

## On Relationships among Chern-Simons Theory, BF Theory and Matrix Model

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(Received January 8, 2008)

Chern-Simons theory on a  $U(1)$  bundle over a Riemann surface  $\Sigma_g$  of genus  $g$  is dimensionally reduced to BF theory with a mass term, which is equivalent to the two-dimensional Yang-Mills on  $\Sigma_g$ . We show that the former is inversely obtained from the latter by the extended matrix T-duality developed in hep-th/0703021. For the case of  $g = 0$  (i.e.  $S^2$ ), the  $U(1)$  bundle represents the lens space  $S^3/\mathbb{Z}_p$ . We find that in this case both the Chern-Simons theory and the BF theory with the mass term are realized in a matrix model. We also construct Wilson loops in the matrix model that correspond to those in the Chern-Simons theory on  $S^3$ .

### §1. Introduction

Some matrix models have been proposed as nonperturbative formulation of superstring or M-theory.<sup>1)–3)</sup> Since the information of topology is relevant for compactification in string theory, it should be included in these matrix models. The topological field theories have been developed to efficiently describe the topological aspects of field theories. It is, therefore, worthwhile to investigate realization of the topological field theories in matrix models.

The large  $N$  reduction<sup>4)</sup> is the first example for realization of field theories in matrix models. It states that a large  $N$  planar gauge theory is equivalent to the matrix model that is its dimensional reduction to zero dimensions unless the  $U(1)^D$  symmetry is broken, where  $D$  denotes the dimensionality of the original gauge theory. However, the  $U(1)^D$  symmetry is in general spontaneously broken in the case of  $D > 2$ . There are two improved versions of the large  $N$  reduced model that preserve the  $U(1)^D$  symmetry. One is the quenched reduced model.<sup>5)–8)</sup> This shares the same idea with the T-duality for D-brane effective theories,<sup>9)</sup> which we call the matrix T-duality in this paper. The other is the twisted reduced model,<sup>10)</sup> which was later rediscovered in the context of the noncommutative field theories. The

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above developments are all concerning gauge theories on flat space-time.

In this paper, as typical examples of the topological field theories, we consider three-dimensional Chern-Simons (CS) theory on a  $U(1)$  bundle over a Riemann surface  $\Sigma_g$  of genus  $g$  and two-dimensional BF theory with a mass term on  $\Sigma_g$  that is obtained by a dimensional reduction of the CS theory on the total space to the base space.<sup>\*)</sup> The BF theory with the mass term is equivalent to the two-dimensional Yang-Mills on  $\Sigma_g$ . Hence, for our purpose, we need to extend the prescription of the large  $N$  reduction or the matrix T-duality to that on curved space.<sup>\*\*) In 23) and 24), the matrix T-duality was extended to the relationship between a Yang-Mills-type gauge theory on the total space of a  $U(1)$  bundle and the Yang-Mills-Higgs theory on its base space that is obtained by dimensionally reducing the original theory with respect to the fiber direction. Also, many works have been done on realization of the gauge theories on fuzzy sphere<sup>13)–16)</sup> by matrix models<sup>17)</sup> and the monopoles on fuzzy sphere.<sup>18)–22)</sup> It was explicitly shown in 23) that if an appropriate continuum limit is taken, the theory expanded around a multi-monopole vacuum of the Yang-Mills-Higgs theory on  $S^2$  is equivalent to the theory expanded around a vacuum with concentric fuzzy spheres of the matrix model that is obtained by dimensionally reducing the original theory on  $S^2$ .</sup>

We show that the CS theory on the  $U(1)$  bundle is inversely obtained from the BF theory with the mass term on the base space by the extended matrix T-duality developed in 24). For the case of  $g = 0$  (i.e.  $S^2$ ), the  $U(1)$  bundle represents the lens space  $S^3/\mathbb{Z}_p$ . We show that in this case both the CS theory and the BF theory with the mass term are realized in the matrix model that is obtained by dimensionally reducing the both theories to zero dimensions. We also construct Wilson loops in the matrix model that correspond to those in the CS theory on  $S^3$ . This is an extension of the work 25) to curved space.

Our study of the relationship between the CS theory and the BF theory with the mass term is motivated by the following observations on the relationship among the matrix models, topological string and lower dimensional topological field theory. The free energy of the hermitian (holomorphic) matrix model in the large  $N$  limit can be regarded as a topological B-model amplitude on a suitable conical Calabi-Yau singularity.<sup>26)–28)</sup> The topological string amplitude obtained from the matrix model can also be applied to derivation of effective superpotential of four-dimensional  $\mathcal{N} = 1$  supersymmetric gauge theory. It is also argued that the  $\mathcal{N} = 1$  (quiver) matrix models “deconstruct” the CS and BF-type theory.<sup>29)</sup> If we now demand a periodicity in the space of the matrix eigenvalues, the Vandermonde measure becomes the unitary one by taking account of mirror images of the eigenvalues (D-branes) in the covering space. This unitarization is reminiscent of the (matrix) T-duality we have mentioned. Indeed, it is shown that the matrix model with unitary measure is equivalent to the partition function of the CS theory, and then its free energy represents a topological A-model amplitude on a mirror Calabi-Yau geometry.<sup>30)</sup> (see also for reviews 31) and 32).) Thus, these topological string amplitudes emerging

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<sup>\*)</sup> This dimensional reduction was suggested in 11).

<sup>\*\*) An interesting approach to description of curved spacetime by matrices was proposed in 12).</sup>

from the matrix models are related with each other by T-duality and mirror symmetry in string theory. On the other hand, two-dimensional Yang-Mills theory and its generalization, which is obtained by a potential deformation of the BF theory (generalized two-dimensional Yang-Mills<sup>33)</sup>), can be interpreted as a matrix model restricted on discrete eigenvalues.<sup>34),35)</sup> In particular, the two-dimensional Yang-Mills theory on  $S^2$  possesses the same kind of the Vandermonde determinant, which is a square of the dimension of a unitary group representation, as the hermitian one matrix model. Using the similar idea of the T-duality, the Vandermonde measure (the dimension of the unitary representation) is replaced with an unitary one, which is a  $q$ -analog of the dimension of the unitary representation. This model is called  $q$ -deformed two-dimensional Yang-Mills theory.<sup>36)</sup> (See also 37) and 38).) The  $q$ -deformed two-dimensional Yang-Mills theory includes non-perturbative aspects of the topological A model string. Actually, we can obtain the partition function of the CS theory on  $S^3$  by extracting a “ground state” configuration from the  $q$ -deformed two-dimensional Yang-Mills theory. These facts on the relationship between various matrix models and lower dimensional solvable gauge theories strongly suggests that there exists T-dual like relationship between two-dimensional Yang-Mills theory on  $S^2$  and the CS theory on  $S^3$ .

This paper is organized as follows. In §2, we show the relationship between the CS theory on the  $U(1)$  bundle over  $\Sigma_g$  and the BF theory with the mass term on  $\Sigma_g$ . In §3, we show the relationship between BF theory on  $S^2$  and the matrix model. We also find how CS theory on  $S^3/\mathbb{Z}_p$  is realized in the matrix model. In §4, we construct the Wilson loops in the matrix model that correspond to those in CS theory on  $S^3$ . Section 5 is devoted to summary and discussion. In Appendix, some formulae concerning the spherical harmonics on  $S^3$ ,  $S^2$  and fuzzy sphere are gathered.

## §2. CS theory vs BF theory with the mass term

We consider a  $U(1)$  bundle  $\mathcal{M}$  over a closed Riemann surface  $\Sigma_g$  of genus  $g$ . The base space  $\Sigma_g$  has a covering  $\mathcal{S}$ , and the total space  $\mathcal{M}$  has a covering  $\{\pi^{-1}(U)|U \in \mathcal{S}\}$ .  $\pi^{-1}(U)$  is diffeomorphic to  $U \times S^1$  by a local trivialization. Thus it is parameterized by  $z^M = (x^\mu, y)$  ( $M = 1, 2, 3$ ,  $\mu = 1, 2$ ), where  $x^\mu$  parameterize the local patch  $U$  and  $y$  parameterizes the  $S^1$  with  $0 \leq y \leq 2\pi R$ . If there is overlap between the local patches  $U$  and  $U'$ , the relation between  $y$  and  $y'$  is given by the transition function  $e^{-\frac{i}{R}v}$  as  $y' = y - v(x)$ . We can assume that the total space is endowed with a metric which can be expressed locally on  $U$  as

$$ds^2 = G_{MN}dz^M dz^N = g_{\mu\nu}(x)dx^\mu dx^\nu + (dy + b_\mu(x)dx^\mu)^2, \quad (2.1)$$

where  $g_{\mu\nu}$  is the metric of the base space.  $b/R$  is the local connection that is viewed as the monopole on the base space and transformed as  $\frac{1}{R}b' = \frac{1}{R}(b + dv)$ . The curvature 2-form  $db/R$  belongs to  $H^2(\Sigma_g, \mathbb{Z})$ . From this fact we can assume that

$$\frac{1}{2\pi R}b_{\mu\nu} = -\frac{p}{V}\sqrt{g}\epsilon_{\mu\nu}, \quad (2.2)$$

where  $b_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$ ,  $p$  is an integer that represents the monopole degree of the  $U(1)$  bundle and  $V$  is the volume of the base space. Note that (2.2) is consistent with the equality

$$\frac{1}{2\pi R} \int_{\Sigma_g} db = -p. \quad (2.3)$$

In the following, all the expressions only make sense locally on  $U$  or  $\pi^{-1}(U)$  unless any remark is made.

