\boldsymbol{Z}V^I = 0$ and $\mathcal{L}_{\rm VL}\left(V_0 L(V^I d_{I\bar{J}}(\boldsymbol{Z}T)^{\bar{J}})\right) = \mathcal{L}_{\rm VL}\left(V_0 \boldsymbol{Z} L(V^I d_{I\bar{J}} T^{\bar{J}})\right)$ vanishes up to total derivatives. This also implies that the action becomes completely independent of the adjoint tensor multiplet $T^{\bar{I}}$.

We now apply the above replacement rules. The central charge transformation is replaced by $\tilde{\boldsymbol{Z}}T^A = \boldsymbol{Z}T^A - (gt_0)^A{}_B T^B$. Because we have

$$(gt_0)^A{}_B \equiv (d^{-1})^{AC}\eta_{CB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (d^{-1})^{\bar{I}K}\eta_{K\beta} \\ (d^{-1})^{\alpha\gamma}\eta_{\gamma J} & 0 & (d^{-1})^{\alpha\gamma}\eta_{\gamma\beta} \end{pmatrix}, \quad T^A = \begin{pmatrix} V^I \\ T^{\bar{I}} \\ T^\alpha \end{pmatrix}, \tag{5·29}$$

$\tilde{\boldsymbol{Z}}T^\alpha$ actually reproduces the rule (5·20), and also

$$\tilde{\boldsymbol{Z}}V^I = \boldsymbol{Z}V^I = 0, \qquad \tilde{\boldsymbol{Z}}T^{\bar{I}} = \boldsymbol{Z}T^{\bar{I}} - (d^{-1})^{\bar{I}K}\eta_{K\beta}T^\beta. \tag{5·30}$$

Accordingly, the primed derivative $\mathcal{D}'_\mu$ applied to T^α also reproduces the rule (5·22), and $\mathcal{D}'_\mu$ applied to V^I undergoes no change: $\mathcal{D}'_\mu V^I = \mathcal{D}_\mu V^I$. Applying the derivative $\mathcal{D}'_\mu$ to $T^{\bar{I}}$ yields

$$\mathcal{D}'_\mu T^{\bar{I}} = \mathcal{D}'^{\rm hom}_\mu T^{\bar{I}} - W^0_\mu (d^{-1})^{\bar{I}K}\eta_{K\beta}T^\beta, \tag{5·31}$$

where $\mathcal{D}'^{\rm hom}_\mu T^{\bar{I}}$ denotes the part homogeneous in $T^{\bar{I}}$. Although it is guaranteed that the component fields of $T^{\bar{I}}$ never appear in the action, the second term $-W^0_\mu (d^{-1})^{\bar{I}K}\eta_{K\beta}T^\beta$ may give a nonvanishing contribution. This in fact happens only in Chern-Simons-like terms, which now read

$$\begin{aligned} &\tfrac{1}{8}e^{-1}\epsilon^{\mu\nu\lambda\rho\sigma} B^A_{\mu\nu}\mathcal{D}'_\lambda B^B_{\rho\sigma} d_{AB} \\ &= \tfrac{1}{8}e^{-1}\epsilon^{\mu\nu\lambda\rho\sigma}\Big\{ B^\alpha_{\mu\nu}\mathcal{D}'_\lambda B^\beta_{\rho\sigma} d_{\alpha\beta} + F^I_{\mu\nu}(W)\mathcal{D}'_\lambda B^{\bar{J}}_{\rho\sigma} d_{I\bar{J}} + B^{\bar{J}}_{\mu\nu}\mathcal{D}'_\lambda F^I_{\rho\sigma}(W) d_{\bar{J}I}\Big\}. \end{aligned} \tag{5·32}$$

