de Sitter Thermodynamics in the Canonical Ensemble

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The existing thermodynamics of the cosmological horizon in de Sitter spacetime is established in the micro-canonical ensemble, while the thermodynamics of black hole horizons is established in the canonical ensemble. Generally in the ordinary thermodynamics and statistical mechanics, both of the micro-canonical and canonical ensembles yield the same equation of state for any thermodynamic system. This implies the existence of a formulation of de Sitter thermodynamics based on the canonical ensemble. This paper reproduces the de Sitter thermodynamics in the canonical ensemble. The procedure is as follows: We put a spherical wall at the center of de Sitter spacetime, which has negligible mass and perfectly reflects the Hawking radiation coming from the cosmological horizon. Then the region enclosed by the wall and horizon settles down to a thermal equilibrium state, for which the Euclidean action is evaluated and the partition function is obtained. The integration constant (subtraction term) of Euclidean action is determined so as to reproduce the equation of state (e.g. entropy-area law) verified already in the micro-canonical ensemble. Our de Sitter canonical ensemble is well-defined in the sense that it preserves the "thermodynamic consistency", which means that the state variables satisfy not only the four laws of thermodynamics but also the appropriate differential relations of state variables with thermodynamic functions; e.g. partial derivatives of the free energy yield the entropy, pressure, and so on. The special role of cosmological constant in de Sitter thermodynamics is also revealed.

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§1. Introduction

Black hole thermodynamics is well established by the Euclidean action method.[1]–[3] Euclidean action method is regarded as one technique to obtain the partition function of canonical ensemble of quantum gravity.[4] (See Appendix A for a brief summary of this method.) In a statistical mechanical sense, the canonical ensemble is suitable for the system under the contact with a heat bath. Indeed, as reviewed in the next section, thermodynamically consistent formulation of canonical ensemble for black holes has been established by York, where the black hole is in a cavity surrounded by a heat bath[2] (see Fig. 1 shown in §2). The region enclosed by the surface of heat bath and the black hole horizon settles down to a thermal equilibrium state and the Euclidean action method is applied to that region.

On the other hand, thermodynamics of cosmological event horizon (CEH) in de Sitter spacetime can be formulated for the region bounded by the CEH solely (the region I in Fig. 2 shown in §2), which is filled with the Hawking radiation of CEH and settles down to a thermal equilibrium state without introducing a heat bath. This region can be regarded as an isolated thermodynamic system without the contact with heat bath. This implies that thermodynamics of CEH can be established in the

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micro-canonical ensemble. Indeed, Hawking and Ross[5] have proposed that, when the thermal equilibrium system of a horizon is isolated and has no contact with heat bath, the Euclidean action $I_E^{(\mathrm{micro})}$ of such system can be interpreted as the number of micro-states W or the density of micro-states[6] of underlying quantum gravity, which satisfies $W = e^{I_E^{(\mathrm{micro})}}$. Then, since $I_E^{(\mathrm{micro})} = \pi r_c^2$ for the isolated system of de Sitter CEH of radius r_c, the entropy-area law can be obtained by Boltzmann's relation, $S_{\mathrm{ds}} := \ln W = \pi r_c^2$, where S_{ds} is the entropy of CEH. The entropy-area law for CEH, which is the equation of state of CEH, has already been verified in the micro-canonical ensemble.

In the ordinary thermodynamics and statistical mechanics, both the micro-canonical and canonical ensembles yield the same equation of state of any thermodynamic system.[6],[7] Then it is expected that we can formulate thermodynamics of CEH in the canonical ensemble, which should satisfy the entropy-area law. This paper aims to reproduce de Sitter thermodynamics in the canonical ensemble. We will introduce, in the next section, a perfectly reflecting spherical wall whose radius is smaller than that of CEH. The wall reflects the Hawking radiation coming from CEH, and plays the role of heat bath. Then the region enclosed by the wall and CEH settles down to a thermal equilibrium state. The Euclidean action of that region is regarded as the partition function, where the integration constant (the so-called subtraction term) of the action should be determined to reproduce the entropy-area law which is the equation of state verified already in the micro-canonical ensemble.

On the other hand, if one wants to regard a theoretical framework as a "thermodynamics", the framework must be *thermodynamically consistent*. The "thermodynamic consistency" means that the state variables satisfy not only the four laws of thermodynamics but also the appropriate differential relations of state variables with thermodynamic functions. Thermodynamic functions are the state variables related to the free energy via the Legendre transformation, for example, the internal energy and enthalpy. Then a derivative of free energy F_o must yield the entropy $S_o \equiv -\partial F_o / \partial T_o$, where T_o is the temperature. There are some other similar differential relations in thermodynamics. Those differential relations are the ones required in the "thermodynamic consistency", and necessary to understand thermodynamic properties of the system under consideration, for example, phase transition, thermodynamic stability and equations of states.

To ensure the thermodynamic consistency of state variables of de Sitter CEH, we refer to Schwarzschild thermodynamics formulated by York,[2] which has precisely established the thermodynamic consistency of the canonical ensemble. For example, the work term in the first law and the notion of *extensive* and *intensive* variables are introduced into our de Sitter thermodynamics with referring to York.[2] Then we can show that the thermodynamic consistency holds for our de Sitter canonical ensemble.

Here let us comment on the difference of Euclidean action method from ordinary statistical mechanics. In the Euclidean action method (see Appendix A), one can obtain the partition function of canonical ensemble without explicitly referring to the micro-states of the system under consideration. Therefore, although the thermo-

dynamic formulation of black hole is established by that method, the micro-states responsible for the black hole entropy still remain unknown at present. Exactly speaking, the black hole thermodynamics at present is still a conjecture, because its micro-states remain unknown. Also the micro-states of CEH are still unknown at present. This paper does not aim to reveal the micro-states of CEH, but aims to construct the thermodynamically consistent canonical ensemble of CEH in de Sitter spacetime.

This paper is organized as follows: In §2, the basic assumptions of de Sitter canonical ensemble are introduced with referring to Schwarzschild canonical ensemble formulated by York, where not only the notion of extensive and intensive variables but also the construction of thermal equilibrium system of CEH under the contact with a heat hath are explained. Furthermore, a special role of cosmological constant in de Sitter thermodynamics is also clarified in §2. Then §3 calculates the Euclidean action, where how to determine the integration constant of the action is also discussed. Section 4 is devoted to the derivation of important state variables and the exhibition of thermodynamic consistency. Section 5 discusses the relation between our Euclidean action and the micro-canonical ensemble proposed already by Hawking and Ross, and also the relation between the cosmological constant and internal energy. Appendix A reviews the Euclidean action method and shows a clear definition of temperature under the contact with a heat bath. Appendix B is for a supplemental explanation about the "pressure" introduced in §4, which is not clearly noted in York's paper.

The Planck unit, $c = G = \hbar = k_B = 1$, is used throughout.

§2. Basic assumptions to make the Euclidean action work well

2.1. *Preliminary*

To make the aim of this section obvious, let us recall the relation between thermodynamics and statistical mechanics.[7] In statistical mechanics, the partition function cannot be expressed as a "function of state variables" unless the appropriate state variables, on which the partition function depends, are specified *a priori*. To understand this, consider for example an ordinary gas in a spherical container of radius R, in which the number of composite particles is N, the mass of one particle is m and the mean velocity of particles is v. The ordinary statistical mechanics, without the help of thermodynamics, yields the partition function $Z_{\text{gas}} = Z_{\text{gas}}(R, N, m, v)$ as simply a function of "parameters", R, N, m and v. Statistical mechanics, solely, cannot determine what combinations of those parameters behave as state variables. To determine it, the first law of thermodynamics is necessary. (Note that the notion of *heat* in the first law is established by purely the argument in thermodynamics, not in statistical mechanics.) Comparing the differential of partition function with the first law results in the identification of partition function with the free energy divided by temperature. Then, since the free energy of ordinary gases is a function of the temperature and volume due to the "thermodynamic" argument, the partition function $Z_{\text{gas}}(R, N, m, v)$ should be rearranged to be a function of temperature and

volume $Z_{\text{gas}}(V, T)$, where $V = (4\pi/3)R^3$ and $T = mv^2$ for ideal gases due to the law of equipartition of energy.[*] (The dependence on N is, for example, $Z_{\text{gas}} \propto N$ for ideal gases.)

It should also be emphasized that the reason why the temperature and volume are regarded as the state variables of the gas is that they are consistent with the four laws of "thermodynamics" and have the appropriate properties as state variable. The appropriate properties are that the state variables are macroscopically measurable, the state variables are distinguished into two categories, *intensive* variables and *extensive* variables, and the extensive variables are additive. Those properties of state variables are specified by purely the argument in thermodynamics, not in statistical mechanics. Therefore, from the above, it is recognized that statistical mechanics cannot yield the partition function as a "function of appropriate state variables" without the help of thermodynamics which specifies the appropriate state variables for the partition function.

Turn our discussion to the Euclidean action method for black hole and de Sitter spacetimes. Since the Euclidean action method is the technique to obtain the "partition function" of the spacetime under consideration (see Appendix A), it is necessary to specify the state variables before calculating the Euclidean action. In this section, we review the canonical ensemble for Schwarzschild spacetime constructed by York.[2] Then, from the Schwarzschild canonical ensemble, we can learn the appropriate state variables for the partition function in de Sitter canonical ensemble, which will be summarized later as the basic assumptions of de Sitter canonical ensemble. Furthermore, the special role of cosmological constant is also clarified. The calculation of Euclidean action is carried out not in this section but in the next section.

2.2. *Schwarzschild canonical ensemble*

In the canonical ensemble for Schwarzschild thermodynamics, there are three key points which imply the basis of the canonical ensemble for de Sitter thermodynamics. The first one is the zeroth law which describes the existence and construction of thermal equilibrium states:

Key point 1 (Zeroth law of black hole) *Place a black hole in a spherical cavity as shown in Fig. 1 and also the observer at the surface of the heat bath. Through the Hawking radiation by black hole and the black body radiation by heat bath, the black hole interacts with the heat bath. Then, by appropriately adjusting the temperature of heat bath, the combined system of black hole and heat bath settles down to a thermal equilibrium state.*

The equilibrium state of black hole under the contact with heat bath is described in the canonical ensemble. And the equilibrium state variables of black hole are defined by the quantities measured at the surface of heat bath where the observer is. Then the "thermodynamically consistent" Schwarzschild canonical ensemble is constructed as follows.