We next consider a gauge theory on the above total space and make a dimensional reduction of the fiber direction to obtain a gauge theory on the base space. The gauge field of the fiber direction,  $A_y$ , is identified with the Higgs field on the base space. The horizontal-vertical decomposition given by the connection 1-form  $\omega = \frac{1}{R}(dy + b)$  tells us how to decompose the gauge fields on the total space into the gauge fields  $a_\mu$  and the Higgs field  $\phi$  on the base space:<sup>24),\*)</sup>

$$\begin{aligned} A_\mu &= a_\mu + b_\mu \phi, \\ A_y &= \phi, \end{aligned} \quad (2.4)$$

where both sides of these equations are assumed to be independent of  $y$ .

We start with the  $U(M)$  CS theory on the total space:

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} d^3z \epsilon^{MNP} \text{Tr} \left( A_M \partial_N A_P + \frac{2}{3} A_M A_N A_P \right). \quad (2.5)$$

By substituting (2.4) into (2.5), we make the dimensional reduction of the CS theory as follows:

$$\begin{aligned} S_{CS} &\rightarrow \frac{kR}{2} \int_{\Sigma_g} d^2x \epsilon^{\mu\nu} \text{Tr} (A_\mu \partial_\nu \phi + \phi \partial_\mu A_\nu + \phi [A_\mu, A_\nu]) \\ &= \frac{kR}{2} \int_{\Sigma_g} d^2x \epsilon^{\mu\nu} \text{Tr} \left( \phi f_{\mu\nu} + \frac{1}{2} b_{\mu\nu} \phi^2 \right) \\ &= \frac{kR}{2} \int_{\Sigma_g} d^2x \text{Tr} \left( \epsilon^{\mu\nu} \phi f_{\mu\nu} - \frac{2\pi R p}{V} \sqrt{g} \phi^2 \right), \end{aligned} \quad (2.6)$$

where  $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu]$  and we have used (2.2) in the last equality. The last line in (2.6) takes the form of the BF theory with the mass term: the first term is the action of the BF theory on  $\Sigma_g$ , where  $\phi$  corresponds to  $B$ , and the second term is a mass term for  $\phi$ . Note that  $k$  in (2.5) must be an integer while such restriction is no longer imposed on  $k$  in (2.6). If we integrate  $\phi$  out, we obtain the two-dimensional Yang-Mills on  $\Sigma_g$ ,

$$S_{YM} = \frac{1}{4g_{YM}^2} \int_{\Sigma_g} d^2x \sqrt{g} \text{Tr} (f_{\mu\nu} f^{\mu\nu}), \quad (2.7)$$

where the coupling constant is given by  $g_{YM}^2 = \frac{2\pi p}{kV}$ .

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<sup>\*)</sup> Equation (4.8) in 24) expressed in the local Lorentz frame is equivalent to (2.4).

The equations of motion for the BF theory with the mass term are

$$\begin{aligned} f_{\mu\nu} + b_{\mu\nu}\phi &= 0, \\ D_\mu\phi &= 0. \end{aligned} \quad (2.8)$$

These equations are solved in the gauge in which  $\phi$  is diagonal as follows:

$$\begin{aligned} \hat{a}_\mu &= -b_\mu\hat{\phi}, \\ \hat{\phi} &= -\frac{i}{pR}\text{diag}(\cdots, \underbrace{n_{s-1}, \cdots, n_{s-1}}_{N_{s-1}}, \underbrace{n_s, \cdots, n_s}_{N_s}, \underbrace{n_{s+1}, \cdots, n_{s+1}}_{N_{s+1}}, \cdots), \end{aligned} \quad (2.9)$$

where  $s$  label the (diagonal) blocks,  $n_s$  must be constant integers due to Dirac's quantization condition for the monopole charge and  $\sum_s N_s = M$ .

In the following, we show that we obtain the  $U(N)$  CS theory on the total space from the  $U(N \times \infty)$  BF theory with the mass term on the base space through the following procedure: we choose a certain background of the  $U(N \times \infty)$  BF theory with the mass term, expand the theory around the background and impose a periodicity condition. The background is given by (2.9) with  $s$  running from  $-\infty$  to  $\infty$ ,  $n_s = ps$  and  $N_s = N$ . We decompose the fields into the backgrounds and the fluctuations as

$$\begin{aligned} a_\mu &\rightarrow \hat{a}_\mu + a_\mu, \\ \phi &\rightarrow \hat{\phi} + \phi. \end{aligned} \quad (2.10)$$

We label the (off-diagonal) blocks by  $(s, t)$  and impose the periodicity (orbifolding) condition on the fluctuations:

$$\begin{aligned} a_\mu^{(s+1, t+1)} &= a_\mu^{(s, t)} \equiv a_\mu^{(s-t)}, \\ \phi^{(s+1, t+1)} &= \phi^{(s, t)} \equiv \phi^{(s-t)}. \end{aligned} \quad (2.11)$$

The fluctuations are gauge-transformed from  $U$  to  $U'$  as<sup>24)</sup>

$$\begin{aligned} a_\mu'^{(s-t)} &= e^{-\frac{i}{R}(s-t)v} a_\mu^{(s-t)}, \\ \phi'^{(s-t)} &= e^{-\frac{i}{R}(s-t)v} \phi^{(s-t)}. \end{aligned} \quad (2.12)$$

We make the Fourier transformation for the fluctuations on each patch to construct the gauge fields on the total space from the fields on the base space:

$$\begin{aligned} A_\mu(x, y) &= \sum_w (a_\mu^{(w)}(x) + b_\mu(x)\phi^{(w)}(x))e^{-i\frac{w}{R}y}, \\ A_y(x, y) &= \sum_w \phi^{(w)}(x)e^{-i\frac{w}{R}y}. \end{aligned} \quad (2.13)$$

We see from (2.12) that the left-hand sides in the above equations indeed transform from a patch to another patch as the gauge fields on the total space. We substitute

(2.10) into the  $U(N \times \infty)$  BF theory with the mass term and use (2.13):

$$\begin{aligned}
 S_{BF} &= \frac{kR}{2} \int_{\Sigma_g} d^2x \epsilon^{\mu\nu} \text{Tr} \left( \phi f_{\mu\nu} + \frac{1}{2} b_{\mu\nu} \phi^2 \right) \\
 &= \frac{kR}{2} \int_{\Sigma_g} d^2x \epsilon^{\mu\nu} \text{tr} \left[ \sum_{s,t} \left( (a_\mu + b_\mu \phi)^{(s,t)} \partial_\nu \phi^{(t,s)} + \phi^{(s,t)} \partial_\mu (a_\nu + b_\nu \phi)^{(t,s)} \right. \right. \\
 &\quad \left. \left. + i \frac{t-s}{R} (a_\mu + b_\mu \phi)^{(s,t)} (a_\nu + b_\nu \phi)^{(t,s)} \right) + 2 \sum_{s,t,u} \phi^{(s,t)} (a_\mu + b_\mu \phi)^{(t,u)} (a_\nu + b_\nu \phi)^{(u,s)} \right] \\
 &= \sum_s \frac{k}{4\pi} \int_{\mathcal{M}} d^3z \epsilon^{MNP} \text{tr} \left( A_M \partial_N A_P + \frac{2}{3} A_M A_N A_P \right), \tag{2.14}
 \end{aligned}$$

where we have ignored a constant term. By dividing an overall factor  $\sum_s$  in the last line to extract a single period, we obtain the CS theory on the total space. In the above procedure, we obtain the theory around the trivial vacuum of CS theory. When  $g \neq 0$  or  $p \neq 1$ , the CS theory on the total space has nontrivial vacua. We can obtain the theories around the nontrivial vacua of CS theory from BF theory on the base space by extending the above procedure in a straightforward way. (See 24.)