Note that in the second term, $\mathcal{D}'_\lambda B^{\bar{J}}_{\rho\sigma} = \mathcal{D}'^{\rm hom}_\lambda B^{\bar{J}}_{\rho\sigma} - W^0_\lambda (d^{-1})^{\bar{J}K}\eta_{K\alpha}B^\alpha_{\rho\sigma}$ by Eq. (5·31). The $\mathcal{D}'^{\rm hom}_\lambda B^{\bar{J}}_{\rho\sigma}$ part, after performing the partial integration, gives the same contribution as the third term, which vanishes by the Bianchi identity $\epsilon^{\mu\nu\lambda\rho\sigma}\mathcal{D}_\lambda F^I_{\rho\sigma}(W) d_{\bar{J}I} = 0$. The contribution of the remaining part, $-W^0_\lambda (d^{-1})^{\bar{J}K}\eta_{K\alpha}B^\alpha_{\rho\sigma}$, in $\mathcal{D}'_\lambda B^{\bar{J}}_{\rho\sigma}$ is seen to give

$$
\begin{aligned}
-\ \tfrac{1}{8}e^{-1}\epsilon^{\mu\nu\lambda\rho\sigma}\Big\{ F^I_{\mu\nu}(W) W^0_\lambda \eta_{I\alpha} B^\alpha_{\rho\sigma} &= B^\alpha_{\mu\nu} W^0_\lambda \eta_{\alpha I} F^I_{\rho\sigma}(W) \\
&= B^\alpha_{\mu\nu} W^0_\lambda d_{\alpha\beta}(gt_0)^\beta{}_I F^I_{\rho\sigma}(W)\Big\}. \qquad (5·33)
\end{aligned}
$$

This doubles the last covariantization term, $B^\alpha_{\mu\nu}\Big(-W^0_\lambda (gt_0)^\alpha{}_I F_{\rho\sigma}{}^I(W)\Big) d_{\alpha\beta}$, of the first term contribution in Eq. (5·32),

$$
\tfrac{1}{8}e^{-1}\epsilon^{\mu\nu\lambda\rho\sigma}B^\alpha_{\mu\nu}\Big\{\mathcal{D}'_\lambda B^\beta_{\rho\sigma} = \Big(\partial_\lambda B^\alpha_{\rho\sigma} - W^I_\lambda (gt_I)^\alpha{}_\beta B^\beta_{\rho\sigma} - W^0_\lambda (gt_0)^\alpha{}_I F_{\rho\sigma}{}^I(W)\Big)\Big\} d_{\alpha\beta}, \qquad (5·34)
$$

and actually reproduces the replacement rule in Eq. (5·24).

Finally, the norm function now is given by

$$
\begin{aligned}
\mathcal{N} \equiv -\sigma^A d_{AB}(gM)^B{}_C\sigma^C &= -\sigma^\alpha d_{\alpha\beta}\Big((gM)^\beta{}_\gamma\sigma^\gamma + \alpha(gt_0)^\beta{}_I M^I\Big) \\
&- M^I d_{I\bar{K}}\Big((gM)^{\bar{K}}{}_{\bar{J}}\sigma^{\bar{J}} + \alpha(gt_0)^{\bar{J}}{}_\alpha\sigma^\alpha\Big) - \sigma^{\bar{J}} d_{\bar{J}K}(gM)^K{}_I M^I. \qquad (5·35)
\end{aligned}
$$

However, because $M^I d_{I\bar{K}}(gM)^{\bar{K}}{}_{\bar{J}}\sigma^{\bar{J}} = \sigma^{\bar{J}} d_{\bar{J}K}(gM)^K{}_I M^I$, due to the G'-invariance of $M^I d_{I\bar{J}}\sigma^{\bar{J}}$ and the relation $(gM)^K{}_I M^I = [gM,\, M]^K = 0$, this reduces to

$$
\mathcal{N} = -\sigma^\alpha d_{\alpha\beta}(gM)^\beta{}_\gamma\sigma^\gamma - \alpha\sigma^\alpha d_{\alpha\beta}(gt_0)^\beta{}_I M^I - \alpha M^I d_{I\bar{K}}(gt_0)^{\bar{J}}{}_\alpha\sigma^\alpha. \qquad (5·36)
$$

If we further substitute the expressions for $(gt_0)^\beta{}_I$ and $(gt_0)^{\bar{J}}{}_\alpha$ given in Eq. (5·29) into Eq. (5·36), the last two terms on the RHS give $2\alpha\sigma^\alpha\eta_{\alpha I}M^I$, and so this $\mathcal{N}$ reproduces that given in (5·21). This completes the proof of the replacement rules given in Eqs. (5·20)–(5·24)

A few comments are in order concerning the result (5·16).

i) First, the action of the tensor multiplets actually gives a contribution to the scalar potential $V_{\rm scalar}$:

$$
V_{\rm scalar} = -\ \tfrac{1}{4}\mathcal{N}_{AB}(gM\sigma)^A(gM\sigma)^B, \qquad (5·37)
$$