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[*] When the number of particles N changes by, for example, a chemical reaction and an exchange of particles with environment, N is also the state variable on which the free energy depends.
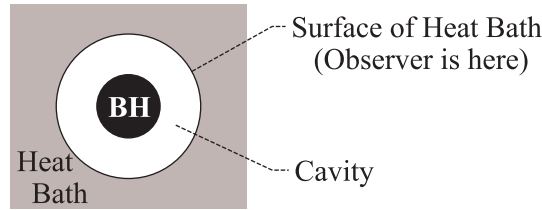
Fig. 1. Schematic image of thermal equilibrium of black hole with heat bath. This is described in the canonical ensemble. State variables of black hole are defined at the surface of heat bath. With those state variables, the consistent thermodynamic formulation is realized using the Euclidean action method.[2]

The second key point is the difference of black hole thermodynamics from thermodynamics of ordinary laboratory systems:

Key point 2 (Peculiar scaling law of black hole) *Extensive and intensive state variables of black hole show a peculiar scaling law: When a length size L (e.g. event horizon radius) is scaled as $L \to \lambda L$ with an arbitrary scaling rate $\lambda\,(>0)$, then the extensive variables X (e.g. entropy) and intensive variables Y (e.g. temperature) are scaled as $X \to \lambda^2 X$ and $Y \to \lambda^{-1} Y$, while the thermodynamic functions Φ (e.g. free energy) are scaled as $\Phi \to \lambda \Phi$. This implies that, because the system size is one of the extensive variables, the thermodynamic system size of equilibrium system constructed in the key point 1 should have the areal dimension. Indeed the surface area of heat bath, $4\pi r_w^2$, behaves as the consistent extensive variable of the system size, where r_w is the radius of the surface of heat bath.*

Here recall that, in thermodynamics of ordinary laboratory systems, the intensive variables remain un-scaled under the scaling of system size, the extensive variables scales as the volume, and the thermodynamic functions are the members of extensive variables. However, as noted in the key point 2, the black hole thermodynamics has the peculiar scaling law of state variables. Although the scaling law differs from that in thermodynamics of ordinary laboratory systems, the peculiar scaling law in black hole thermodynamics retains the thermodynamic consistency as noted in the next key point.

The third key point is the similarity of black hole thermodynamics with thermodynamics of ordinary laboratory systems:

Key point 3 (Euclidean action method and thermodynamic consistency) *The free energy F_{BH} of black hole is yielded by the Euclidean action method, where the integration constant (the so-called subtraction term) of the action integral is determined with referring to flat spacetime. The action integral is evaluated in the region, $2M < r < r_w$, which is in thermal equilibrium as noted in the key point 1. (Here M is the mass parameter.) Then, as for the ordinary thermodynamics, this free energy is expressed as a function of two independent state variables, temperature and system size:*

$$F_{\mathrm{BH}}(T_{\mathrm{BH}}, 4\pi r_w^2)\,, \qquad (2\cdot1)$$

where the intensive variable $T_{\mathrm{BH}} := \left(8\pi M \sqrt{1 - 2M/r_w}\right)^{-1}$ is the Hawking temper-

ature measured by the observer at r_w, and the factor $\sqrt{1 - 2M/r_w}$ is the so-called Tolman factor[8] which expresses the gravitational redshift affecting the Hawking radiation propagating from the black hole horizon to the observer. This Hawking temperature is obtained by Eq. (A·9) of Appendix A. In order to let T_{BH} and $4\pi r_w^2$ be independent variables in F_{BH}, the mass parameter M and the heat bath radius r_w are regarded as two independent variables. Then the thermodynamic consistency holds as follows: The entropy S_{BH} and the "surface pressure" σ_{BH} are defined by

$$S_{\mathrm{BH}} := -\frac{\partial F_{\mathrm{BH}}(T_{\mathrm{BH}}, 4\pi r_w^2)}{\partial T_{\mathrm{BH}}} = \frac{A_{\mathrm{BH}}}{4} \quad , \quad \sigma_{\mathrm{BH}} := -\frac{\partial F_{\mathrm{BH}}(T_{\mathrm{BH}}, 4\pi r_w^2)}{\partial (4\pi r_w^2)} , \qquad (2\cdot 2)$$

where σ_{BH} has the dimension of force par unit area because the system size $4\pi r_w^2$ has the dimension of area. See Appendix B for a detail explanation of thermodynamic meaning of σ_{BH}. (The temperature of heat bath should be adjusted to be T_{BH} in the key point 1.) These differential relations among the free energy, entropy and surface pressure are the same with those obtained in thermodynamics of ordinary laboratory systems. Furthermore, as for the ordinary thermodynamics, the internal energy and the other thermodynamic functions are defined by the Legendre transformation of the free energy; for example the internal energy E_{BH} is

$$E_{\mathrm{BH}}(S_{\mathrm{BH}}, 4\pi r_w^2) := F_{\mathrm{BH}} + T_{\mathrm{BH}} S_{\mathrm{BH}} . \qquad (2\cdot 3)$$

The enthalpy, Gibbs energy and so on are also defined by the Legendre transformation. Then the differential relations among those thermodynamic functions and the other state variables also hold, for example $T_{\mathrm{BH}} \equiv \partial E_{\mathrm{BH}}/\partial S_{\mathrm{BH}}$. Furthermore, with the state variables obtained above, we can check that the first, second and third laws of thermodynamics hold for black holes.

The above three key points hold also for the other single-horizon black hole spacetimes, and those black hole thermodynamics has already been established.[2],[3]

Here let us remark about the heat bath introduced in the key point 1. In York's consistent black hole thermodynamics,[2] the heat bath is essential to establish the thermodynamic consistency in the canonical ensemble as explained below: Generally in thermodynamics, as noted in the key point 3, thermodynamic functions are defined as a function of two independent state variables. Especially the free energy should be expressed as a function of the temperature and the extensive state variable which represents the system size. This thermodynamic requirement is satisfied by introducing the heat bath, which gives us two independent variables; the mass parameter M and the radius of heat bath r_w. These two independent variables makes it possible to define the temperature T_{BH} and the surface area $4\pi r_w^2$ as the two independent state variables in free energy $F_{\mathrm{BH}}(T_{\mathrm{BH}}, 4\pi r_w^2)$. Therefore the heat bath is necessary to establish manifestly the thermodynamic consistency.

Concerning the heat bath, let us make another comment here. It is possible to take the limit $r_w \to \infty$ after constructing the consistent black hole thermodynamics with the heat bath of finite r_w. Here one may think that the limit $r_w \to \infty$ corresponds to the micro-canonical ensemble, since state variables are expressed as functions of only one parameter M and the heat bath seems to disappear (run away
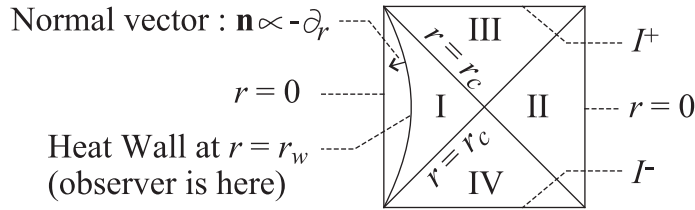
Fig. 2. Penrose diagram of de Sitter spacetime. $I^{\pm}$ is the future/past null infinity. r_c is the CEH radius. The heat wall of radius r_w is placed at the center of the region surrounded by CEH. Our observer is at the wall. This wall reflects perfectly the Hawking radiation, and the observer sees that the region enclosed by the wall and CEH is in a thermal equilibrium state.

to infinitely distant region). However, it should be emphasized that, generally in statistical mechanics, the micro-canonical ensemble is not some limiting case of the canonical ensemble. Therefore the limit $r_w \to \infty$ does not mean to consider the micro-canonical ensemble (the system without heat bath), but it is just the large limit of the cavity size in the canonical ensemble. Hence the black hole thermodynamics have been established in the canonical ensemble.

2.3. *Basic assumptions of de Sitter canonical ensemble*

Let us proceed to the introduction of basic assumptions of de Sitter canonical ensemble. Those assumptions give us the way to construct thermal equilibrium state under the contact with heat bath, and specifies the appropriate state variables for the partition function. Furthermore, to make our discussion exact logically, the use of Euclidean action method is also listed up as one assumption in this paper.[*]

The cosmological event horizon (CEH) in de Sitter spacetime is a spherically symmetric null hyper-surface defined as the boundary of causal past of the observer's world line.[9] The observer detects the Hawking radiation of thermal spectrum emitted by the CEH.[9] This implies that the observer can regard the CEH as an object in thermal equilibrium. It is expected that a thermodynamically consistent canonical ensemble of CEH can be constructed in a way similar to that of the Schwarzschild canonical ensemble. Then, referring to the key point 1 of Schwarzschild thermodynamics, the first basic assumption of de Sitter canonical ensemble is the zeroth law:

Assumption 1 (Zeroth law of CEH) *Place a spherically symmetric thin wall of radius r_w at the center of the region surrounded by the CEH as shown in Fig. 2. The wall is smaller than CEH, $r_w < r_c$, where r_c is the CEH radius. The mass energy of this wall is negligible, and the geometry of the region $r_w < r < r_c$ is of de Sitter spacetime. Let the wall reflect perfectly the Hawking radiation, and we call this perfectly reflecting wall the "heat wall" hereafter. We put our observer at the heat wall. Then this observer sees that the region enclosed by the heat wall and CEH settles down to a thermal equilibrium state. The equilibrium state variables of CEH*

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[*] In this paper, we take the standpoint that the Euclidean action method is simply one (promising) way of obtaining the partition function of spacetimes. At present, since we do not know the complete quantum gravity theory, the use of Euclidean action is understood as one assumption.

are defined by the quantity measured at the heat wall where the observer is.

Since this equilibrium state of CEH has the contact with the heat wall, we expect that de Sitter thermodynamics is obtained in the canonical ensemble. However before calculating the partition function, as explained in §2.1, we have to specify the appropriate state variables for the partition function, the temperature and extensive variable of system size. To do so, it is necessary to clarify the notion of extensivity and intensivity, which is the significant scaling property of state variables under the scaling of system size. The scaling behavior provides a guideline to specify the state variable of system size, and also provides the basis to adopt the temperature defined by Eq. (A·9) of Appendix A as an intensive state variable (see §4.8).