### §3. BF theory with the mass term vs matrix model

For the case of  $g = 0$ , the base space is  $S^2$  and the total space is the lens space  $S^3/\mathbb{Z}_p$ . In this section, we show that the BF theory with the mass term on  $S^2$  is realized in the matrix model that is its dimensional reduction to zero dimensions. Combining this result with the result in the previous section, we find that the CS theory on  $S^3/\mathbb{Z}_p$  is realized in the matrix model.

One needs two patches to describe the lens space  $S^3/\mathbb{Z}_p$ : the patch I is specified by  $0 \leq \theta < \pi$  and the patch II is specified by  $0 < \theta \leq \pi$ . We adopt the following metric for  $S^3/\mathbb{Z}_p$ :

$$ds^2 = \frac{1}{\mu^2} \left( d\theta^2 + \sin^2 \theta d\varphi^2 + \left( \frac{1}{p} d(\psi \pm \varphi) + (\cos \theta \mp 1) d\varphi \right)^2 \right), \tag{3.1}$$

where  $0 \leq \theta \leq \pi$ ,  $0 \leq \varphi \leq 2\pi$  and  $0 \leq \psi \leq 4\pi$ . The upper sign is taken in the patch I while the lower sign in the patch II. The radius of  $S^3/\mathbb{Z}_p$  is  $2/\mu$  and that of  $S^2$  is  $1/\mu$ .  $R$  is given by  $\frac{2}{p\mu}$  and  $b_\theta = 0$ ,  $b_\varphi = \frac{1}{\mu}(\cos \theta \mp 1)$ . Note that for  $p = 1$ , (3.1) takes the well-known form of the metric of  $S^3$ :

$$ds^2 = \frac{1}{\mu^2} (d\theta^2 + \sin^2 \theta d\varphi^2 + (d\psi + \cos \theta d\varphi)^2). \tag{3.2}$$

It is convenient to rewrite the BF theory with the mass term on  $S^2$  by using the three-dimensional flat-space notation. We define vector fields in terms of the gauge fields and the Higgs field on  $S^2$ :

$$\vec{y} = -i \left( \frac{1}{\mu} \phi \vec{e}_r + a_\theta \vec{e}_\varphi - \frac{1}{\sin \theta} a_\phi \vec{e}_\theta \right), \tag{3.3}$$

where  $\vec{e}_r = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$  and  $\vec{e}_\theta = \frac{\partial \vec{e}_r}{\partial \theta}$ ,  $\vec{e}_\varphi = \frac{1}{\sin \theta} \frac{\partial \vec{e}_r}{\partial \varphi}$ . We also introduce the angular momentum operator in three-dimensional flat space,

$$\vec{L}^{(0)} = -i\vec{e}_\phi \partial_\theta + i \frac{1}{\sin \theta} \vec{e}_\theta \partial_\phi. \quad (3.4)$$

The BF theory with the mass term on  $S^2$  takes the form

$$S_{BF} = \frac{2k}{p\mu} \int d\theta d\varphi \text{Tr} \left( -\frac{1}{2\mu} \sin \theta \phi^2 + \phi(\partial_\theta a_\varphi - \partial_\varphi a_\theta + [a_\theta, a_\varphi]) \right). \quad (3.5)$$

It is rewritten in terms of  $y_i$  and  $L_i^{(0)}$  as

$$S_{BF} = -\frac{1}{g_{BF}^2} \int \frac{d\Omega_2}{4\pi} \text{Tr} \left\{ y_i \left( \frac{1}{2} y_i + \frac{i}{2} \epsilon_{ijk} L_j^{(0)} y_k + \frac{i}{6} \epsilon_{ijk} [y_j, y_k] \right) \right\}, \quad (3.6)$$

where  $g_{BF}^2 = \frac{p}{8\pi k}$ . By making a replacement  $y_i \rightarrow \hat{y}_i + y_i$ , where  $\hat{y}_i$  denote the background for  $y_i$  corresponding to (2.9), we expand the theory around (2.9):

$$S_{BF} = -\frac{1}{g_{BF}^2} \int \frac{d\Omega_2}{4\pi} \sum_{s,t} \text{tr} \left\{ y_i^{(s,t)} \left( \frac{1}{2} y_i^{(t,s)} + \frac{i}{2} \epsilon_{ijk} L_j^{(q_{ts})} y_k^{(t,s)} + \frac{i}{6} \epsilon_{ijk} [y_j, y_k]^{(t,s)} \right) \right\}, \quad (3.7)$$

where  $q_{ts} = (n_t - n_s)/2$  and we have ignored a constant term.  $\vec{L}^{(q)}$  is the angular momentum operator in the presence of a monopole with the magnetic charge  $q$  at the origin, which takes the form<sup>39)</sup>

$$\vec{L}^{(q)} = \vec{L}^{(0)} - q \frac{\cos \theta \mp 1}{\sin \theta} \vec{e}_\theta - q \vec{e}_r, \quad (3.8)$$

where the upper sign is taken in the patch I and the lower sign in the patch II.

We obtain a matrix model by dropping the derivatives in (3.6):

$$S_{mm} = -\frac{1}{g_{mm}^2} \text{Tr} \left\{ X_i \left( \frac{1}{2} X_i + \frac{i}{6} \epsilon_{ijk} [X_j, X_k] \right) \right\}, \quad (3.9)$$

where  $X_i$  are  $M \times M$  hermitian matrices. Note that this matrix model is nothing but the Dijkgraaf-Vafa matrix model which reproduces the effective superpotential of  $\mathcal{N} = 1^*$  theory.<sup>(28),(40),(41)</sup> We will find a relationship between the BF theory with the mass term on  $S^2$  and the above matrix model. There the Higgs branch of  $\mathcal{N} = 1^*$  theory, which is classified by an irreducible decomposition of  $SU(2)$  representation, plays an important role in the equivalence.

A general solution to the equations of motion of the matrix model is given by

$$\hat{X}_i = L_i, \quad (3.10)$$

where  $L_i$  are the representation matrices of the  $SU(2)$  generators which are in general

reducible, and are decomposed into irreducible representations:

$$L_i = \left( \begin{array}{c} \cdots \\ \cdots \\ L_i^{[j_{s-1}]} \quad \cdots \quad N_{s-1} \\ \cdots \\ L_i^{[j_{s-1}]} \\ \cdots \\ L_i^{[j_s]} \quad \cdots \quad N_s \\ \cdots \\ L_i^{[j_s]} \\ \cdots \\ L_i^{[j_{s+1}]} \quad \cdots \quad N_{s+1} \\ \cdots \\ L_i^{[j_{s+1}]} \\ \cdots \end{array} \right), \quad (3.11)$$

where  $L_i^{[j]}$  are the spin  $j$  representation matrices of  $SU(2)$  and  $M = \sum_s N_s(2j_s + 1)$ . By making a replacement  $X_i \rightarrow \hat{X}_i + X_i$ , we expand the theory around (3.11):

$$S_{mm} = -\frac{1}{g_{mm}^2} \sum_{s,t} \text{tr} \left\{ X_i^{(s,t)} \left( \frac{1}{2} X_i^{(t,s)} + \frac{i}{2} \epsilon_{ijk} L_j \circ X_k^{(t,s)} + \frac{i}{6} \epsilon_{ijk} [X_j, X_k]^{(t,s)} \right) \right\}, \quad (3.12)$$

where  $L_i \circ$  is defined by

$$L_i \circ X_j^{(s,t)} = L_i^{[j_s]} X_j^{(s,t)} - X_j^{(s,t)} L_i^{[j_t]}, \quad (3.13)$$

and we have ignored a constant term.