Note that $gM\sigma^A$ vanishes for $A = I$, because $gM\sigma^I = [gM, M]^I = 0$. Since $gM\sigma^\alpha = M^I(gt_I)^\alpha{}_\beta\sigma^\beta + \alpha(gt_0)^\alpha{}_J M^J$, this potential contains terms quadratic, cubic and quartic in M^I ($I = 0, 1, 2, \cdots$) (recall that $M^0 = \alpha$). Only the first terms quadratic in the M (which are also quadratic in the σ) was found by Günaydin and Zagermann,[7] while the other terms exist only when the tensor-vector mixing $(gt_0)^\alpha{}_J$ is introduced. Note, however, that this mixing comes from the tensor-vector mixing *mass term.* The general tensor-vector mixing terms $(gt_I)^\alpha{}_J$ for $I \neq 0$ in the G' transformation, as was introduced by Bergshoeff et al.,[9] can be eliminated by the field redefinitions, as we saw above, and then play no role here.

Table II. Comparison of the multiplets. The numbers outside and inside the parentheses denote the off-shell and on-shell degrees of freedom, respectively, of each component field.

multiplets	large tensor $\boldsymbol{T}$	16+16	tensor gauge $\boldsymbol{A}$	8+8	vector $\boldsymbol{V}$	8+8
constraints	$\boldsymbol{L}(\boldsymbol{V}_*\boldsymbol{T})=0$		$\boldsymbol{L}(\boldsymbol{V}^0\boldsymbol{A})=0$		$\boldsymbol{Z}\boldsymbol{V}=0$	
components	σ	1(1)	σ	1(1)	M	1(1)
	τ^i	8(4)	τ^i	8(4)	Ω^i	8(4)
	$\hat{B}_{ab}$	10(3)	$A_{\mu\nu}$	6(3)	W_μ	4(3)
	X^{ij}	3(0)			Y^{ij}	3(0)
	$Z\sigma$	1(0)	$Z\sigma$	1(0)		
	$Z\tau^i$	8(0)				
	$Z^2\sigma$	1(0)				

ii) The action (5·16) does not contain the kinetic terms for the vector multiplets V^I, because we have inserted the central charge transformation operator $\boldsymbol{Z}$ in the initial Lagrangian $\mathcal{L}_{\rm VL}(V^0L(T^\alpha \boldsymbol{Z} T^\beta)d_{\alpha\beta})$. As done in Ref. 11), the general kinetic terms for the vector multiplets can be given by an action of the form*)

$$-\mathcal{L}_{\rm VL}(V^IL(V^JV^K))c_{IJK}. \tag{5·38}$$

If we add this action of vector kinetic terms to the above tensor action (5·8), the resultant component expression is still given by (5·16), provided that the 'norm function' $\mathcal{N}$ there is understood to be

$$\mathcal{N} \equiv c_{I\alpha\beta}M^I\sigma^\alpha\sigma^\beta + 2\eta_{\alpha I}\alpha M^I\sigma^\alpha + c_{IJK}M^IM^JM^K, \tag{5·39}$$

and the following Chern-Simons term is added to the previous Chern-Simons-like term (5·24):

$$\begin{aligned}\mathcal{L}_{\text{C-S}} = \ & \tfrac{1}{8}c_{IJK}\epsilon^{\lambda\mu\nu\rho\sigma}W^I_\lambda\Big(F^J_{\mu\nu}(W)F^K_{\rho\sigma}(W) + \ \tfrac{1}{2}g[W_\mu, W_\nu]^JF^K_{\rho\sigma}(W) \\ & \qquad + \ \tfrac{1}{10}g^2[W_\mu,W_\nu]^J[W_\rho,W_\sigma]^K\Big)\,. \end{aligned}\tag{5·40}$$

iii) As seen from the action (5·16), the tensor fields $B^\alpha_{\mu\nu}$ possess the first-order kinetic term (5·24), as well as the mass term, and therefore they are 'self-dual' tensor fields of the type (1·1), as discussed in Ref. 1). The number of the on-shell modes of $B^\alpha_{\mu\nu}$ for each α is ${}_{(5-1)}C_2/2 = 3$. We also see that $Z\sigma$, $Z^2\sigma$ and $Z\tau^i$ are all non-propagating auxiliary fields, so that the on-shell modes in the large tensor multiplet T^α for each α are $1(\sigma)+3(B_{\mu\nu})$ bosons plus $4(\tau^i)$ fermions. The on-shell and off-shell mode counting is summarized in Table II for the three multiplets: large tensor multiplet, tensor gauge multiplet and vector multiplet. Note also that the number of these large tensor multiplets T^α appearing in the action is always *even*, because the coefficient $d_{\alpha\beta}$ of the kinetic term must be antisymmetric under $\alpha \leftrightarrow \beta$. (The symmetric part of $d_{\alpha\beta}$, if any, could add only a total derivative term to the Lagrangian.)