Concerning the extensivity and intensivity, note that, in general, thermodynamics seems to be a very universal formalism which can be applied to any system if it is in thermal equilibrium. This implies that the scaling behavior of state variables shown in the key point 2 of Schwarzschild thermodynamics is common to any horizon system. Then the second assumption of de Sitter canonical ensemble is as follows:

Assumption 2 (Scaling law and system size of CEH) *State variables of CEH are classified into three categories, extensive variables, intensive variables and thermodynamic functions, and those state variables satisfy the same scaling law as explained in the key point 2: When a length size L (e.g. CEH radius) is scaled as $L \to \lambda L$ with an arbitrary scaling rate $\lambda (> 0)$, then the extensive variables X (e.g. system size) and intensive variables Y (e.g. temperature) are scaled as $X \to \lambda^2 X$ and $Y \to \lambda^{-1} Y$, while the thermodynamic functions Φ (e.g. free energy) are scaled as $\Phi \to \lambda \Phi$. This implies that the thermodynamic system size of the equilibrium system constructed in assumption 1 should have the areal dimension, and the surface area of heat wall, $A_{\mathrm{ds}} := 4\pi r_w^2$, behaves as the extensive variable of system size. (Indeed, it will be shown later that thermodynamically consistent formulation of de Sitter canonical ensemble can be established with this A_{ds}.)*

The concrete functional form of free energy cannot be determined by only the assumption 2. But, as implied by the key point 3 of Schwarzschild thermodynamics, the Euclidean action method can yield the concrete form of free energy. Hence the third assumption of de Sitter canonical ensemble is the declaration of using the Euclidean action method:

Assumption 3 (Euclidean action and state variables of CEH) *Euclidean action I_E of a thermal equilibrium state of CEH yields the partition function of canonical ensemble by Eq. (A·13) of Appendix A, where the integration in I_E is calculated over the region enclosed by the heat wall and CEH ($r_w < r < r_c$). And the free energy $F_{\mathrm{ds}}(T_{\mathrm{ds}}, A_{\mathrm{ds}})$ of CEH is defined by Eq. (A·14), where T_{ds} is the temperature of CEH determined by Eq. (A·9) and A_{ds} is the area of heat wall discussed in the assumption 2. Once F_{ds} is determined, any state variable of CEH is defined from F_{ds} in the same way with the state variable in thermodynamics of ordinary laboratory systems. For example, CEH entropy S_{ds} is defined by $S_{\mathrm{ds}} := -\partial F_{\mathrm{ds}}/\partial T_{\mathrm{ds}}$.*

The definition of Euclidean action I_E is shown in Eq. (A·7) of Appendix A, where the Lorentzian action I_L is used. The Lorentzian Einstein-Hilbert action I_L

is

$$I_L := \frac{1}{16\pi} \int_{\mathcal{M}} dx^4 \sqrt{-g} \, (\mathcal{R} - 2\,\Lambda) + \frac{1}{8\,\pi} \int_{\partial\mathcal{M}} dx^3 \sqrt{h}\, K + I_{L\mathrm{sub}} \,, \qquad (2\cdot4)$$

where $\mathcal{M}$ is the region enclosed by the heat wall and CEH in de Sitter spacetime ($r_w < r < r_c$), $\mathcal{R}$ is the Ricci scalar, Λ is the cosmological constant, g is the determinant of the metric, h and K in the second term are respectively the determinant of first fundamental form (induced metric) and the trace of second fundamental form (extrinsic curvature) of the boundary $\partial\mathcal{M}$ (which is the world volume of heat wall), and $I_{L\mathrm{sub}}$ is the integration constant of I_L which is sometimes called the subtraction term. The second term $\int_{\partial M}$ in Eq. (2·4) is required to eliminate the second derivatives of metric from the action.[10] The third term $I_{L\mathrm{sub}}$ does not contribute to the Einstein equation obtained by $\delta I_L = 0$, and is specified in the working hypothesis 2 explained below. The Einstein equation for de Sitter spacetime yields the relation $\mathcal{R} = 4\,\Lambda$. Using this I_L, the Euclidean action I_E for our thermal system constructed in assumption 1 is calculated in the next section.

In §4, we will suggest that the assumptions 1, 2 and 3 construct a candidate of the "thermodynamically consistent" de Sitter canonical ensemble. Let us emphasize that, exactly and logically speaking, once these assumptions are adopted, we also have to adopt supplemental working hypotheses as explained in the next subsection.

2.4. *Supplemental working hypotheses*

When we require the thermodynamic consistency, the differential relations like Eq. (2·2) must be explicitly satisfied. And note that the free energy F_{ds} should be a function of two independent variables T_{ds} and A_{ds}. These require the existence of two independent state variables. On the other hand, we have only two parameters Λ and r_w in our equilibrium system constructed in assumption 1. Hence we have to adopt the following working hypothesis:

Working Hypothesis 1 (Two independent variables) *To construct de Sitter canonical ensemble in a thermodynamically consistent way, we require that the cosmological constant Λ is an independent "working variable". Then we have two independent variables Λ and r_w which can ensure the free energy to be a function of two independent state variables. When we regard non-variable Λ as the physical situation, it is obtained by the "constant Λ process" in the "generalized" de Sitter thermodynamics in which Λ is regarded as a working variable to ensure the thermodynamic consistency.*

A verification of this working hypothesis will be discussed in §5. This working hypothesis can be combined with the assumption 3 to obtain a "complete free energy" as a function of two independent state variables. Here, in order to emphasize the role of Λ as a working variable, we divide the requirement of "complete free energy" into the assumption 3 and the working hypothesis 1. The assumption 3 and working hypothesis 1 may be combined to form one assumption.

Here let us comment on this working hypothesis. This paper is not the first which requires the variable Λ. For example, in order to consider a thermodynamic formalism for Schwarzschild-de Sitter black hole, it has been reported that Λ has to be regarded as an variable, otherwise thermodynamic consistency is lost.[11] The

variable Λ seems to be a practical idea/assumption to treat CEH in thermodynamic framework.

Under the working hypothesis 1, the Euclidean action is to be expressed as a function of two "independent" state variables, temperature and surface area of heat wall (see the assumption 2). Of course, also the integration constant $I_{L\mathrm{sub}}$ in Eq. (2·4) should be determined. Here recall that, as mentioned in §1 (third paragraph), the entropy-area law should be reproduced from our Euclidean action of de Sitter canonical ensemble, since this law is the equation of state verified already in de Sitter micro-canonical ensemble.[5),6)] This requirement gives us the guiding principle to determine $I_{L\mathrm{sub}}$, which we summarize in the following working hypothesis:

Working Hypothesis 2 (Consistency with the micro-canonical ensemble)
The entropy-area law, which is the equation of state verified already in the micro-canonical ensemble, should be reproduced in our de Sitter canonical ensemble. In other words, the integration constant $I_{L\mathrm{sub}}$ in the action I_L should be determined so as to reproduce the entropy-area law. To do so, in calculating the Euclidean action of de Sitter spacetime, we set for the time being,

$$I_{E\mathrm{sub}} = \alpha_w\, I_E^{(\mathrm{flat})}, \qquad\qquad (2\cdot 5)$$

where $I_{E\mathrm{sub}}$ is the integration constant in de Sitter Euclidean action, $I_E^{(\mathrm{flat})}$ is the Euclidean action of flat spacetime ($\Lambda = 0$) whose concrete form will be shown in §3, and α_w is a dimensionless factor. In order not to let $I_{E\mathrm{sub}}$ affect the variational principle in obtaining Einstein equation, the factors α_w and $I_E^{(\mathrm{flat})}$ should be expressed by only the quantity determined at the boundary of Euclidean de Sitter space (at the heat wall). Indeed, as will be shown in §3, $I^{(\mathrm{flat})}$ is expressed by only such quantity. The concrete form of α_w will be obtained in §4.3.

Let us emphasize that Eq. (2·5) is just a working hypothesis including the unknown factor α_w. It seems that, even if one starts calculations of de Sitter Euclidean action with $I_{E\mathrm{sub}}$ which is proportional (not to $I_E^{(\mathrm{flat})}$ but) to some other action integral determined by only the boundary of Euclidean space, then the requirement of preserving the entropy-area law results in the same form with our $I_{E\mathrm{sub}}$ obtained in §4.3. The essence of the working hypothesis 2 is not Eq. (2·5) but the preservation of entropy-area law which is the equation of state verified already in the micro-canonical ensemble. This working hypothesis 2 is based on the statistical mechanical requirement that both the micro-canonical and canonical ensembles yield the same equation of state for any thermodynamic system.[6),7)]

§3.   Euclidean action for canonical ensemble

We calculate the Euclidean action in this section. But, for the completeness of discussion, let us summarize the Lorentzian and Euclidean de Sitter metric before calculating the Euclidean action.

3.1. *Euclidean de Sitter space*

The Lorentzian de Sitter metric in static chart is

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\Omega^2\,, \tag{3·1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$ is the line element on a unit two-sphere, and

$$f(r) := 1 - H^2\,r^2 \quad,\quad 3\,H^2 = \Lambda\,. \tag{3·2}$$

The Penrose diagram is shown in Fig. 2. The static chart with $0 \le r < H^{-1}$ covers the region I or II shown in Fig. 2. The notion of CEH is observer dependent.[9] For the observer whose world line is confined in the region I, the radius r_c of CEH is given by $f(r_c) = 0$,

$$r_c = \frac{1}{H}\,. \tag{3·3}$$

A Killing vector $\xi \propto \partial_t$ becomes null at the CEH. This means that the CEH is the Killing horizon of ξ. Surface gravity κ of the Killing horizon, CEH, is defined by the relation, $\nabla_\xi\,\xi = \kappa\,\xi$ at the CEH. The value of κ depends on the normalization of ξ by definition.[12] When the Killing vector is normalized as $\xi = \partial_t$, we get

$$\kappa = H\,. \tag{3·4}$$

This κ is the surface gravity of CEH measured by the observer at the central world line $r = 0$, because t is the proper time of the central observer and the norm $\xi^2 = -f(r)$ is -1 at $r = 0$.

The global chart of Lorentzian de Sitter spacetime is introduced by the coordinate transformation from (t, r, θ, φ) to (W, X, θ, φ),

$$W - X := e^{H\,(t-r^*)} \quad,\quad W + X := -e^{-H\,(t+r^*)}\,, \tag{3·5}$$

where $dr^* := dr/f(r)$ which means $r^* = (1/2H)\,\ln|(1 + H\,r)/(1 - H\,r)|$. This transformation yields

$$ds^2 = \frac{1}{H^2\,(W^2 - X^2 - 1)^2}\left[-dW^2 + dX^2 + \frac{\left(W^2 - X^2 + 1\right)^2}{4}\,d\Omega^2\right]\,. \tag{3·6}$$

The coordinate transformation (3·5) implies the range of coordinates, $X < W < -X$ and $X < 0$. By extending it to the range $-\infty < W < \infty$ and $-\infty < X < \infty$, the global chart covers the whole region of maximally extended de Sitter spacetime, I, II, III and IV shown in Fig. 2.

Here concerning the relation of the two charts, note that the Penrose diagram shown in Fig. 2 is depicted according to the global chart (3·6). Under the coordinate transformation (3·5), the direction of timelike Killing vector $\xi := \partial_t$ is future pointing in the region I, and past pointing in the region II. Therefore, throughout this paper, we consider the region I in using the static chart.[*]

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[*] If the signature of exponent in the transformation (3·5) is opposite, $W - X = \exp[-H(t-r^*)]$ and $W + X = -\exp[H(t+r^*)]$, then the direction of ∂_t becomes past pointing in the region I, and future pointing in the region II.