In what follows, we show that the theory around (2.9) of the BF theory with the mass term is equivalent to the theory around (3.11) with  $2j_s + 1 = N_0 + n_s$  of the matrix model in the  $N_0 \rightarrow \infty$  limit. For this purpose, we make a harmonic expansion of (3.7) and (3.12). In (3.7), we expand the fields in terms of the monopole vector spherical harmonics  $\tilde{Y}_{Jmqi}^\rho$  defined in (A.10) and (A.18) as

$$y_i^{(s,t)} = \sum_{\rho=0,\pm 1} \sum_{\tilde{Q} \geq |q_{st}|} \sum_{m=-Q}^Q y_{Jm\rho}^{(s,t)} \tilde{Y}_{Jmqi}^\rho, \quad (3.14)$$

where  $Q = J + \frac{(1+\rho)\rho}{2}$  and  $\tilde{Q} = J - \frac{(1-\rho)\rho}{2}$ . Substituting (3.14) into (3.7) yields

$$S_{BF} = -\frac{1}{g_{BF}^2} \text{tr} \left( \frac{1}{2} \sum_{s,t} \rho(J+1) y_{Jm\rho}^{(s,t)\dagger} y_{Jm\rho}^{(s,t)} \right)$$



$$+ \frac{i}{3} \sum_{s,t,u} \mathcal{E}_{J_1 m_1 q_{st} \rho_1 J_2 m_2 q_{tu} \rho_2 J_3 m_3 q_{us} \rho_3} y_{J_1 m_1 \rho_1}^{(s,t)} y_{J_2 m_2 \rho_2}^{(t,u)} y_{J_3 m_3 \rho_3}^{(u,s)} \Bigg), \quad (3.15)$$

where (A.20), (A.22) and (A.23) were used. In (3.12), we expand the matrices in terms of fuzzy vector spherical harmonics  $\hat{Y}_{Jm(j_s j_t)i}^\rho$  as

$$X_i^{(s,t)} = \sum_{\rho=0,\pm 1} \sum_{\tilde{Q} \geq |j_s - j_t|}^{j_s + j_t} \sum_{m=-Q}^Q X_{Jm\rho}^{(s,t)} \otimes \hat{Y}_{Jm(j_s j_t)i}^\rho. \quad (3.16)$$

Since  $j_s + j_t = N_0 + \frac{n_s + n_t}{2} - 1$ ,  $N_0$  plays a role of the ultraviolet cutoff. Note also that  $j_s - j_t = (n_s - n_t)/2 = q_{st}$ . Substituting (3.16) into (3.12) yields

$$\begin{aligned} S_{mm} = & -\frac{N_0}{g_{mm}^2} \text{tr} \left( \frac{1}{2} \sum_{s,t} \rho(J+1) X_{Jm\rho}^{(s,t)\dagger} X_{Jm\rho}^{(s,t)} \right. \\ & \left. + \frac{i}{3} \sum_{s,t,u} \hat{\mathcal{E}}_{J_1 m_1 (j_s j_t) \rho_1 J_2 m_2 (j_t j_u) \rho_2 J_3 m_3 (j_u j_s) \rho_3} X_{J_1 m_1 \rho_1}^{(s,t)} X_{J_2 m_2 \rho_2}^{(t,u)} X_{J_3 m_3 \rho_3}^{(u,s)} \right), \end{aligned} \quad (3.17)$$

where (A.20), (A.22) and (A.23) were again used. In the  $N_0 \rightarrow \infty$  limit, the ultraviolet cutoff goes to infinity and  $\hat{\mathcal{E}}_{J_1 m_1 (j_s j_t) \rho_1 J_2 m_2 (j_t j_u) \rho_2 J_3 m_3 (j_u j_s) \rho_3}$  reduces to  $\mathcal{E}_{J_1 m_1 q_{st} \rho_1 J_2 m_2 q_{tu} \rho_2 J_3 m_3 q_{us} \rho_3}$  as shown in Appendix. Hence, in the limit in which  $N_0 \rightarrow \infty$  and  $g_{mm} \rightarrow \infty$  such that  $g_{mm}^2/N_0 = g_{BF}^2$ , (3.17) agrees with (3.15) under the identification  $X_{Jm\rho}^{(s,t)} = y_{Jm\rho}^{(s,t)}$ . We have proven our statement.

Combining the above result with the result in the previous section, we see that the  $U(N)$  CS theory around the trivial vacuum on  $S^3/\mathbb{Z}_p$  is realized in the matrix model as follow. In (3.11), we make  $s$  run from  $-\infty$  to  $\infty$  and put  $N_s = N$  and  $2j_s + 1 = N_0 + ps$ . We impose the periodicity condition:  $X_{Jm\rho}^{(s+1,t+1)} = X_{Jm\rho}^{(s,t)} = X_{Jm\rho}^{(s-t)}$ . We take the limit in which  $N_0 \rightarrow \infty$  and  $g_{mm} \rightarrow \infty$  such that  $g_{mm}^2/N_0 = \frac{p}{8\pi k}$  and divide the overall factor  $\sum_s$  out. Thus we obtain the CS theory around the trivial vacuum on  $S^3/\mathbb{Z}_p$ . Indeed, if we expand the gauge fields  $A_i$  expressed in the local Lorentz frame in terms of the vector spherical harmonics on  $S^3$  defined in (A.9) and (A.18) as

$$A_i = \sum_{Jm\tilde{m}} A_{Jm\tilde{m}} Y_{Jm\tilde{m}i}^\rho, \quad (3.18)$$

we obtain a harmonic expansion of the CS theory on  $S^3/\mathbb{Z}_p$  as follows:

$$\begin{aligned} S_{CS} = & \frac{8\pi k}{p\mu^3} \int \frac{d\Omega_3}{2\pi^2/p} \epsilon^{ijk} \text{Tr} \left( \frac{1}{2} A_i \nabla_j A_k + \frac{1}{3} A_i A_j A_k \right) \\ = & \frac{8\pi k}{p\mu^3} \text{Tr} \left( \frac{1}{2} \mu \rho(J+1) A_{Jm\tilde{m}\rho}^\dagger A_{Jm\tilde{m}\rho} \right. \\ & \left. + \frac{1}{3} \mathcal{E}_{J_1 m_1 \tilde{m}_1 \rho_1 J_2 m_2 \tilde{m}_2 \rho_2 J_3 m_3 \tilde{m}_3 \rho_3} A_{J_1 m_1 \tilde{m}_1 \rho_1} A_{J_2 m_2 \tilde{m}_2 \rho_2} A_{J_3 m_3 \tilde{m}_3 \rho_3} \right), \end{aligned} \quad (3.19)$$

where  $\tilde{m}$ ,  $\tilde{m}_1$ ,  $\tilde{m}_2$ ,  $\tilde{m}_3$  are restricted to  $\frac{p}{2}\mathbb{Z}$ . If we compare (3·19) with (3·17), we see that the relation between the modes is given by

$$A_{Jm\tilde{m}\rho} = i\mu X_{Jm\rho}^{(2\tilde{m}/p)}. \quad (3\cdot20)$$

It is now easy to obtain the CS theory around a nontrivial vacuum on  $S^3/\mathbb{Z}_p$  from the matrix model.

#### §4. Description of Wilson loop in CS on $S^3$ by matrices

Let  $z^M(\sigma)$  parameterize a closed loop on  $S^3$ , where  $M = \theta, \varphi, \psi$ ,  $0 \leq \sigma \leq 1$  and  $z^M(0) = z^M(1) = z^M$ . Then we consider a Wilson loop on  $S^3$ , which takes the form

$$\begin{aligned} W &= \text{Tr} \left[ P \exp \left( \int_0^1 A_M(z(\sigma)) \frac{dz^M(\sigma)}{d\sigma} d\sigma \right) \right] \\ &= \text{Tr} \left[ P \exp \left( \int_0^1 A_i(z(\sigma)) e_M^i(z(\sigma)) \frac{dz^M(\sigma)}{d\sigma} d\sigma \right) \right], \end{aligned} \quad (4\cdot1)$$

where  $e_M^i$  ( $i = 1, 2, 3$ ) is the right-invariant 1-form on  $S^3$  defined in appendix. We divide the loop into  $n$  small bits denoted by  $\Delta z_a^M$  ( $a = 1, \dots, n$ ) and take the  $n \rightarrow \infty$  limit. By definition the bits satisfy  $\sum_{a=1}^n \Delta z_a^M = 0$ . The Wilson loop (4·1) is rewritten as

$$\begin{aligned} W &= \text{Tr} \left[ \prod_{a=1}^n \left( 1 + A_{i_a}(z + \sum_{b=1}^{a-1} \Delta z_b) e_{M_a}^{i_a}(z + \sum_{c=1}^{a-1} \Delta z_c) \Delta z_a^{M_a} \right) \right] \\ &= \text{Tr} \left[ \left( 1 + A_{i_1}(z) e_{M_1}^{i_1}(z) \Delta z_1^{M_1} \right) \left( 1 + A_{i_2}(z + \Delta z_1) e_{M_2}^{i_2}(z + \Delta z_1) \Delta z_2^{M_2} \right) \cdots \right. \\ &\quad \cdots \left( 1 + A_{i_{n-1}}(z - \Delta z_{n-1} - \Delta z_n) e_{M_{n-1}}^{i_{n-1}}(z - \Delta z_{n-1} - \Delta z_n) \Delta z_{n-1}^{M_{n-1}} \right) \\ &\quad \left. \left( 1 + A_{i_n}(z - \Delta z_n) e_{M_n}^{i_n}(z - \Delta z_n) \Delta z_n^{M_n} \right) \right]. \end{aligned} \quad (4\cdot2)$$

We expand the gauge fields in terms of the scalar spherical harmonics on  $S^3$  defined in (A·9):<sup>\*</sup>

$$A_i(z) = \sum_{Jm\tilde{m}} Y_{Jm\tilde{m}}(z) A_{Jm\tilde{m}i}. \quad (4\cdot3)$$