*) Precisely speaking, the VL action formula $\mathcal{L}_{\rm VL}(V^IL(V^JV^K))$ applies only to the case in which the index I is that of an Abelian vector multiplet. Nevertheless, one can construct an invariant action corresponding to the form (5·38) as long as $c_{IJK}V^IV^JV^K$ is G'-invariant. (See Ref. 11) for details.)

§6. An invariant action for A and a duality relation

The tensor gauge (small tensor) multiplet is just a special tensor multiplet that satisfies the stronger constraint (4·1), so that we can apply to it the embedding into the linear multiplet formula (3·6) and then the VL action formula, to obtain an invariant action for a tensor gauge multiplet A:

$$\mathcal{L}_A = -\mathcal{L}_{\rm VL}(V^0 L(A\,A)). \tag{6·1}$$

When computing the explicit form of this action, we need to use the complicated component expression of the constraints (4·1). However, we can avoid this, as in the large tensor multiplet case of the previous section. Since, for any quantity X,

$$X = X\Big|_{*\text{-free}} + X\Big|_{*\text{-terms}} \tag{6·2}$$

holds trivially, we apply it to $X = -\mathcal{L}_{\rm VL}(V^0 L(A\,A)) + 2\mathcal{L}_{\rm TL}(AL(V^0 A))$ and use the identity

$$\mathcal{L}_{\rm VL}(V^0 L(A\,A))\Big|_{*\text{-free}} = \mathcal{L}_{\rm TL}(AL(V^0 A))\Big|_{*\text{-free}}, \tag{6·3}$$

as before. Then we have

$$\begin{aligned}
&\mathcal{L}_A = \mathcal{L}_{\rm VL}(V^0 L(A\,A))\Big|_{*\text{-free}} - 2\mathcal{L}_{\rm TL}(AL(V^0 A)) + \varDelta\mathcal{L}(*\text{-terms}),\\
&\varDelta\mathcal{L}(*\text{-terms}) \equiv [2\mathcal{L}_{\rm TL}(AL(V^0 A)) - \mathcal{L}_{\rm VL}(V^0 L(A\,A))]\Big|_{*\text{-terms}}.
\end{aligned} \tag{6·4}$$

The reason we have considered the combination $2\mathcal{L}_{\rm TL}(AL(V^0A)) - \mathcal{L}_{\rm VL}(V^0 L(A\,A))$ is that the $\boldsymbol{Z}(Z\sigma)$ and $\boldsymbol{Z}\tau$ terms are contained in $\mathcal{L}_{\rm VL}(V^0 L(A\,A))$ twice as many times as in $\mathcal{L}_{\rm TL}(AL(V^0 A))$.

To see the content of this action, let us examine only the bosonic part for simplicity. With the aid of Eq. (6·4), the bosonic part can be read as follows:

$$\begin{aligned}
e^{-1}\mathcal{L}_A\Big|_{\rm boson} = \alpha\left(\tfrac{1}{4}B_{ab}B^{ab} + \tfrac{1}{2}\mathcal{D}'_a\sigma\mathcal{D}'^a\sigma - \left(\frac{\sigma}{\alpha}\right)^2 Y_{0ij}Y_0^{ij}\right) + \tfrac{1}{2}\mathcal{D}'_a(\sigma^2)\,\mathcal{D}^a\alpha\\
+ (\tfrac{1}{8}R(M) - \tfrac{1}{4}D - \tfrac{3}{2}v^2)\alpha\sigma^2 - F_{ab}(W^0)v^{ab}\,\sigma^2\\
+ \tfrac{1}{2}\alpha(\alpha^2 - (W^0_b)^2)(Z\sigma)^2 + \tfrac{1}{8}e^{-1}\epsilon^{\lambda\mu\nu\rho\sigma}W^0_\lambda B_{\mu\nu}B_{\rho\sigma}.
\end{aligned} \tag{6·5}$$