Next we proceed to the construction of Euclidean de Sitter space. As explained in Appendix A, we apply the Wick rotations $t \to -i\tau$ in the static chart and $W \to -iw$ in the global chart to obtain the Euclidean de Sitter space. These Wick rotations are equivalent, because the transformation (3·5), $W = e^{-Hr^*} \sinh(Ht)$, implies that the imaginary time w in global chart is defined by $w := e^{-Hr^*} \sin(H\tau)$, where τ is the imaginary time in the static chart. The Euclidean metric in the static chart is

$$ds_E^2 = f(r)\, d\tau^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega^2 \,. \tag{3·7}$$

The Euclidean metric in the global chart is

$$ds_E^2 = \frac{1}{H^2 \left(w^2 + X^2 + 1\right)^2} \left[dw^2 + dX^2 + \frac{\left(w^2 + X^2 - 1\right)^2}{4}\, d\Omega^2 \right]. \tag{3·8}$$

About the global chart, we get from the coordinate transformation (3·5),

$$w^2 + X^2 = \frac{1 - Hr}{1 + Hr}\,. \tag{3·9}$$

Because of $w^2 + X^2 \geq 0$, the Euclidean de Sitter space corresponds to the region I (or II), $0 < r < r_c$, in the Lorentzian de Sitter spacetime. Because the Lorentzian de Sitter spacetime is regular at $r = 0$ and $r = r_c$, the Euclidean de Sitter space is also regular at those points. To examine the regularity of Euclidean space, we make use of the metric in static chart (3·7). The regularity at $r = 0$ is rather obvious, since the metric near $r = 0$ is flat and regular, $ds_E^2 \simeq d\tau^2 + dr^2 + r^2\, d\Omega^2$.

To examine the regularity of Euclidean space at $r = r_c$, let us define a coordinate y and a function $b(y)$ by

$$y^2 := r_c - r \quad, \quad b(y) := \sqrt{f(r_c - y^2)}. \tag{3·10}$$

We get $b'(y) := db(y)/dy = 2H^2\, r\, y/b(y)$ which denotes

$$\lim_{y \to 0} b'(y) = 2H \lim_{y \to 0} \frac{y}{b(y)} = 2H \frac{1}{\displaystyle\lim_{y \to 0} b'(y)}\,. \tag{3·11}$$

This means $b'(0) = \sqrt{2H}$, and near the CEH, $f \simeq [b(0) + b'(0)\, y]^2 = 2H\, y^2$. Therefore the Euclidean metric near the CEH is

$$ds_E^2 \simeq \frac{2}{H} \left[y^2\, d(H\tau)^2 + dy^2 \right] + \frac{1}{H^2}\, d\Omega^2 \,. \tag{3·12}$$

It is obvious that the Euclidean de Sitter space is regular at CEH if the imaginary time has the period β defined by

$$0 \leq \tau < \beta := \frac{2\pi}{H}\,. \tag{3·13}$$

Throughout our discussion, the imaginary time τ has the period β.

3.2. *Euclidean action for "thermodynamically consistent" de Sitter thermodynamics*

Let us proceed to the calculation of Euclidean action I_E defined by Eq. (A·7). To obtain I_E, we should specify the Lorentzian action I_L given in Eq. (2·4). The integral region $\mathcal{M}$ in I_L is the Lorentzian region which forms the thermal equilibrium state constructed in the assumption 1. Therefore $\mathcal{M}$ is expressed by $r_w < r < r_c$, and its boundary $\partial\mathcal{M}$ is at the heat wall, $r = r_w$. There is another boundary at $r = r_c$ in the Lorentzian region $\mathcal{M}$. However we do not need to consider it, because, as shown at the end of the previous subsection, the points at $r = r_c$ in *Euclidean* space do not form a boundary but are the regular points when the imaginary time τ has the period (3·13). Then the first fundamental form h_{ij} $(i, j = 0, 2, 3)$ of $\partial\mathcal{M}$ in the static chart (in Lorentzian de Sitter spacetime) is

$$ds^2\big|_{r=r_w} = h_{ij}\, dx^i\, dx^j = -f_w\, dt^2 + r_w^2\, d\Omega^2\,, \tag{3.14}$$

where

$$f_w := f(r_w) = 1 - H^2\, r_w^2\,. \tag{3.15}$$

Here, since $\mathcal{M}$ is the region enclosed by the heat wall and CEH, the direction of unit normal vector $\boldsymbol{n}$ to $\partial\mathcal{M}$ is pointing towards the "center" $r = 0$, which means $\boldsymbol{n} \propto -\partial_r$ (see Fig. 2). Then the second fundamental form of $\partial\mathcal{M}$ in the static chart is

$$K_{ij} = -\sqrt{f_w}\, \mathrm{diag}\left[\, H^2\, r_w\,,\, r_w\,,\, r_w\, \sin^2\theta\,\right]\,, \tag{3.16}$$

where diag denotes the diagonal matrix form.

On the other hand, the second fundamental form $K_{ij}^{(\mathrm{flat})}$ of a spherically symmetric timelike hyper-surface $r = r_w$ in Minkowski spacetime is obtained by setting $H = 0$ in Eq. (3·16),

$$K_{ij}^{(\mathrm{flat})} = -\mathrm{diag}\left[\, 0\,,\, r_w\,,\, r_w\, \sin^2\theta\,\right]\,, \tag{3.17}$$

where the normal vector $\boldsymbol{n}^{(\mathrm{flat})}$ to this surface is set to be $\boldsymbol{n}^{(\mathrm{flat})} \propto -\partial_r$ as that in K_{ij}. For Minkowski spacetime $\mathcal{R} = 0$ and $\Lambda = 0$, and its Lorentzian action $I^{(\mathrm{flat})}$ is expressed by only the surface term in Eq. (2·4),

$$I^{(\mathrm{flat})} = \frac{1}{8\pi} \int_{\partial\mathcal{M}} dx^3\, \sqrt{h}\, K^{(\mathrm{flat})}\,. \tag{3.18}$$

In applying this $I^{(\mathrm{flat})}$ to Eq. (2·5) of working hypothesis 2, the integral element $\sqrt{h}$ in $I^{(\mathrm{flat})}$ should be that of de Sitter spacetime, because the background spacetime on which the integral is calculated is the de Sitter spacetime.

From the above, applying the Wick rotation $t \to -i\tau$ to the Lorentzian action I_L of de Sitter spacetime, we obtain the Euclidean action I_E via Eqs. (A·7) and (2·5),

$$\begin{aligned}
I_E &= \frac{3\, H^2}{8\,\pi} \int_{\mathcal{M}_E} dx_E^4\, \sqrt{g_E} + \frac{1}{8\,\pi} \int_{\partial\mathcal{M}} dx_E^3\, \sqrt{h_E}\, \left(\, K_E + \alpha_w\, K_E^{(\mathrm{flat})}\,\right) \\
&= \frac{\pi}{H^2}\, \left(\, 1 - 2\, H\, r_w\, f_w - 2\, \alpha_w\, H\, r_w\, \sqrt{f_w}\,\right)\,,
\end{aligned} \tag{3.19}$$

where the relation for de Sitter spacetime $\mathcal{R} = 4\Lambda = 12H^2$ is used in the first equality, Q_E is the quantity Q evaluated on the Euclidean de Sitter space, and $\mathcal{M}_E$ is the Euclidean region expressed by $0 \leq \tau < \beta$, $r_w \leq r \leq r_c$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. This I_E corresponds to $I_E[g_{E\,cl}]$ in Eq. (A·14) of Appendix A, which yields the partition function for the thermal equilibrium of spacetime with neglecting quantum fluctuations.

A comment on the limit $r_w \to 0$ of this I_E will be given in §5 after specifying the form of α_w in the next section.

§4. de Sitter thermodynamics in the canonical ensemble

In this section, principal state variables of CEH are calculated successively, and the "thermodynamic consistency" is also shown explicitly.

4.1. *Temperature*

As explained in Eq. (A·9) of Appendix A, the temperature T_{ds} of CEH is defined by the integral in imaginary time direction at the heat wall,

$$T_{\mathrm{ds}} := \left[\int_0^\beta \sqrt{f_w}\, d\tau \right]^{-1} = \frac{H}{2\pi\sqrt{f_w}} \,, \tag{4·1}$$

where β is defined in Eq. (3·13) and f_w is in Eq. (3·15). Note that, as implied by Eq. (3·4), $H/2\pi$ coincides with the Hawking temperature measured at $r = 0$,[9] and the factor $\sqrt{f_w}$ in Eq. (4·1) is the Tolman factor which expresses the gravitational redshift affecting the Hawking radiation propagating from CEH to the observer.[8] This T_{ds} is the temperature measured by the observer at heat wall.

4.2. *Surface area and homothetic variation of the system*

In the ordinary laboratory systems, the size of the system under consideration is its volume. For de Sitter spacetime, the volume cannot be defined uniquely, since the choice of spatial slice in the region $\mathcal{M}$ of $r_w < r < r_c$ is not determined in a natural way. However, an area can be uniquely and naturally assigned to our system. It is the surface area A_{ds} of the heat wall.

The above discussion supports the assumption 2 in which A_{ds} is regarded as the state variable. Therefore, we adopt A_{ds} as the extensive state variable of system size,

$$A_{\mathrm{ds}} := 4\pi r_w^2 \,. \tag{4·2}$$

Here we have to note that, as explained in Appendix B, the variation of system size is restricted to homothetic variations. For our system, the homothetic variation is the spherical variation due to the spherical symmetry.