Then the gauge fields at  $z + \Delta z_1$  are evaluated as

$$\begin{aligned} A_i(z + \Delta z_1) &= \sum_{Jm\tilde{m}} Y_{Jm\tilde{m}}(z + \Delta z_1) A_{Jm\tilde{m}i} \\ &= \sum_{Jm\tilde{m}} e^{\Delta z_1^{M_1} \partial_{M_1}} Y_{Jm\tilde{m}}(z) A_{Jm\tilde{m}i} \end{aligned}$$

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<sup>\*</sup> In this section, we expand the vector fields in terms of the scalar harmonics to make the discussion simpler.

$$\begin{aligned}
&= \sum_{Jm\tilde{m}} e^{i\mu\Delta z_1^{M_1} e_{M_1}^{i_1}(z)\mathcal{L}_{i_1}} Y_{Jm\tilde{m}}(z) A_{Jm\tilde{m}i} \\
&= \sum_{Jmm_1\tilde{m}} Y_{Jm_1\tilde{m}}(z) \langle Jm_1 | e^{i\mu\Delta z_1^{M_1} e_{M_1}^{i_1}(z)J_{i_1}} | Jm \rangle A_{Jm\tilde{m}i}. \quad (4.4)
\end{aligned}$$

Here  $\mathcal{L}_i$  are the Killing vectors that obey the  $SU(2)$  algebra and equal  $-\frac{i}{\mu}e_i^M\partial_M$ , where  $e_i^M$  are the inverse of  $e_M^i$ .  $\langle Jm_1 | e^{i\mu\Delta z_1^{M_1} e_{M_1}^{i_1}(z)J_{i_1}} | Jm \rangle$  is the matrix element of the spin  $J$  representation for an  $SU(2)$  element  $e^{i\mu\Delta z_1^{M_1} e_{M_1}^{i_1}(z)J_{i_1}}$ . The gauge fields at  $z + \Delta z_1 + \Delta z_2$  are evaluated as

$$\begin{aligned}
A_i(z + \Delta z_1 + \Delta z_2) &= \sum_{Jm\tilde{m}} e^{i\mu\Delta z_2^{M_2} e_{M_2}^{i_2}(z+\Delta z_1)\mathcal{L}_{i_2}} Y_{Jm\tilde{m}}(z + \Delta z_1) A_{Jm\tilde{m}i} \\
&= \sum_{Jmm_2\tilde{m}} Y_{Jm_2\tilde{m}}(z + \Delta z_1) \langle Jm_2 | e^{i\mu\Delta z_2^{M_2} e_{M_2}^{i_2}(z+\Delta z_1)J_{i_2}} | Jm \rangle A_{Jm\tilde{m}i} \\
&= \sum_{Jmm_1m_2\tilde{m}} Y_{Jm_1\tilde{m}}(z) \langle Jm_1 | e^{i\mu\Delta z_1^{M_1} e_{M_1}^{i_1}(z)J_{i_1}} | Jm_2 \rangle \\
&\quad \times \langle Jm_2 | e^{i\mu\Delta z_2^{M_2} e_{M_2}^{i_2}(z+\Delta z_1)J_{i_2}} | Jm \rangle A_{Jm\tilde{m}i}. \quad (4.5)
\end{aligned}$$

Similarly we can evaluate  $A_{i_a}(z + \sum_{b=1}^{a-1} \Delta z_b)$  and express the Wilson loop (4.2) in terms of the spherical harmonics at  $z$ . Due to homogeneity of  $S^3$ , we can consider a set of Wilson loops starting and ending at  $z$  such that  $e_{M_a}^{i_a}(z + \sum_{b=1}^{a-1} \Delta z_b) \Delta z_a^{M_a}$  is independent of  $z$ . Then we average the Wilson loops over  $S^3$ :

$$\tilde{W} = \int \frac{d\Omega_3}{2\pi^2} W, \quad (4.6)$$

where the integration acts only on products of  $Y_{Jm\tilde{m}}(z)$ . Note that  $\langle W \rangle = \langle \tilde{W} \rangle$ .

Correspondingly, we can consider the Wilson loop in the matrix model:

$$\begin{aligned}
\hat{W} &= \frac{1}{TN_0} \text{Tr} \left[ P \exp \left( i\mu \int_0^1 X_i e_M^i(z(\sigma)) \frac{dz^M(\sigma)}{d\sigma} d\sigma \right) \right] \\
&= \frac{1}{TN_0} \text{Tr} \left[ \prod_{a=1}^n \left( 1 + i\mu X_{i_a} e_{M_a}^{i_a}(z + \sum_{b=1}^{a-1} \Delta z_b) \Delta z_a^{M_a} \right) \right], \quad (4.7)
\end{aligned}$$

where  $T = \sum_s 1$ . Here we decompose  $X_i$  as  $X_i \rightarrow L_i + X_i$ , where  $L_i$  are given in (3.11) with  $s$  running from  $-\infty$  to  $\infty$ ,  $2j_s + 1 = N_0 + s$ ,  $N_s = N$ ,  $N_0 \rightarrow \infty$ , and the periodicity condition  $X_i^{(s,t)} = X_i^{(s-t)}$  imposed. We rewrite (4.7) as

$$\hat{W} = \frac{1}{TN_0} \text{Tr} \left[ \prod_{a=1}^n \left( 1 + i\mu X \cdot e(z + \sum_{b=1}^{a-1} \Delta z_b) \cdot \Delta z_a \right) e^{i\mu L \cdot e(z + \sum_{c=1}^{a-1} \Delta z_c) \cdot \Delta z_a} \right], \quad (4.8)$$

where  $L \cdot e \cdot \Delta x = L_i e_M^i \Delta x^M$  and so on. We further evaluate (4.8) as follows:

$$\hat{W} = \frac{1}{TN_0} \text{Tr} \left[ (1 + i\mu X \cdot e(z) \cdot \Delta z_1) e^{i\mu L \cdot e(z) \Delta z_1} (1 + i\mu X \cdot e(z + \Delta z_1) \cdot \Delta z_2) e^{-i\mu L \cdot e(z) \cdot \Delta z_1} \right]$$

$$\begin{aligned}
& e^{i\mu L \cdot e(z) \cdot \Delta z_1} e^{i\mu L \cdot e(z + \Delta z_1) \cdot \Delta z_2} (1 + i\mu X \cdot e(z + \Delta z_1 + \Delta z_2) \cdot \Delta z_3) \\
& \times e^{-i\mu L \cdot e(z + \Delta z_1) \cdot \Delta z_2} e^{-i\mu L \cdot e(z) \cdot \Delta z_1} \\
& \dots \\
& e^{i\mu L \cdot e(z) \cdot \Delta z_1} \dots e^{i\mu L \cdot e(z - \Delta z_{n-1} - \Delta z_n) \cdot \Delta z_{n-1}} (1 + i\mu X \cdot e(z - \Delta z_n) \cdot \Delta z_n) \\
& \times e^{-i\mu L \cdot e(z - \Delta z_{n-1} - \Delta z_n) \cdot \Delta z_{n-1}} \dots e^{-i\mu L \cdot e(z) \cdot \Delta z_1} \\
& e^{i\mu L \cdot e(z) \cdot \Delta z_1} \dots e^{i\mu L \cdot e(z - \Delta z_n) \cdot \Delta z_n} \Big]. \tag{4.9}
\end{aligned}$$

Note that the factor  $e^{i\mu L \cdot e(z) \cdot \Delta z_1} \dots e^{i\mu L \cdot e(z - \Delta z_n) \cdot \Delta z_n}$  appearing in the last line of (4.9) equals the identity if it is invariant under any deformation of the loop. In order to see this invariance, we consider two paths which start at  $z$  and end at  $z + \Delta z + \Delta z'$ : (1)  $z \rightarrow z + \Delta z \rightarrow z + \Delta z + \Delta z'$  and (2)  $z \rightarrow z + \Delta z' \rightarrow z + \Delta z + \Delta z'$ . We associate  $e^{i\mu L \cdot e(z) \cdot \Delta z} e^{i\mu L \cdot e(z + \Delta z) \cdot \Delta z'}$  and  $e^{i\mu L \cdot e(z) \cdot \Delta z'} e^{i\mu L \cdot e(z + \Delta z') \cdot \Delta z}$  with (1) and (2), respectively. The difference between these quantities is evaluated up to  $\mathcal{O}((\Delta z)^3)$  as

$$\begin{aligned}
& e^{i\mu L \cdot e(z) \cdot \Delta z} e^{i\mu L \cdot e(z + \Delta z) \cdot \Delta z'} - e^{i\mu L \cdot e(z) \cdot \Delta z'} e^{i\mu L \cdot e(z + \Delta z') \cdot \Delta z} \\
& = i\mu (\partial_M e_M^i(z) - \partial_{M'} e_M^i(z) - \mu f_{ijj'} e_M^j(z) e_M^{j'}(z)) L_i \Delta z^M \Delta z'^{M'}. \tag{4.10}
\end{aligned}$$