This expression is not yet the final one, since $B_{\mu\nu}$ should be rewritten in terms of the tensor gauge field $A_{\mu\nu}$. As derived in the Appendix, the terms containing $B_{\mu\nu}$ are written as

$$\begin{aligned}
&\tfrac{1}{4}\alpha\hat{B}_{ab}\hat{B}^{ab} + \tfrac{1}{8}\epsilon^{abcde}W^0_a\hat{B}_{bc}\hat{B}_{de}\\
&= \frac{1}{4\alpha}\left\{\frac{1}{3}(\mathcal{H}_{abc})^2 + \frac{1}{\alpha^2-(W^0)^2}\Big[\left(\mathcal{H}_{abc}W^{0c}\right)^2 - \tfrac{1}{6}\epsilon^{abcde}\alpha\mathcal{H}_{abc}\mathcal{H}_{def}W^{0f}\Big]\right\},
\end{aligned} \tag{6·6}$$

where $\mathcal{H}_{abc}$ is the quantity introduced in Eq. (4·14), and the boson part is essentially the field strength of the tensor gauge field $A_{\mu\nu}$:

$$\mathcal{H}_{\lambda\mu\nu}\Big|_{\rm boson} \equiv 3\partial_{[\lambda}A_{\mu\nu]} - \tfrac{1}{2}\epsilon_{\lambda\mu\nu\rho\sigma}\left(4v^{\rho\sigma}\alpha\sigma + \hat{F}^{\rho\sigma}(W^0)\sigma\right). \tag{6·7}$$

Thus we see that $\mathcal{L}_A$ contains the kinetic terms of the tensor gauge field $A_{\mu\nu}$ as well as the scalar field σ correctly.

Finally, we show that this tensor gauge multiplet is dual to the vector multiplet. To show this, let us consider the following action for a tensor gauge multiplet A and a large tensor multiplet T which is G'-neutral:

$$\mathcal{L} = \mathcal{L}_{\mathrm{VL}}(V^0 L(T^2)) + 2\mathcal{L}_{\mathrm{VL}}(V^0 L(AT)). \tag{6·8}$$

If we first integrate out A, or equivalently use equations of motion resulting from $\delta S/\delta A = 0$, we find that the tensor multiplet T reduces to a vector multiplet V. This can be seen as follows. We attach the superscript T to the component fields of the large tensor multiplet as $T = (\sigma^T, \tau^T, B^T_{ab}, Z\sigma^T, \cdots)$ to distinguish them from those of the tensor gauge multiplet $A = (\sigma, \tau, A_{\mu\nu}, Z\sigma)$. If we concentrate on the bosonic parts, we have

$$\begin{aligned}\mathcal{L}_{\mathrm{VL}}(V^0 L(AT)) \sim &- \tfrac{1}{8}\epsilon^{\mu\nu\lambda\rho\sigma} B^T_{\mu\nu}\partial_\lambda A_{\rho\sigma} - \tfrac{1}{2}\alpha(\alpha^2 - (W^0)^2)(Z\sigma^T)(Z\sigma) \\ &+ \tfrac{1}{2}\alpha\left[(\alpha^2 - (W^0)^2)(Z^2\sigma^T) + (\text{terms containing } Z\sigma^T)\right]\sigma, \end{aligned}\tag{6·9}$$

so that the equations of motion $\delta S/\delta A = 0$ yield

$$\left(\frac{\delta S}{\delta(Z\sigma)},\ \frac{\delta S}{\delta\sigma},\ \frac{\delta S}{\delta A_{\mu\nu}}\right) = 0 \ \Rightarrow\ \left(Z\sigma^T,\ Z^2\sigma^T,\ \epsilon^{\mu\nu\lambda\rho\sigma}\partial_\lambda B^T_{\mu\nu}\right) = 0. \tag{6·10}$$

Thus, the tensor multiplet has vanishing central charge, and $B_{\mu\nu}$ satisfies the Bianchi identity, so that it can be expressed by a vector field. The fermionic part should also reduce to that of a vector multiplet by supersymmetry. If we substitute the solution $T = V$ back into the action (6·8), then, using the identity $\mathcal{L}_{\mathrm{VL}}(V^0 L(AV)) = 0$, we obtain

$$\mathcal{L} = \mathcal{L}_{\mathrm{VL}}(V^0 L(VV)), \tag{6·11}$$

which is the action for the vector multiplet V. [All the component fields of the central charge vector multiplet V^0, except the gravi-photon, are eliminated by the dilatation $\boldsymbol{D}$ and special supersymmetry $\boldsymbol{S}$ gauge fixing or, otherwise, by non-propagating auxiliary fields.]