4.3. *Choice of α_w of working hypothesis 2*

This subsection determines the form of α_w which appears in Eq. (3·19). As required in the working hypothesis 2, α_w is a dimensionless factor composed of the quantity determined at the heat wall. This implies that α_w is a function of the

parameter,

$$x := H \, r_w \,, \tag{4.3}$$

which means $f_w = 1 - x^2$. This x is regarded as the dimensionless quantity determined at the heat wall. Then our Euclidean action I_E is expressed as

$$I_E = \frac{\pi}{H^2} \left(1 - 2 \, x \, f_w - 2 \, x \, \alpha_w(x) \, \sqrt{f_w} \right) \,. \tag{4.4}$$

In order to determine $\alpha_w(x)$ so as to preserve the entropy-area law which is the equation of state verified already in the micro-canonical ensemble,[5),6)] we need the free energy. The free energy $F_{\rm ds}$ of CEH is obtained via Eq. (A·14) of Appendix A,

$$F_{\rm ds}(T_{\rm ds}, A_{\rm ds}) := -\frac{1}{2H\sqrt{f_w}} \left(1 - 2 \, x \, f_w - 2 \, x \, \alpha_w(x) \, \sqrt{f_w} \right) \,, \tag{4.5}$$

where H and x are regarded as functions of $T_{\rm ds}$ and $A_{\rm ds}$. Then, by the assumption 3, the entropy $S_{\rm ds}$ of CEH is defined by

$$S_{\rm ds} := -\frac{\partial F_{\rm ds}(T_{\rm ds}, A_{\rm ds})}{\partial T_{\rm ds}} = -\frac{\partial_H F_{\rm ds}}{\partial_H T_{\rm ds}} = \frac{\pi}{H^2} \, D \left[\, \alpha_w(x) \, \right] \,, \tag{4.6}$$

where

$$D \left[\, \alpha_w(x) \, \right] := -2 \, x^2 \, f_w^{3/2} \, \frac{d\alpha_w}{dx} - \left(1 - 2 \, x^3 \right) \, f_w + x^2 \,. \tag{4.7}$$

Therefore, to preserve the entropy-area law, $\alpha_w(x)$ has to be a solution of the first-order differential equation, $D \left[\, \alpha_w(x) \, \right] = 1$. Then we obtain

$$\alpha_w(x) = \left(\frac{1}{x} - 1 \right) \, \sqrt{f_w} + k_w \,, \tag{4.8}$$

where k_w is the integration constant. The value of k_w cannot be determined at present. However, as will be shown in §4.6, by the existence of surface pressure at a limit $r_w \to 0$, we find k_w is zero,

$$k_w = 0 \,. \tag{4.9}$$

Although the verification of this value is shown later, we proceed our calculations with setting $k_w = 0$ for simplicity of discussion. Then, adopting Eq. (4·9), the Euclidean action (3·19) is determined,

$$I_E = -\frac{\pi}{H^2} \left(1 - 2 \, x^2 \right) \,. \tag{4.10}$$

A comment on the limit $I_E \to -\pi/H^2$ as $r_w \to 0$ will be given in §5.

4.4. *Free energy, entropy and second law*

The previous subsection has given us the free energy $F_{\rm ds}$ and the entropy $S_{\rm ds}$. The free energy is

$$F_{\rm ds}(T_{\rm ds}, A_{\rm ds}) = \frac{1 - 2 \, x^2}{2 \, H \, \sqrt{f_w}} \,, \tag{4.11}$$

where x is defined in Eq. (4·3). As noted at the working hypothesis 1, F_{ds} should be regarded as a function of T_{ds} and A_{ds}. And the entropy is

$$S_{\mathrm{ds}} = \frac{\pi}{H^2} \,. \qquad (4·12)$$

Concerning the entropy, it should be noted that, exactly speaking, the second law is not a "theorem" proven by some other assumptions, but the basic assumption which cannot be proven in the framework of thermodynamics. The best way to believe the second law is to "check" the statement of the law for as many processes as we can. For the black hole thermodynamics, the generalized second law is checked for some representative processes.[13] For de Sitter thermodynamics, we need to check the generalized second law for as many processes as we can. However let us expect that the generalized second law holds also for the CEH in de Sitter spacetime, since our basic assumptions 1, 2, and 3 are the natural extension of consistent black hole thermodynamics.

4.5.   *Internal energy*

The internal energy E_{ds} of CEH is defined by the argument of ordinary statistical mechanics,

$$E_{\mathrm{ds}} := - \left. \frac{\partial \ln Z_{\mathrm{cl}}}{\partial (1/T_{\mathrm{ds}})} \right|_{A_{\mathrm{ds}}=\mathrm{const}} = \frac{\partial (F_{\mathrm{ds}}/T_{\mathrm{ds}})}{\partial (1/T_{\mathrm{ds}})} = F_{\mathrm{ds}} + T_{\mathrm{ds}} \, S_{\mathrm{ds}}, \qquad (4·13)$$

where Eq. (A·14) of Appendix A is used at the second equality and the definition of S_{ds} in Eq. (4·6) is used at the third equality. The third equality, $E_{\mathrm{ds}} = F_{\mathrm{ds}} + T_{\mathrm{ds}} \, S_{\mathrm{ds}}$, is the Legendre transformation between $F_{\mathrm{ds}}(T_{\mathrm{ds}}, A_{\mathrm{ds}})$ and E_{ds}. This implies that E_{ds} is a function of S_{ds} and A_{ds}, which is consistent with the ordinary thermodynamic argument that the internal energy is a function of extensive state variables. Then we get

$$E_{\mathrm{ds}}(S_{\mathrm{ds}}, A_{\mathrm{ds}}) := \frac{1}{H} \, \sqrt{f_w} \,, \qquad (4·14)$$

where H and x are regarded as functions of S_{ds} and A_{ds} via Eqs. (4·2), (4·3) and (4·12). The origin of internal energy E_{ds} will be discussed in §5.

4.6.   *Surface pressure*

As explained in Appendix B, the conjugate state variable to A_{ds} is the surface pressure σ_{ds} defined by

$$\sigma_{\mathrm{ds}} := - \frac{\partial F_{\mathrm{ds}}(T_{\mathrm{ds}}, A_{\mathrm{ds}})}{\partial A_{\mathrm{ds}}} = - \frac{(\partial_H F_{\mathrm{ds}})\,(\partial_{r_w} T_{\mathrm{ds}}) - (\partial_{r_w} F_{\mathrm{ds}})\,(\partial_H T_{\mathrm{ds}})}{(\partial_H A_{\mathrm{ds}})\,(\partial_{r_w} T_{\mathrm{ds}}) - (\partial_{r_w} A_{\mathrm{ds}})\,(\partial_H T_{\mathrm{ds}})} \,, \qquad (4·15)$$

where Eq. (C) of Appendix C is used at the second equality.

As mentioned in §4.3, we determine the integration constant k_w in this subsection. To do so, we calculate σ_{ds} without setting k_w zero. From Eqs. (4·5) and (4·8), the free energy $\tilde{F}$ with non-zero k_w is $\tilde{F} = F_{\mathrm{ds}} + k_w \, r_w$, where F_{ds} is shown in Eq. (4·11). Then we obtain from Eq. (4·15),

$$\sigma_{\mathrm{ds}} = \frac{H}{8\,\pi\,\sqrt{f_w}} - \frac{k_w}{8\,\pi\,r_w} \,. \qquad (4·16)$$

It is obvious that, if $k_w \neq 0$, then the surface pressure diverges in the limit $r_w \to 0$. Hence, when we require the existence of finite σ_{ds} for thermal equilibrium states of CEH with arbitrarily small heat wall, it is natural to set $k_w = 0$. Then, with adopting this choice $k_w = 0$, we obtain

$$\sigma_{\mathrm{ds}} = \frac{H}{8\,\pi\,\sqrt{f_w}} = \frac{1}{4}\,T_{\mathrm{ds}}\,. \tag{4.17}$$

4.7. *First and third laws*

By definitions of S_{ds} and σ_{ds},

$$dF_{\mathrm{ds}}(T_{\mathrm{ds}}, A_{\mathrm{ds}}) = -S_{\mathrm{ds}}\,dT_{\mathrm{ds}} - \sigma_{\mathrm{ds}}\,dA_{\mathrm{ds}}\,. \tag{4.18}$$

Then the first law holds automatically via the Legendre transformation in Eq. (4.13),

$$dE_{\mathrm{ds}}(S_{\mathrm{ds}}, A_{\mathrm{ds}}) = d(F_{\mathrm{ds}} + T_{\mathrm{ds}}\,S_{\mathrm{ds}}) = T_{\mathrm{ds}}\,dS_{\mathrm{ds}} - \sigma_{\mathrm{ds}}\,dA_{\mathrm{ds}}\,. \tag{4.19}$$

Next, to discuss the third law, note that T_{ds} is monotonically increasing as a function of r_w as shown by $\partial_{r_w} T_{\mathrm{ds}}(H, r_w) = H^3\,r_w/(2\,\pi\,f_w^{3/2}) > 0$. The minimum value of T_{ds} as a function of r_w is given at $r_w = 0$, $T_{\mathrm{ds}}|_{r_w \to 0} = H/2\,\pi$. This denotes the zero-temperature state is achieved only by the process $H \to 0$ and $r_w \to 0$. However it is obvious from Eq. (4.14) that the infinite energy is required to realize the process $H \to 0$. The infinite energy supply is unphysical. Hence the third law holds in the sense that the zero-temperature state cannot be achieved by any physical process.

4.8. *Scaling law, Euler relation and Λ as a "hidden" variable*

This subsection demonstrates the consistency of the scaling law, which is introduced in the assumption 2, with the other assumptions 1 and 3.

Let us consider the scaling of length size,

$$r_w \to \lambda\,r_w \quad , \quad r_c \to \lambda\,r_c\,. \tag{4.20}$$

The scaling of CEH radius denotes $H \to \lambda^{-1}\,H$. Then from Eqs. (4.1) and (4.17), we get the scaling of intensive variables,

$$T_{\mathrm{ds}} \to \frac{1}{\lambda}\,T_{\mathrm{ds}} \quad , \quad \sigma_{\mathrm{ds}} \to \frac{1}{\lambda}\,\sigma_{\mathrm{ds}}\,. \tag{4.21}$$

From Eqs. (4.2) and (4.12), the scaling of extensive variables is

$$A_{\mathrm{ds}} \to \lambda^2\,A_{\mathrm{ds}} \quad , \quad S_{\mathrm{ds}} \to \lambda^2\,S_{\mathrm{ds}}\,. \tag{4.22}$$

From Eqs. (4.11) and (4.14), we get the scaling of thermodynamic functions,

$$F_{\mathrm{ds}} \to \lambda\,F_{\mathrm{ds}} \quad , \quad E_{\mathrm{ds}} \to \lambda\,E_{\mathrm{ds}}\,. \tag{4.23}$$

Therefore we find that the scaling law of assumption 2 is consistent with the assumptions 1 and 3. Concerning this consistency, the definition of temperature in

Eq. (A·9) should be emphasized. It is obvious that the temperature has the dimension of the inverse of length size by definition. Hence, the scaling law of assumption 2 is necessary to adopt Eq. (A·9) as the temperature which should be intensive.