This vanishes thanks to the Maurer-Cartan equation. This fact indicates that the factor  $e^{-i\mu L \cdot e(z) \cdot \Delta z_1} \dots e^{-i\mu L \cdot e(z - \Delta z_n) \cdot \Delta z_n}$  associated with a closed path equals the identity. Eventually, the Wilson loop (4.7) takes the form

$$\begin{aligned}
\hat{W} &= \frac{1}{TN_0} \text{Tr} \left[ \prod_{a=1}^n e^{i\mu L \cdot e(z) \cdot \Delta z_1} \dots e^{i\mu L \cdot e(z + \sum_{b=1}^{a-2} \Delta x_b) \cdot \Delta z_{a-1}} \right. \\
& \times \left. \left( 1 + i\mu X \cdot e(z + \sum_{c=1}^{a-1} \Delta z_c) \cdot \Delta z_a \right) e^{-i\mu L \cdot e(z + \sum_{d=1}^{a-2} \Delta z_d) \cdot \Delta z_{a-1}} \dots e^{-i\mu L \cdot e(z) \cdot \Delta z_1} \right]. \tag{4.11}
\end{aligned}$$

We expand the  $(s, t)$  block of  $X_i$  in terms of the fuzzy scalar spherical harmonics as

$$X_i^{(s,t)} = \sum_{Jm} \hat{Y}_{Jm}^{(jsjt)} X_{Jmi}^{(s,t)}. \tag{4.12}$$

We evaluate an expression appearing in the Wilson loop:

$$\begin{aligned}
e^{i\mu L \cdot e(z) \cdot \Delta z_1} X_i^{(s,t)} e^{-i\mu L \cdot e(z) \cdot \Delta z_1} &= \sum_{Jm} e^{i\mu L \cdot e(z) \cdot \Delta z_1} \hat{Y}_{Jm}^{(jsjt)} e^{-i\mu L \cdot e(z) \cdot \Delta z_1} X_{Jmi}^{(s,t)} \\
&= \sum_{Jmm_1} \hat{Y}_{Jm_1}^{(jsjt)} \langle Jm_1 | e^{i\mu \Delta z_1 \cdot e(z) \cdot J} | Jm \rangle X_{Jmi}^{(s,t)}. \tag{4.13}
\end{aligned}$$

We evaluate another expression:

$$e^{i\mu L \cdot e(z) \cdot \Delta z_1} e^{i\mu L \cdot e(z + \Delta z_1) \cdot \Delta z_2} X_i^{(s,t)} e^{-i\mu L \cdot e(z + \Delta z_1) \cdot \Delta z_2} e^{-i\mu L \cdot e(z) \cdot \Delta z_1}$$

$$\begin{aligned}
&= \sum_{Jmm_2} e^{i\mu L \cdot e(z) \cdot \Delta z_1} \hat{Y}_{Jm_2}^{(j_s j_t)} e^{-i\mu L \cdot e(z) \cdot \Delta z_1} \langle Jm_2 | e^{i\mu \Delta z_2 \cdot e(z + \Delta z_1) \cdot J} | Jm \rangle X_{Jma}^{(s,t)} \\
&= \sum_{Jmm_1 m_2 \tilde{m}} \hat{Y}_{Jm_1}^{(j_s j_t)} \langle Jm_1 | e^{i\mu \Delta z_1 \cdot e(z) \cdot J} | Jm_2 \rangle \\
&\quad \times \langle Jm_2 | e^{i\mu \Delta z_2 \cdot e(z + \Delta z_1) \cdot J} | Jm \rangle X_{Jmi}^{(s,t)}.
\end{aligned} \tag{4.14}$$

In this way, we can express the Wilson loop (4.11) in terms of the fuzzy spherical harmonics.

Using the formulae in appendix, we can easily show that for arbitrary  $K$  in the  $N_0 \rightarrow \infty$  limit

$$\frac{1}{TN_0} \text{Tr}(\hat{Y}_{J_1 m_1(j_{s_1} j_{t_1})} \cdots \hat{Y}_{J_K m_K(j_{s_K} j_{t_K})}) \rightarrow \int \frac{d\Omega_3}{2\pi^2} Y_{J_1 m_1 \tilde{m}_1}(x) \cdots Y_{J_K m_K \tilde{m}_K}(x), \tag{4.15}$$

where  $t_\alpha = s_{\alpha+1}$ ,  $t_K = s_1$  and  $j_{s_\alpha} - j_{t_\alpha} = (s_\alpha - t_\alpha)/2 = \tilde{m}_\alpha$ . Then, by comparing (4.4) with (4.13) and (4.5) with (4.14) and using (3.20), we conclude that

$$\hat{W} \rightarrow \tilde{W} \tag{4.16}$$

in the  $N_0 \rightarrow \infty$  limit.

## §5. Summary and discussion

In this paper, we first found the relationship between the CS theory on the total space of the  $U(1)$  bundle over  $\Sigma_g$  and the BF theory with the mass term on  $\Sigma_g$ . We showed that the former with the  $U(N)$  gauge symmetry is obtained by expanding the latter with  $U(N \times \infty)$  gauge symmetry around the background (2.9) with  $s$  running from  $-\infty$  to  $\infty$ ,  $n_s = ps$  and  $N_s = N$  and the periodicity condition (2.11) imposed. We next restricted ourselves to the case of  $g = 0$  and found the relationship between the BF theory with the mass term and the matrix model (3.9). We showed that the theory around each background of the former is equivalent to the theory around a certain background of the latter. By combining the above two findings, we found that the CS theory on  $S^3/\mathbb{Z}_p$  is equivalent to the theory around the background (3.11) of the matrix model in the  $N_0 \rightarrow \infty$  limit, where  $s$  runs from  $-\infty$  to  $\infty$ ,  $2j_s + 1 = N_0 + ps$ ,  $N_s = N$  and the periodicity condition is imposed. We also constructed the Wilson loops in the matrix model that correspond to those in the CS theory on  $S^3$ .

It is important to see whether BF theory with mass term on  $\Sigma_g$  with  $g \neq 0$  is realized in a matrix model and the CS theory on the  $U(1)$  bundle over  $\Sigma_g$  with  $g \neq 0$  is further realized in the matrix model.

The equivalences we found are classical ones. It is not obvious that the equivalences hold at the quantum level.\*<sup>)</sup> We expect from the following discussion that

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\*<sup>)</sup> For the studies of quantum corrections in the related models, see 17), 42)–44) and references therein.

this is the case. The phenomenon which induces the mass term for the Higgs field via the compactification on the non-trivial  $U(1)$  fiber bundle is equivalent to the moduli stabilization by flux (see e.g. 45) and 46) and references therein) and the  $\Omega$ -background in 47). These background fluxes and compactifications lift up the flat directions of the Higgs fields thanks to the induced mass term. If this moduli is also stabilized at the quantum level and the localization mechanism works, we can evaluate exactly the partition function by counting the isolated (BPS) vacua to show that the relationships among the CS theory, the BF theory and the matrix model hold at the quantum level. In the context of the large  $N$  reduced model, the moduli stabilization means that we need no quenching prescription. Indeed, the works 48) and 49) suggest that the moduli stabilization and the localization mechanism work in the relationship between the BF theory and the matrix model at the quantum level. We would like to discuss this point in the near future.

In (3.7), we ignored the constant term, which depend on the background and takes the form

$$S_{BF}^{(\text{bg})} = -\frac{\pi k}{p} \sum_s N_s n_s^2. \quad (5.1)$$

On the other hand, in (3.12), we ignored the following constant term:

$$\begin{aligned} S_{mm}^{(\text{bg})} &= -\frac{4\pi k}{3pN_0} \text{Tr}(L_i^2) \\ &= -\frac{4\pi k}{3pN_0} \sum_s N_s (2j_s + 1) j_s (j_s + 1) \\ &= -\frac{\pi k}{p} \left( \frac{1}{3} M N_0^2 + N_0 \sum_s N_s n_s + \sum_s N_s n_s^2 - \frac{1}{3} M + \mathcal{O}\left(\frac{1}{N_0}\right) \right). \end{aligned} \quad (5.2)$$

We see that (5.1) and (5.2) coincide in the  $N_0 \rightarrow \infty$  limit up to a constant independent of the background as far as we fix the first Chern class  $\sum_s N_s n_s$  of the background on  $S^2$ . This fact would be relevant when we sum up over the backgrounds in the path integral.