It may be necessary to add an explanation of the identity $\mathcal{L}_{\mathrm{VL}}(V^0 L(AV)) = 0$. This follows from the more general identity

$$\mathcal{L}_{\mathrm{VL}}(V^0 L(V_1 T)) = \mathcal{L}_{\mathrm{VL}}(V_1 L(V^0 T)), \tag{6·12}$$

which holds for any G'-neutral (Abelian) vector multiplet V_1 and any tensor multiplet, a large one T or a small one A. Then, applying this, we have

$$\mathcal{L}_{\mathrm{VL}}(V^0 L(AV)) = \mathcal{L}_{\mathrm{VL}}(V L(V^0 A)) = 0, \tag{6·13}$$

using the constraint $L(V^0 A) = 0$.

If we instead integrate T out first in the initial action (6·8), then we obtain

$$\mathcal{L} = \mathcal{L}(V^0 L((T+A)^2)) - \mathcal{L}(V^0 L(AA)) \Rightarrow -\mathcal{L}(V^0 L(AA)), \tag{6·14}$$

the action for a tensor gauge multiplet. The first term, $\mathcal{L}(V^0L((T+A)^2))$, represents the non-propagating 'mass' terms for the tensor multiplet $T' \equiv T + A$ and can be eliminated.

This duality can be shown in the opposite way if we start from the action

$$\mathcal{L} = -\mathcal{L}(V^0L(T^2)) + 2\mathcal{L}(V^0L(VT)). \tag{6·15}$$

Then, integrating V out first yields the equation of motion $L(V^0T) = 0$ by (6·12), so that the tensor multiplet T reduces to a tensor gauge multiplet A. Substituting this solution $T = A$ back into the action yields the tensor gauge multiplet action $-\mathcal{L}(V^0L(AA))$, since $\mathcal{L}(V^0L(VA)) = 0$. Integrating out T first, on the other hand, gives the vector action $\mathcal{L}(V^0L(VV))$.

Acknowledgements

The authors would like to thank Sorin Cucu, Stefan Vandoren and Antoine Van Proeyen for valuable discussions. T. K. is supported in part by a Grant-in-Aid for Scientific Research (No. 13640279) from the Japan Society for the Promotion of Science. T. K. and K. O. are supported in part by a Grant-in-Aid for Scientific Research on Priority Areas (No. 12047214) and a Grant-in-Aid for Scientific Research (No. 01350), respectively, from the Ministry of Education, Culture, Sports, Science and Technology, Japan. The work of K. O. is supported in part by the Japan Society for the Promotion of Science under the Pre-doctoral Research Program.

Appendix A
—— *Solving Eq. (4·13)* ——

Let us solve an equation of the form (4·13):

$$3W_{[a}\hat{B}_{bc]} + \tfrac{1}{2}M\epsilon_{abcde}\hat{B}^{de} = \mathcal{H}_{abc}. \tag{A·1}$$

We define the following two operations $\mathcal{P}_{\|}$ and $\mathcal{R}_{\perp}$ on rank 2 antisymmetric tensors T_{ab}:

$$\begin{aligned} \mathcal{P}_{\|}T_{ab} &\equiv \tfrac{W_aW^c}{W^2}T_{cb} + T_{ac}\tfrac{W^cW_b}{W^2}, \\ \mathcal{R}_{\perp}T_{ab} &\equiv \tfrac{1}{2M}\epsilon_{abcde}W^cT^{de}. \end{aligned} \tag{A·2}$$

Then $\mathcal{P}_{\|}$ and $(M^2/W^aW_a)\mathcal{R}_{\perp}^2$ are 'longitudinal' and transverse projection operators, respectively:

$$\mathcal{P}_{\|}^2 = \mathcal{P}_{\|}, \qquad \lambda^{-1}\mathcal{R}_{\perp}^2 + \mathcal{P}_{\|} = 1, \qquad \mathcal{R}_{\perp}\mathcal{P}_{\|} = 0, \qquad \lambda \equiv \tfrac{W^2}{M^2}. \tag{A·3}$$