Furthermore, to show a more robust consistency of the scaling law, recall that the internal energy is a function of S_{ds} and A_{ds}. Then the above scaling law implies

$$\lambda E_{\mathrm{ds}}(S_{\mathrm{ds}}, A_{\mathrm{ds}}) = E_{\mathrm{ds}}(\lambda^2 S_{\mathrm{ds}}, \lambda^2 A_{\mathrm{ds}}) . \tag{4·24}$$

This denotes that $E_{\mathrm{ds}}(S_{\mathrm{ds}}, A_{\mathrm{ds}})$ is the homogeneous expression of degree 1/2. By the partial differential of this equation with respect to λ, we get the Euler relation,

$$\frac{1}{2} E_{\mathrm{ds}}(S_{\mathrm{ds}}, A_{\mathrm{ds}}) = T_{\mathrm{ds}} S_{\mathrm{ds}} - \sigma_{\mathrm{ds}} A_{\mathrm{ds}} . \tag{4·25}$$

Note that the concrete functional forms of state variables are not used in deriving this Euler relation. The Euler relation (4·25) is obtained from the scaling behavior (4·24) and the differential relations implied by the first law, $T_{\mathrm{ds}} \equiv \partial E_{\mathrm{ds}}(S_{\mathrm{ds}}, A_{\mathrm{ds}})/\partial S_{\mathrm{ds}}$ and $\sigma_{\mathrm{ds}} \equiv -\partial E_{\mathrm{ds}}(S_{\mathrm{ds}}, A_{\mathrm{ds}})/\partial A_{\mathrm{ds}}$. On the other hand, we can check that the concrete forms of state variables T_{ds}, S_{ds}, σ_{ds} and A_{ds} obtained in previous subsections satisfy the relations (4·24) and (4·25). Hence the state variables obtained so far are completely consistent with the scaling law of assumption 2.

Finally in this subsection, recall that the cosmological constant Λ is regarded as an independent variable in the working hypothesis 1. It is obvious that the assumption 2, via the relation $\Lambda = 3 H^2$, excludes the "bare Λ" from state variables, since Λ is neither intensive nor extensive. Hence the bare Λ cannot be a "state variable" but a "hidden variable" in the consistent de Sitter thermodynamics.

4.9. *Heat capacity and thermal stability*

The thermodynamic consistency of our de Sitter canonical ensemble has been clearly checked so far. This subsection researches the thermal stability of CEH. The appropriate quantity to consider thermal stability is the heat capacity.

The representative heat capacity may be the heat capacity $C_{A_{\mathrm{ds}}}$ at constant A_{ds}. Since A_{ds} depends only on r_w, $C_{A_{\mathrm{ds}}}$ describes the response of temperature to the energy supply into CEH with fixing the position of observer r_w. $C_{A_{\mathrm{ds}}}$ is defined by

$$C_{A_{\mathrm{ds}}} := T_{\mathrm{ds}} \frac{\partial S_{\mathrm{ds}}(T_{\mathrm{ds}}, A_{\mathrm{ds}})}{\partial T_{\mathrm{ds}}} = T_{\mathrm{ds}} \frac{\partial_H S_{\mathrm{ds}}}{\partial_H T_{\mathrm{ds}}} , \tag{4·26}$$

where T_{ds} and A_{ds} are regarded as independent variables. By Eqs. (4·1) and (4·12), we get

$$C_{A_{\mathrm{ds}}} = -\frac{2 \pi}{H^2} f_w . \tag{4·27}$$

Obviously the heat capacity $C_{A_{\mathrm{ds}}}$ is negative definite. When the energy is supplied to (extracted from) the CEH, then the temperature T_{ds} decreases (increases). This denotes the CEH is thermally unstable. However, when we consider the constant Λ process as the physical one, this heat capacity $C_{A_{\mathrm{ds}}}$ is the capacity for unphysical process, since $C_{A_{\mathrm{ds}}}$ is defined by the derivative with respect to H as seen in Eq. (4·26). The thermal instability due to negative $C_{A_{\mathrm{ds}}}$ seems to be unphysical.

When we consider the constant Λ process as the physical one, the heat capacity C_Λ at constant Λ is of interest. It is defined by

$$C_\Lambda := T_{\mathrm{ds}} \frac{\partial S_{\mathrm{ds}}(T_{\mathrm{ds}}, H)}{\partial T_{\mathrm{ds}}} = T_{\mathrm{ds}} \frac{\partial_{r_w} S_{\mathrm{ds}}}{\partial_{r_w} T_{\mathrm{ds}}} \,, \tag{4.28}$$

where T_{ds} and Λ are regarded as independent variables. Then, since S_{ds} is independent of r_w, we find

$$C_\Lambda = 0 \,. \tag{4.29}$$

Since C_Λ is not negative but *zero*, thermal equilibrium of CEH is not thermally unstable but thermally *marginal* stable for constant Λ process.

Since the constant Λ process means the variation of only the position of observer r_w with fixing the CEH radius $r_c = \sqrt{3/\Lambda}$, the vanishing heat capacity (4.29) means that the observer's position r_w can change without a heat supply to the CEH. Here note that, Eq. (4.29) does not imply that changing r_w has no thermodynamic effect on the CEH. For example, changing r_w in constant Λ process gives rise to the change of surface pressure, $\partial \sigma_{\mathrm{ds}}/\partial r_w \neq 0$. The vanishing heat capacity at constant Λ process (4.29) means simply the disappearance of heat supply in changing r_w with fixing Λ. This is a peculiar thermodynamic property of the CEH.

4.10. *Surface compressibility and mechanical stability*

Let us research the mechanical stability of our thermal equilibrium system of CEH. As explained in Appendix B, the appropriate quantity to consider the mechanical stability may be the isothermal surface compressibility $\kappa_{T_{\mathrm{ds}}}$ defined by $\kappa_{T_{\mathrm{ds}}} := A_{\mathrm{ds}}^{-1} \partial A_{\mathrm{ds}}(T_{\mathrm{ds}}, \sigma_{\mathrm{ds}})/\partial \sigma_{\mathrm{ds}}$. However, since σ_{ds} is proportional to T_{ds} as shown in Eq. (4.17), the definition of $\kappa_{T_{\mathrm{ds}}}$ becomes meaningless. Then, instead of $\kappa_{T_{\mathrm{ds}}}$, let us consider the "isentropic" surface compressibility $\kappa_{S_{\mathrm{ds}}}$ defined by

$$\kappa_{S_{\mathrm{ds}}} := \frac{1}{A_{\mathrm{ds}}} \frac{\partial A_{\mathrm{ds}}(S_{\mathrm{ds}}, \sigma_{\mathrm{ds}})}{\partial \sigma_{\mathrm{ds}}} \,, \tag{4.30}$$

where S_{ds} and σ_{ds} are regarded as independent variables. Since S_{ds} depends only on Λ as shown in Eq. (4.12), $\kappa_{S_{\mathrm{ds}}}$ is equivalent to the surface compressibility at constant Λ, and therefore it seems to be the physical quantity. By Eqs. (4.2), (4.12) and (4.17), we get

$$\kappa_{S_{\mathrm{ds}}} = \frac{1}{A_{\mathrm{ds}}} \frac{\partial_{r_w} A_{\mathrm{ds}}}{\partial_{r_w} \sigma_{\mathrm{ds}}} = \frac{16 \pi f_w^{3/2}}{H x^2} \,. \tag{4.31}$$

Obviously $\kappa_{S_{\mathrm{ds}}}$ is positive definite. When the surface A_{ds} increases, the surface pressure σ_{ds} also increases. If we take the same criterion of mechanical stability as York[2] (see the end of Appendix B), then the positivity of $\kappa_{S_{\mathrm{ds}}}$ implies that our thermal equilibrium system is mechanically stable.

§5. Summary and discussion

Referring to the "thermodynamically consistent" black hole canonical ensemble formulated by York,[2] we have introduced the basic assumptions 1–3 to construct

the thermodynamically consistent de Sitter canonical ensemble. The need of those assumptions was discussed in §2.1. The central technique is the Euclidean action method introduced by Gibbons and Hawking,[4] in which the choice of integration constant is important. The integration constant in our Euclidean action (4·10) has been determined so as to preserve the entropy-area law which is the equation of state verified already in the micro-canonical ensemble.[5),6)] Then as shown in §4, we have constructed the thermodynamically consistent de Sitter canonical ensemble, which describes a stable equilibrium state as discussed in §§4.9 and 4.10 with the help of Appendix B. It is also found in §4.8 that, due to the assumption 2, when Λ is regarded as a working variable, the bare Λ cannot be a state variable but a "hidden variable" in the consistent de Sitter thermodynamics.

As mentioned in the working hypothesis 1, we regard Λ as a working variable to obtain thermodynamically consistent de Sitter canonical ensemble. The validity of this working hypothesis 1 can be recognized simply by the following fact: The entropy $S_{\mathrm{ds}} = 3\pi/\Lambda$ depends only on Λ as already verified in the micro-canonical ensemble,[5),6)] and consequently the definition of S_{ds} in Eq. (4·6) is expressed by using the derivatives of F_{ds} and T_{ds} with respect to Λ. The derivative with respect to Λ requires implicitly the variable Λ. Hence, in order to calculate the entropy in the canonical ensemble, it is necessary to adopt the working hypothesis 1. *The canonical ensemble of de Sitter spacetime constructs the "generalized" thermodynamics in which Λ behaves as a working variable, and the physical process is described by the constant Λ process.*

Note that, in de Sitter micro-canonical ensemble proposed by Hawking and Ross,[5] the number of states W is equal to $\exp I_E^{(\mathrm{micro})}$, where $I_E^{(\mathrm{micro})}$ is the Euclidean action of de Sitter space without heat wall. Because the heat wall is not introduced and there is no boundary in the Euclidean de Sitter space for the micro-canonical ensemble, $I_E^{(\mathrm{micro})}$ consists of only the bulk term (first term) of I_L and we obtain $I_E^{(\mathrm{micro})} = \pi r_c^2$. (Then, as mentioned in §1, Boltzmann's relation yields the entropy-area law, $S_{\mathrm{ds}} := \ln W = \pi r_c^2$.) On the other hand, as shown in this paper, the partition function of de Sitter canonical ensemble is the Euclidean action I_E obtained in Eq. (4·10). When we consider an arbitrarily small heat wall $r_w \to 0$, the action I_E takes the limit value $I_E \to -\pi r_c^2$ as $r_w \to 0$. This limiting value is different from $I_E^{(\mathrm{micro})}$ by the negative signature. Here one may naively think that our I_E should coincide with $I_E^{(\mathrm{micro})}$ at this limit. This naive requirement seems reasonable from the point of view of spacetime geometry, but is not necessarily reasonable from the point of view of statistical mechanics, because the coincidence of I_E with $I_E^{(\mathrm{micro})}$ at the limit $r_w \to 0$ means that the micro-canonical ensemble is some limiting case of the canonical ensemble. In statistical mechanics, the micro-canonical ensemble is not some limiting case of the canonical ensemble. In de Sitter thermodynamics, the limit $r_w \to 0$ is just the case of an arbitrarily small heat wall and not the case without heat wall. Therefore, in statistical mechanical sense, there seems to be no reason to require that I_E coincides with $I_E^{(\mathrm{micro})}$ at the limit $r_w \to 0$. (Concerning this discussion, see also the end of §2.2 in which the limiting case of large heat bath for black hole thermodynamics is summarized.)