Once we verify the relationship between the CS theory on  $S^3/\mathbb{Z}_p$  and the matrix model at the quantum level, we hope that using the Wilson loops in the matrix model constructed in §4 we can compute the knot invariants. We expect wide application of the Wilson loops constructed in §4, since they are independent of the theory we consider. For instance, it was suggested in 23) that  $\mathcal{N} = 4$  super Yang Mills on  $R \times S^3$  is realized in the plane wave matrix model in the same manner as the CS theory on  $S^3$  is realized in the matrix model. Namely, we can construct the Wilson loops in the plane wave matrix model that correspond to those in  $\mathcal{N} = 4$  super Yang Mills on  $R \times S^3$ . In particular, by including the six scalars in the Wilson loops, we can construct the half-BPS Wilson loops on  $R \times S^3$  50) in term of the matrices.

## Acknowledgements

K.O. would like to thank T. Higaki, K. Takenaga and S. Watamura for useful discussions and comments. A.T. would like to thank Humboldt University for hospitality, where part of this work was done. The work of G.I. is supported in part by the JSPS Research Fellowship for Young Scientists. The work of K.O. and A.T. is supported in part by Grant-in-Aid for Scientific Research (Nos. 19740120 and 19540294) from the Ministry of Education, Culture, Sports, Science and Technology, respectively.

## Appendix A

### — Spherical Harmonics —

In this appendix, we review the properties of the spherical harmonics on  $S^3$ ,  $S^2$  and fuzzy sphere summarized in 23) and 51), and add some new formulae. We regard  $S^3$  as the  $SU(2)$  group manifold. We parameterize an element of  $SU(2)$  in terms of the Euler angles as

$$g = e^{-i\varphi J_3} e^{-i\theta J_2} e^{-i\psi J_3}, \quad (\text{A}\cdot 1)$$

where  $J_i$  satisfy  $[J_i, J_j] = i\epsilon_{ijk}J_k$ . The isometry of  $S^3$  is  $SO(4) = SU(2) \times SU(2)$ , and these two  $SU(2)$ 's act on  $g$  from left and right, respectively. We construct the right-invariant 1-forms:

$$dgg^{-1} = -i\mu e^i J_i, \quad (\text{A}\cdot 2)$$

where  $2/\mu$  corresponds to the radius of  $S^3$ . They are explicitly given by

$$\begin{aligned} e^1 &= \frac{1}{\mu}(-\sin\varphi d\theta + \sin\theta \cos\varphi d\psi), \\ e^2 &= \frac{1}{\mu}(\cos\varphi d\theta + \sin\theta \sin\varphi d\psi), \\ e^3 &= \frac{1}{\mu}(d\varphi + \cos\theta d\psi), \end{aligned} \quad (\text{A}\cdot 3)$$

and satisfy the Maurer-Cartan equation

$$de^i - \frac{\mu}{2}\epsilon_{ijk}e^j \wedge e^k = 0. \quad (\text{A}\cdot 4)$$

The metric constructed from  $e_M^i$  ( $M = \theta, \varphi, \psi$ ) agrees with (3.2). The Killing vectors dual to  $e^i$  are given by

$$\mathcal{L}_i = -\frac{i}{\mu}e_i^M \partial_M, \quad (\text{A}\cdot 5)$$

where  $e_i^M$  are inverse of  $e_M^i$ . The explicit form of the Killing vectors is

$$\mathcal{L}_1 = -i \left( -\sin\varphi \partial_\theta - \cot\theta \cos\varphi \partial_\varphi + \frac{\cos\varphi}{\sin\theta} \partial_\psi \right),$$

$$\begin{aligned}\mathcal{L}_2 &= -i \left( \cos \varphi \partial_\theta - \cot \theta \sin \varphi \partial_\varphi + \frac{\sin \varphi}{\sin \theta} \partial_\psi \right), \\ \mathcal{L}_3 &= -i \partial_\varphi.\end{aligned}\tag{A.6}$$

$\mathcal{L}_i$  satisfy the  $SU(2)$  algebra.

In the following expressions, the upper sign is taken in the patch I ( $0 \leq \theta < \pi$ ) and the lower sign in the patch II ( $0 < \theta \leq \pi$ ). Since  $S^3$  is a  $U(1)$  bundle over  $S^2$  and  $y = (\psi \pm \varphi)/\mu$ , the angular momentum operator in the monopole background with the monopole charge  $q$  is obtained by making a replacement in (A.6):

$$\frac{1}{\mu} \partial_y \rightarrow -iq.\tag{A.7}$$

The result is

$$\begin{aligned}L_1^{(q)} &= i (\sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi) - q \frac{1 \mp \cos \theta}{\sin \theta} \cos \varphi, \\ L_2^{(q)} &= i (-\cos \varphi \partial_\theta + \cot \theta \sin \varphi \partial_\varphi) - q \frac{1 \mp \cos \theta}{\sin \theta} \sin \varphi, \\ L_3^{(q)} &= -i \partial_\varphi \mp q,\end{aligned}\tag{A.8}$$

which satisfy the  $SU(2)$  algebra and agree with (3.4).

The scalar spherical harmonics on  $S^3$  are given by

$$Y_{Jm\tilde{m}}(\Omega_3) = (-1)^{J-\tilde{m}} \sqrt{2J+1} \langle J - \tilde{m} | g^{-1} | Jm \rangle.\tag{A.9}$$

The monopole scalar spherical harmonics<sup>39)</sup> are expressed in terms of the scalar spherical harmonics on  $S^3$ :

$$\tilde{Y}_{Jmq}(\Omega_2) = e^{-iq(\psi \pm \varphi)} Y_{Jmq}(\Omega_3).\tag{A.10}$$

The fuzzy scalar spherical harmonics are given by

$$\hat{Y}_{Jm}^{(jj')} = \sqrt{N_0} \sum_{r,r'} (-1)^{-j+r'} C_{jr\ j'-r'}^{Jm} |jr\rangle \langle j'r'|.\tag{A.11}$$

These spherical harmonics possess the following properties.

Basis of  $SU(2)$  algebra

$$\begin{aligned}\mathcal{L}_\pm Y_{Jm\tilde{m}} &= \sqrt{(J \mp m)(J \pm m + 1)} Y_{Jm\tilde{m}}, \\ \mathcal{L}_3 Y_{Jm\tilde{m}} &= m Y_{Jm\tilde{m}}, \\ L_\pm^{(q)} \tilde{Y}_{Jmq} &= \sqrt{(J \mp m)(J \pm m + 1)} \tilde{Y}_{Jmq}, \\ L_3^{(q)} \tilde{Y}_{Jmq} &= m \tilde{Y}_{Jmq}, \\ L_\pm \circ \hat{Y}_{Jm}^{(jj')} &= \sqrt{(J \mp m)(J \pm m + 1)} \tilde{Y}_{Jm}^{(jj')}, \\ L_3 \circ \hat{Y}_{Jm}^{(jj')} &= m \tilde{Y}_{Jm}^{(jj')},\end{aligned}\tag{A.12}$$



where  $L_i \circ$  are defined by  $L_i \circ \hat{Y}_{Jm}^{(jj')} = L_i^{[j]} \hat{Y}_{Jm}^{(jj')} - \hat{Y}_{Jm}^{(jj')} L_i^{[j']}$  and satisfy the  $SU(2)$  algebra.