Taking the dual of the given equation (A·1), we rewrite it as

$$\tilde{\mathcal{H}}_{ab} = M\hat{B}_{ab} + \tfrac{1}{2}\epsilon_{abcde}W^c\hat{B}^{de} = M(1+\mathcal{R}_{\perp})\hat{B}_{ab}. \tag{A·4}$$

Then we can solve it as

$$\hat{B}_{ab} = \tfrac{1}{M}(1+\mathcal{R}_{\perp})^{-1}\tilde{\mathcal{H}}_{ab} = \tfrac{1}{M(1-\lambda)}(1-\mathcal{R}_{\perp}-\lambda\mathcal{P}_{\|})\tilde{\mathcal{H}}_{ab}, \tag{A·5}$$

yielding Eq. (4·15). Using this we have

$$\begin{aligned}
&\tfrac{1}{4}M\hat{B}_{ab}\hat{B}^{ab} + \tfrac{1}{8}\epsilon^{abcde}W_a\hat{B}_{bc}\hat{B}_{de} \\
&\quad = \tfrac{1}{4}M\hat{B}_{ab}(1+\mathcal{R}_\perp)\hat{B}^{ab} = \tfrac{1}{4M}\tilde{\mathcal{H}}_{ab}(1+\mathcal{R}_\perp)^{-1}\tilde{\mathcal{H}}^{ab} \\
&\quad = \tfrac{1}{4(M^2-W^2)}\tilde{\mathcal{H}}_{ab}(1-\mathcal{R}_\perp-\lambda\mathcal{P}_\parallel)\tilde{\mathcal{H}}^{ab},
\end{aligned} \tag{A·6}$$

which gives Eq. (6·6) in the text.

References

1) K. Pilch, P. K. Townsend and van Nieuwenhuizen, Phys. Lett. B **136** (1984), 38.
2) E. Cremmer, "Supergravities in 5 dimensions," in *Superspace and supergravity*, ed. S. Hawking and M. M. Roček (Cambridge University Press, 1980).
A. H. Chamseddine and H. Nicolai, Phys. Lett. B **96** (1980), 89.
3) J. Maldacena, Adv. Theor. Math. Phys. **2** (1998), 231, hep-th/9711200.
For a review, see O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, hep-th/9905111.
4) P. Hořava and E. Witten, Nucl. Phys. B **460** (1996), 506, hep-th/9510209; Nucl. Phys. B **475** (1996), 94, hep-th/9603142.
5) I. Antoniadis, Phys. Lett. B **246** (1990), 377.
J. D. Lykken, Phys. Rev. D **54** (1996), 3693.
N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B **429** (1998), 263.
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B **436** (1998), 257.
6) L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999), 3370, hep-ph/9905221; Phys. Rev. Lett. **83** (1999), 4690, hep-th/9906064.
7) M. Günaydin and M. Zagermann, Nucl. Phys. B **572** (2000), 131, hep-th/9912027.
8) M. Günaydin, G. Sierra and P. K. Townsend, Nucl. Phys. B **242** (1984), 244; Nucl. Phys. B **253** (1985), 573.
9) E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, R. Halbersma, S. Vandoren and A. Van Proeyen, hep-th/0205230.
10) T. Kugo and K. Ohashi, Prog. Theor. Phys. **104** (2000), 835, hep-ph/0006231.
11) T. Kugo and K. Ohashi, Prog. Theor. Phys. **105** (2001), 323, hep-ph/0010288.
12) T. Fujita and K. Ohashi, Prog. Theor. Phys. **106** (2001), 221, hep-th/0104130.
13) T. Fujita, T. Kugo and K. Ohashi, Prog. Theor. Phys. **106** (2001), 671, hep-th/0104130.
14) T. Kugo and K. Ohashi, Prog. Theor. Phys. **108** (2002), 203, hep-th/0203276.
15) E. Bergshoeff, S. Cucu, M. Derix, T. de Wit, R. Halbersma and A. Van Proeyen, J. High Energy Phys. **06** (2001), 051, hep-th/0104113.
16) E. Bergshoeff, E. Sezgin and A. Van Proeyen, Nucl. Phys. B **264** (1986), 653.