Finally let us discuss the origin of internal energy E_{ds}. For the first, recall the Schwarzschild thermodynamics formulated by York.[2] The internal energy E_{BH} in Schwarzschild thermodynamics is related to its mass parameter M by

$$M = E_{\text{BH}} - \frac{{E_{\text{BH}}}^2}{2\,r_w}\,, \tag{5.1}$$

where r_w is the outer-most radius of cavity shown in Fig. 1. The second term ${E_{\text{BH}}}^2/(2r_w)$ can be interpreted as the self-gravitational potential energy of black hole. Then E_{BH} is interpreted as the "bare" mass energy of the *black hole in cavity*, while M is the "net" mass energy including the self-gravitational potential. It seems reasonable to consider that the origin of internal energy E_{BH} is the mass of black hole. The mass M as the origin of energy E_{BH} can be clearly exhibited in the large cavity limit $r_w \to \infty$. In this limit we have $E_{\text{BH}}|_{r_w \to \infty} = M$, which manifestly shows that E_{BH} is originated from M.

Then turn our discussion to de Sitter thermodynamics. Let us consider the small heat wall limit $r_w \to 0$, which seems to correspond to the large cavity limit in Schwarzschild thermodynamics, since the heat wall is most distant from CEH. In this limit we have

$$\lim_{r_w \to 0} E_{\text{ds}} = \frac{1}{H}\,. \tag{5.2}$$

This may show that the origin of internal energy E_{ds} is the cosmological constant $\Lambda (= 3H^2)$. In the framework of classical general relativity, the de Sitter spacetime is a vacuum spacetime which includes no energy source. However, in the de Sitter thermodynamics which includes essentially the quantum gravitational effects, Λ may be interpreted as a kind of energy source which is responsible to the energy E_{ds}. Also Λ may be responsible to the entropy S_{ds}.

Acknowledgements

Appendix A
—— *Euclidean Action as a Partition Function of Spacetime* ——

The Euclidean action method for systems including gravity is originally introduced by Gibbons and Hawking[4] in an analogy with the thermal field theory of matter fields in flat spacetime. For the first, thermal fields in flat spacetime is summarized. Then its generalization by Gibbons and Hawking is briefly reviewed.

A.1. *Thermal fields in flat spacetime*

The thermal field theory is the statistical mechanics of quantum fields in thermal equilibrium.[14] The partition function for the canonical ensemble of a field ϕ in

Minkowski spacetime is defined by the path integral,

$$Z_{\text{flat}} := \int \mathcal{D}\phi \, e^{I_E[\phi]} \,, \tag{A·1}$$

where $\mathcal{D}\phi$ is a normalized measure of path integral and $I_E[\phi]$ is the Euclidean action of ϕ defined by

$$I_E[\phi] := i \times \text{Lorentzian action with replacing } t \text{ by } -i\,\tau \,, \tag{A·2}$$

where the Lorentzian metric signature is $(-\,+\,+\,+)$, the time coordinate t in the Minkowski spacetime is of ordinary Cartesian coordinates (the time-time component of metric is -1), and the replacement of real time t by imaginary time τ is called the *Wick rotation*. By the Wick rotation $t \to -i\,\tau$, the metric in evaluating $I_E[\phi]$ becomes that of flat Euclidean space with signature $(+\,+\,+\,+)$. The "direction" of Wick rotation on complexified time plane is "clockwise" $t \to -i\,\tau$ (not "counterclockwise" $t \to +i\,\tau$) in order to make Z_{flat} correspond to the partition function of (grand-)canonical ensemble in quantum statistics.[14] In the path integral in Eq. (A·1), an appropriate boundary condition is also given to ϕ in order to realize a thermal equilibrium state. At least, because thermal equilibrium state is static, a periodic boundary condition in the imaginary time direction is required,

$$\phi(\tau) = \phi(\tau + \beta) \,, \tag{A·3}$$

where β is the imaginary time period.[*] With this condition, it has already been known[14] that Z_{flat} corresponds to the partition function of canonical ensemble of equilibrium temperature T_{flat} defined by

$$T_{\text{flat}} := \frac{1}{\beta} \,. \tag{A·4}$$

Z_{flat} describes thermal equilibrium state of ϕ of equilibrium temperature T_{flat} in Minkowski spacetime, and the free energy F_{flat} of the equilibrium state is obtained,

$$F_{\text{flat}} = -T_{\text{flat}} \ln Z_{\text{flat}} \,. \tag{A·5}$$

A.2. *Curved spacetime and thermal fields on it*

In curved spacetime, we consider a thermal equilibrium state of the combined system of spacetime and matter field. For the canonical ensemble of our combined system, it is usually assumed that the partition function Z is obtained by replacing flat metric in Eq. (A·1) with curved one,[4]

$$Z := \int \mathcal{D}g_E \cdot \mathcal{D}\phi \, e^{I_E[g_E, \phi]} \,, \tag{A·6}$$

where

$$I_E[g_E, \phi] := i \times I_L[g, \phi] \text{ with Wick rotation } t \to -i\,\tau \,, \tag{A·7}$$

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[*] When the periodic boundary condition in imaginary time is not required, the path integral in Eq. (A·1) describes an ordinary transition amplitude of ϕ in ordinary quantum field theory of zero temperature.

where g_E is the Euclidean metric of signature $(++++)$ obtained from the Lorentzian metric g by the Wick rotation $t \to -i\tau$, and $I_L[g, \phi]$ is the sum of Lorentzian Einstein-Hilbert action and Lorentzian matter action. Here since the spacetime metric g is also assumed to be quantum metric, g_E appears as an integral variable in the path integral (A·6). To consider equilibrium states of spacetime with matter field, the periodic boundary condition in imaginary time is required for not only ϕ but also g_E,

$$g_{E\,\mu\nu}(\tau) = g_{E\,\mu\nu}(\tau + \beta). \tag{A·8}$$

Here the equilibrium temperature of g and ϕ is not defined simply by β^{-1}, because the spacetime is curved. Instead of the simple inverse β^{-1}, the temperature T should be defined by the proper length in the Euclidean space of g_E in the imaginary time direction,

$$T := \left[\int_0^\beta \sqrt{g_{E\,\tau\tau}}\, d\tau \right]^{-1}. \tag{A·9}$$

Since the metric component $g_{E\,\tau\tau}$ is a function of spacetime coordinates, the integral in Eq. (A·9) becomes a function of spatial coordinates. Therefore it is important to specify where the temperature is defined. Here note that the Euclidean action method is for the canonical ensemble. This implies the existence of a heat bath whose temperature coincides with the temperature of the system under consideration, since the system is in a thermal equilibrium with the heat bath. Therefore it is reasonable to evaluate $g_{E\,\tau\tau}$ in Eq. (A·9) at the contact surface of the system with the heat bath. The contact surface is the boundary of the spacetime region. Hence the temperature T should be evaluated at the spacetime boundary.

In the path integral in Eq. (A·6), the metric and matter field are not necessarily solutions of classical Einstein equation and field equations. However when the field ϕ is weak enough, the dominant contribution would come from the classical solutions, $g_{\rm cl}$ and $\phi_{\rm cl}$, and we can expand as

$$g_{\mu\nu} = g_{{\rm cl}\,\mu\nu} + \delta g_{\mu\nu} \quad , \quad \phi = \phi_{\rm cl} + \delta\phi, \tag{A·10}$$

where δg and $\delta\phi$ describe quantum/statistical fluctuations of metric and matter. In spacetimes with event horizon, this expansion seems reasonable since the Hawking temperature is usually very low and the matter field ϕ of Hawking radiation is weak. Then the Euclidean action becomes

$$I_E[g_E, \phi] = I_E[g_{E\,{\rm cl}}, \phi_{\rm cl}] + I_E[\delta g_E] + I_E[\delta\phi] + \text{higher order terms}, \tag{A·11}$$

where $g_{E\,{\rm cl}}$ is the Euclidean metric obtained from $g_{\rm cl}$, and the second and third terms are quadratic in fluctuations by definition of classical field equations. The partition function becomes

$$\ln Z = I_E[g_{E\,{\rm cl}}, \phi_{\rm cl}] + \ln \int \mathcal{D}(\delta g_E)\, e^{I_E[\delta g_E]} + \ln \int \mathcal{D}(\delta\phi)\, e^{I_E[\delta\phi]} + \cdots. \tag{A·12}$$

The leading term $I_E[g_{E\,{\rm cl}}, \phi_{\rm cl}]$ includes only the classical solutions. Hence the partition function $Z_{\rm cl}$ of the thermal equilibrium state of background classical spacetime and matter is defined by

$$\ln Z_{\rm cl} := I_E[g_{E\,{\rm cl}}, \phi_{\rm cl}]. \tag{A·13}$$

The state variables obtained from $Z_{\rm cl}$ describe thermal equilibrium states of background spacetime and matter. For spacetimes with event horizon, $Z_{\rm cl}$ is interpreted as the partition function of the event horizon. For empty background spacetimes ($\phi_{\rm cl} = 0$) like Schwarzschild and de Sitter spacetimes, $Z_{\rm cl}$ is determined by only classical metric, $\ln Z_{\rm cl} = I_E[g_{E\,\rm cl}]$. This $I_E[g_{E\,\rm cl}]$ describes the canonical ensemble of the thermal equilibrium states of the background classical spacetime, where the thermal equilibrium is achieved by the interaction with the quantum fluctuations of metric. Then the free energy of those classical background spacetimes is determined by

$$F = -T \, \ln Z_{\rm cl} = -T \, I_E[g_{E\,\rm cl}] \,, \tag{A.14}$$

where T is defined by Eq. (A·9) with replacing g_E by $g_{E\,\rm cl}$. This T is the equilibrium temperature of the event horizon.

Appendix B
—— *Surface Area, Surface Pressure and Mechanical Stability* ——

In the consistent thermodynamic formulation of Schwarzschild black hole by York,[2] the surface area of heat bath is the consistent state variable of system size (see Fig. 1). As shown in §2, the consistency of the surface area of heat bath as a state variable is also retained in our de Sitter thermodynamics. However, for the ordinary laboratory systems, it is not the area but the volume which behaves as the consistent state variable of system size. Therefore, for the purpose of a consistent formulation of de Sitter thermodynamics (and also of black hole thermodynamics), it is necessary to summarize the physical meanings of the surface area and the thermodynamic conjugate state variable to it.

The physical meaning of surface area (and surface pressure and isothermal surface compressibility defined below) may not be clearly explained in the original work of consistent black hole thermodynamics by York.[2] We expect this appendix may be helpful to consider not only de Sitter thermodynamics but also black hole thermodynamics.