Complex conjugate

$$\begin{aligned} (Y_{Jm\tilde{m}})^\dagger &= (-1)^{m-\tilde{m}} Y_{J-m-\tilde{m}}, & (\tilde{Y}_{Jmq})^\dagger &= (-1)^{m-q} \tilde{Y}_{J-m-q}, \\ (\hat{Y}_{Jm}^{(jj')})^\dagger &= (-1)^{m-(j-j')} \hat{Y}_{J-m}^{(j'j)}. \end{aligned} \quad (\text{A.13})$$

Orthonormal relation

$$\begin{aligned} \int \frac{d\Omega_3}{2\pi^2} (Y_{J'm'\tilde{m}'}^\dagger)^\dagger Y_{Jm\tilde{m}} &= \delta_{JJ'} \delta_{mm'} \delta_{\tilde{m}\tilde{m}'}, \\ \int \frac{d\Omega_2}{4\pi} (Y_{J'm'q}^\dagger)^\dagger Y_{Jmq} &= \delta_{JJ'} \delta_{mm'}, \\ \frac{1}{N_0} \text{tr}((\hat{Y}_{J'm'}^{(jj')})^\dagger \hat{Y}_{Jm}^{(jj')}) &= \delta_{JJ'} \delta_{mm'}. \end{aligned} \quad (\text{A.14})$$

Integral of the product of three harmonics

$$\begin{aligned} &\int \frac{d\Omega_3}{2\pi^2} (Y_{J_1 m_1 \tilde{m}_1})^\dagger Y_{J_2 m_2 \tilde{m}_2} Y_{J_3 m_3 \tilde{m}_3} \\ &= \sqrt{\frac{(2J_2+1)(2J_3+1)}{2J_1+1}} C_{J_2 m_2 J_3 m_3}^{J_1 m_1} C_{J_2 \tilde{m}_2 J_3 \tilde{m}_3}^{J_1 \tilde{m}_1} \equiv C_{J_2 m_2 \tilde{m}_2 J_3 m_3 \tilde{m}_3}^{J_1 m_1 \tilde{m}_1}, \\ &\int \frac{d\Omega_2}{4\pi} (\tilde{Y}_{J_1 m_1 q_1})^\dagger \tilde{Y}_{J_2 m_2 q_2} \tilde{Y}_{J_3 m_3 q_3} = C_{J_2 m_2 q_2 J_3 m_3 q_3}^{J_1 m_1 q_1}, \\ &\frac{1}{N_0} \text{tr}((\hat{Y}_{J_1 m_1}^{(j'j)})^\dagger \hat{Y}_{J_2 m_2}^{(j'j'')} \hat{Y}_{J_3 m_3}^{(j''j)}) \\ &= (-1)^{J_1+2J_3-j+j'-2j''} \sqrt{N_0(2J_2+1)(2J_3+1)} C_{J_2 m_2 J_3 m_3}^{J_1 m_1} \begin{Bmatrix} J_1 & J_2 & J_3 \\ j'' & j & j' \end{Bmatrix} \\ &\equiv \hat{C}_{J_2 m_2(j'j'') J_3 m_3(j''j)}^{J_1 m_1(j'j)}. \end{aligned} \quad (\text{A.15})$$

There is a formula for the asymptotic relations between the  $6-j$  symbols and the  $3-j$  symbols. If  $R \gg 1$ , one obtains

$$\left\{ \begin{matrix} a & b & c \\ d+R & e+R & f+R \end{matrix} \right\} \approx \frac{(-1)^{a+b+c+2(d+e+f+R)}}{\sqrt{2R}} \begin{pmatrix} a & b & c \\ e-f & f-d & d-e \end{pmatrix}. \quad (\text{A.16})$$

Using this formula, one sees that in the  $N_0 \rightarrow \infty$  limit

$$\hat{C}_{J_2 m_2(j'j'') J_3 m_3(j''j)}^{J_1 m_1(j'j)} \rightarrow C_{J_2 m_2 q_2 J_3 m_3 q_3}^{J_1 m_1 q_1} \quad (\text{A.17})$$

with the identification  $j' - j = q_1$ ,  $j' - j'' = q_2$ ,  $j'' - j = q_3$ .

The vector spherical harmonics on  $S^3$ ,  $S^2$  and the fuzzy sphere are defined in terms of the scalar spherical harmonics as

$$Y_{Jm\tilde{m}i}^\rho = i^\rho \sum_{n,p} U_{in} C_{\tilde{Q}p 1n}^{Qm} Y_{\tilde{Q}p\tilde{m}}, \quad \tilde{Y}_{Jmqi}^\rho = i^\rho \sum_{n,p} U_{in} C_{\tilde{Q}p 1n}^{Qm} \tilde{Y}_{\tilde{Q}pq},$$

$$\hat{Y}_{Jm(jj')i}^\rho = i^\rho \sum_{n,p} U_{in} C_{\hat{Q}_p 1n}^{Qm} \hat{Y}_{\hat{Q}_p}^{(jj')}, \quad (\text{A}\cdot 18)$$

where the unitary matrix  $U$  is given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \quad (\text{A}\cdot 19)$$

The vector spherical harmonics possess the following properties.

Action of the  $SU(2)$  generators

$$\begin{aligned} \frac{1}{\mu} \epsilon_{ijk} \nabla_j Y_{Jm\tilde{m}k}^\rho &= i \epsilon_{ijk} \mathcal{L}_j Y_{Jm\tilde{m}k}^\rho + Y_{Jm\tilde{m}i}^\rho = \rho(J+1) Y_{Jm\tilde{m}i}^\rho, \\ i \epsilon_{ijk} L_j^{(q)} Y_{Jmqk}^\rho + Y_{Jmqi}^\rho &= \rho(J+1) Y_{Jmqi}^\rho, \\ i \epsilon_{ijk} L_j \circ \hat{Y}_{Jm(jj')k}^\rho + \hat{Y}_{Jm(jj')i}^\rho &= \rho(J+1) \hat{Y}_{Jm(jj')i}^\rho. \end{aligned} \quad (\text{A}\cdot 20)$$

Complex conjugate

$$\begin{aligned} (Y_{Jm\tilde{m}i}^\rho)^\dagger &= (-1)^{m-\tilde{m}+1} Y_{J-m-\tilde{m}i}^\rho, \quad (Y_{Jmqi}^\rho)^\dagger = (-1)^{m-q+1} Y_{J-m-qi}^\rho, \\ (\hat{Y}_{Jm(jj')i}^\rho)^\dagger &= (-1)^{m-(j-j')+1} \hat{Y}_{J-m(j'j)i}^\rho. \end{aligned} \quad (\text{A}\cdot 21)$$

Orthonormal relation

$$\begin{aligned} \int \frac{d\Omega_3}{2\pi^2} (Y_{J'm'\tilde{m}'i}^{\rho'})^\dagger Y_{Jm\tilde{m}i}^\rho &= \delta_{JJ'} \delta_{mm'} \delta_{\tilde{m}\tilde{m}'} \delta_{\rho\rho'}, \\ \int \frac{d\Omega_2}{4\pi} (\tilde{Y}_{J'm'qi}^{\rho'})^\dagger \tilde{Y}_{Jmqi}^\rho &= \delta_{JJ'} \delta_{mm'} \delta_{\rho\rho'}, \\ \frac{1}{N_0} \text{tr}((\hat{Y}_{J'm'(j'j)i}^{\rho'})^\dagger \hat{Y}_{Jm(j'j)i}^\rho) &= \delta_{JJ'} \delta_{mm'} \delta_{\rho\rho'}. \end{aligned} \quad (\text{A}\cdot 22)$$

Integral of the product of three harmonics

$$\begin{aligned} \int \frac{d\Omega_3}{2\pi^2} \epsilon_{ijk} Y_{J_1 m_1 \tilde{m}_1 i}^{\rho_1} Y_{J_2 m_2 \tilde{m}_2 j}^{\rho_2} Y_{J_3 m_3 \tilde{m}_3 k}^{\rho_3} &\equiv \mathcal{E}_{J_1 m_1 \tilde{m}_1 \rho_1 \ J_2 m_2 \tilde{m}_2 \rho_2 \ J_3 m_3 \tilde{m}_3 \rho_3}, \\ \int \frac{d\Omega_2}{4\pi} \epsilon_{ijk} \tilde{Y}_{J_1 m_1 q_1 i}^{\rho_1} \tilde{Y}_{J_2 m_2 q_2 j}^{\rho_2} \tilde{Y}_{J_3 m_3 q_3 k}^{\rho_3} &= \mathcal{E}_{J_1 m_1 q_1 \rho_1 \ J_2 m_2 q_2 \rho_2 \ J_3 m_3 q_3 \rho_3}, \\ \epsilon_{ijk} \frac{1}{N_0} \text{tr}(\hat{Y}_{J_1 m_1 (jj')i}^{\rho_1} \hat{Y}_{J_2 m_2 (j'j'')j}^{\rho_2} \hat{Y}_{J_3 m_3 (j''j)k}^{\rho_3}) &\equiv \hat{\mathcal{E}}_{J_1 m_1 (jj') \rho_1 \ J_2 m_2 (j'j'') \rho_2 \ J_3 m_3 (j''j) \rho_3}. \end{aligned} \quad (\text{A}\cdot 23)$$

One can compute  $\mathcal{E}$  and  $\hat{\mathcal{E}}$  using (A·15). Their explicit forms are given in 23). In the limit  $N_0 \rightarrow \infty$ ,  $\hat{\mathcal{E}}_{J_1 m_1 (jj') \rho_1 \ J_2 m_2 (j'j'') \rho_2 \ J_3 m_3 (j''j) \rho_3} \rightarrow \mathcal{E}_{J_1 m_1 q_1 \rho_1 \ J_2 m_2 q_2 \rho_2 \ J_3 m_3 q_3 \rho_3}$  with the identification  $j - j' = q_1$ ,  $j' - j'' = q_2$ ,  $j'' - j = q_3$ .

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