In this appendix we consider an ordinary laboratory system of volume V_o and surface area A_o. If the system is a gas in a container, V_o and A_o are respectively the volume and surface area of the container. The state variable of system size is V_o, which is extensive by definition. In general, the set of numerical values of independent state variables are uniquely determined to each thermal equilibrium state. The independent state variables are regarded as the "coordinate" in the state space of thermal equilibrium states. For example, the numerical value of the set (T_o, V_o) at a thermal equilibrium state is different from that at the other thermal equilibrium state, where T_o is the temperature of the system under consideration. However, the surface area A_o cannot be the state variable, because we can deform the geometrical shape of the system with keeping the "coordinate" (T_o, V_o) unchanged. Furthermore, while V_o is the extensive variable, A_o is never extensive since $A \to \nu^{2/3} A_o$ under the volume scaling $V_o \to \nu V_o$.

However if the variation of the system size is restricted to the *homothetic varia-*

tion, there is one-to-one correspondence between V_o and A_o. For example when the system is spherically symmetric, the one-to-one correspondence is realized under the spherical variation of that system. Hence, if the variation of system size is homothetic, the surface area A_o instead of the volume V_o is regarded as a kind of state variable of the system size which has irregular scaling behavior for ordinary laboratory systems. However for the black hole and de Sitter thermodynamics as explained in §2, the extensive variables X are scaled as $X \to \nu^{2/3} X$ under the volume scaling of the scaling rate ν. Therefore the surface area A_o for black hole and de Sitter CEH seems to be the extensive state variable of system size which possesses consistent scaling property (see §4.8).

Next, let us discuss the state variable which is thermodynamically conjugate to the system size. When the volume V_o is chosen as the state variable of the size of ordinary laboratory systems, the conjugate intensive state variable to V_o is the pressure P_o. It can be defined via the free energy $F_o(T_o, V_o)$ of the system, $P_o := -\partial F_o(T_o, V_o)/\partial V_o$. Keeping this relation in mind, for the case that the surface area A_o is chosen as the state variable of system size, let us define a state variable σ_o conjugate to A_o as

$$\sigma_o := -\left. \frac{\partial F_o(T_o, A_o)}{\partial A_o} \right|_{Homo}, \tag{B·1}$$

where $F_o(T_o, A_o) := F_o(T_o, V_o(A_o))$ with regarding V_o as a function of A_o, and the restriction "*Homo*" means that the variation of system size is restricted to the homothetic ones. Let us call σ_o the *surface pressure* of the system after York.[2] Then we get from the definition of the surface pressure,

$$\sigma_o = -\left. \frac{\partial F(T_o, V_o)}{\partial V_o} \frac{dV_o(A_o)}{dA_o} \right|_{Homo} = P_o L_o, \tag{B·2}$$

where $L_o := dV_o/dA_o|_{Homo}$. Here L_o is the length scale determined by the volume variation per unit area variation in the homothetic way, and we call L_o the *homothetic scaling rate*. Therefore, the surface pressure σ_o is the work done by the system onto its environment per unit surface area.

Here note that σ_o is not the surface "tension" of a membrane. To explain it, recall that there are two versions of the ordinary surface tension. One is for the membrane system whose system size is spatially two dimensional, and another is the interface between two phases in a phase equilibrium system. The former example is a rubber sheet, and the latter example is the surface of a drop of water in a phase equilibrium with the surrounding vapor. Let us discuss the difference of σ_o from the surface tensions of these two examples: For the first, in thermodynamics of rubber sheet, the system size is definitely the area of the sheet and its tension is defined by the differential of its free energy by the area. However our system in this appendix is not the membrane like a rubber sheet but the ordinary gas in a container of "volume" V_o. Hence, although σ_o has the same dimension as the surface tension, the thermodynamic meaning of σ_o is not the same with the surface tension of membrane systems.

Next, in thermodynamics of the water drop in a phase equilibrium with surrounding vapor, the surface tension σ_{int} of the interface between water drop and

surrounding vapor is related to the difference between pressures of water and vapor via the *Laplace relation*, $P_{\text{water}} - P_{\text{vapor}} = 2\,\sigma_{\text{int}}/L_{\text{int}}$, where L_{int} is the radius of drop (with assuming the spherical symmetry of drop for simplicity). If $P_{\text{vapor}} = 0$, then one may think σ_o is similar to P_{water} under the identification of L_{int} with $2L_o$. However it should be emphasized that the *phase equilibrium* is necessary to obtain the Laplace relation (see, for example, Kondepudi and Prigogine[7]). Our system, the gas in a container, is the system of one phase and not in phase equilibrium. Furthermore, the gas of our system is in a mechanical equilibrium with the material composing the container which has a *non-zero* pressure. Therefore, exactly speaking, the interpretation of σ_o as the surface tension, which is based on the equilibrium of two phases, seems to be inappropriate. Although the similarity between σ_o and the surface tension may be recognized as mentioned above, we do not call σ_o the surface tension, but call it the *surface pressure*.

From the above we recognize that the implication of σ_o about mechanical stability of the system should be researched in a way independent of the surface tension. One may consider a mechanical stability of the system via the surface pressure in an analogy with ordinary pressure. When (T_o, P_o) are chosen as independent state variables, the state variable useful to consider the mechanical stability is the isothermal compressibility, $\kappa_{T_o}^{(V_o)} := -P_o^{-1}\,\partial V_o(T_o, P_o)/\partial P_o$, where V_o is regarded as a function of T_o and P_o via the equations of states. Then the mechanical stability is described by the positivity, $\kappa_{T_o}^{(V_o)} > 0$. In an analogy with $\kappa_{T_o}^{(V_o)}$, we define the *isothermal surface compressibility*,

$$\kappa_{T_o}^{(A_o)} := \frac{1}{A_o}\left.\frac{\partial A_o(T_o, \sigma_o)}{\partial \sigma_o}\right|_{Homo}, \tag{B·3}$$

where A_o is regarded as a function of T_o and σ_o via the equations of states. Here, following York's black hole thermodynamics,[2] we take the signature $+$ on the right-hand side. Since Eq. (B·3) is just a formal definition in an analogy with ordinary one $\kappa_{T_o}^{(V_o)}$, it is not clear at present if the mechanical stability of the system is described by the positivity $\kappa_{T_o}^{(A_o)} > 0$ or not.[*]

To get some insight into the mechanical stability, let us examine two examples, the ideal gas and the radiation field in a spherical container of radius r_o. Due to the spherical symmetry, the homothetic variation is the spherical variation and the homothetic scaling rate becomes, $L_o = r_o/2$. For the first consider the ideal gas. Its equations of states are $P_o V_o = N_o T_o$ and $E_o = N_o C_v T_o$, where N_o is the number of molecules, E_o is the internal energy and C_v is the specific heat at constant volume. Then we get the ordinary isothermal compressibility $\kappa_{T_o}^{(V_o)} = P_o^{-1} > 0$, which denotes the ideal gas is mechanically stable. On the other hand, we obtain the surface pressure $\sigma_o = (3/2)\,N_o T_o/A_o > 0$, and the isothermal surface compressibility $\kappa_{T_o}^{(A_o)} = -\sigma_o^{-1} < 0$. Therefore, one may think that the mechanical stability of any system is described by $\kappa_{T_o}^{(A_o)} < 0$. However it is not always true as shown below.

The second example is the radiation field in a spherical cavity of radius r_o. Its

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[*] If the surface pressure is a kind of the ordinary surface tension of membrane systems, then the positivity $\kappa_{T_o}^{(A_o)} > 0$ means the system is mechanically stable.

equations of states are $P_o V_o = E_o/3$ and $E_o = a T_o^4 V_o$, where a is the radiation density constant. When we consider the ordinary isothermal compressibility $\kappa_{T_o}^{(V_o)}$, the set (T_o, P_o) should be the independent state variables. However, because of the relation $P_o = a T_o^4/3$ implied by the equations of states, T_o cannot be independent of P_o, and consequently $\kappa_{T_o}^{(V_o)}$ cannot be defined for the radiation field. But it is usually expected that the radiation field is one of the thermodynamically normal and mechanically stable systems. On the other hand, we can define σ_o and $\kappa_{T_o}^{(A_o)}$ for the radiation field. They are $\sigma_o = a r_o T_o^4/6 > 0$ and $\kappa_{T_o}^{(A_o)} = 2/\sigma_o > 0$. The isothermal surface compressibility for radiation field has an opposite signature to that of ideal gas. Hence we cannot definitely determine the criterion of mechanical stability by the signature of $\kappa_{T_o}^{(A_o)}$.

However in the black hole thermodynamics formulated by York,[2] referring to the radiation field, it is assumed that the positivity of isothermal surface compressibility becomes the criterion of mechanical stability of the systems for which the ordinary isothermal compressibility is not defined. For the thermal equilibrium of black hole with heat bath (see Fig. 1), the ordinary isothermal compressibility cannot be defined, because the volume of the system cannot be defined uniquely by the same reason as de Sitter spacetime (see §4.2). Then York[2] suggests that the thermal equilibrium state of black hole with heat bath is mechanically stable (unstable) for sufficiently small (large) heat bath by showing the positivity (negativity) of the isothermal surface compressibility.

Appendix C
—— *Useful Differential Formulas* ——

This appendix shows the differential formula used in §4. Let F be a function of g and h, $F = F(g, h)$. And consider the case that g and h are also functions of x and y, $g = g(x, y)$ and $h = h(x, y)$. Then define $F(x, y)$ by

$$F(x, y) := F(g(x, y), h(x, y)). \qquad (\text{C·1})$$

Let us aim to express the partial derivatives $\partial_g F(g, h)$ and $\partial_h F(g, h)$ by using the derivatives with respect to x and y. By standard differential calculus, we get

$$\frac{\partial F(x, y)}{\partial x} = \frac{\partial F(g, h)}{\partial g} \frac{\partial g(x, y)}{\partial x} + \frac{\partial F(g, h)}{\partial h} \frac{\partial h(x, y)}{\partial x},$$
$$\frac{\partial F(x, y)}{\partial y} = \frac{\partial F(g, h)}{\partial g} \frac{\partial g(x, y)}{\partial y} + \frac{\partial F(g, h)}{\partial h} \frac{\partial h(x, y)}{\partial y}. \qquad (\text{C·2})$$

Regarding these equations as the algebraic ones of $\partial_g F(g, h)$ and $\partial_h F(g, h)$, we obtain the formulas,

$$\frac{\partial F(g, h)}{\partial g} = \frac{(\partial_x F)(\partial_y h) - (\partial_y F)(\partial_x h)}{(\partial_x g)(\partial_y h) - (\partial_y g)(\partial_x h)},$$
$$\frac{\partial F(g, h)}{\partial h} = \frac{(\partial_x F)(\partial_y g) - (\partial_y F)(\partial_x g)}{(\partial_x h)(\partial_y g) - (\partial_y h)(\partial_x g)}. \qquad (\text{C·3})$$

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