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# Integrals of a Lotka-Volterra System of Infinite Species

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A Lotka-Volterra system of infinite species is introduced. Each of the infinite species is represented by a point on a unit circle. The probability density on the circle is given by the solution of the Lotka-Volterra system. Infinite number of conserved quantities are given for the system.

Here we introduce a continuous version of the previous result[1)] on a Lotka-Volterra system of $2s+1$ variables which has $s+1$ conserved quantities. A Lotka-Volterra system of $n$ variables

$$\frac{d}{dt}P_i=P_i(P_{i-1}-P_{i+1})$$

with $P_{i+n}=P_i$ for each integer $i$, has soliton solutions.[2)~6)] Toda lattice[7)] is well known because of its soliton solutions. Toda lattice of $2m$ variables has $m$ conserved quantities.[8)~10)] In the discrete version of our system[1),11)~16)] each of $2s+1$ species interacts with the other $2s$ species as

$$\frac{d}{dt}P_i=P_i\Big(\sum_{j=1}^{s}P_{i-j}-\sum_{j=1}^{s}P_{i+j}\Big) \tag{1}$$

with $P_{i+2s+1}=P_i$ for each integer $i$. The $s+1$ conserved quantities for Eq. (1) are given in the previous paper by a combinatorial proof. The conserved quantities for the case $s=2$ are

$$P_1+P_2+P_3+P_4+P_5=I_0\,,$$

$$P_1P_2P_4+P_2P_3P_5+P_3P_4P_1+P_4P_5P_2+P_5P_1P_3=I_1$$

and

$$P_1P_2P_3P_4P_5=I_2\,.$$

Consider the continuous version of Eq. (1),

$$\frac{d}{dt}P(x,t)=P(x,t)\Big(\int_{x-\pi}^{x}P(y,t)dy-\int_{x}^{x+\pi}P(y,t)dy\Big) \tag{2}$$

with $P(x,t)=P(x+2\pi,t)$ for each $x$.

The conserved quantities are very naturally introduced to this equation. Consider two points on a unit circle $P$ and $Q$ whose coordinates are $(\cos x, \sin x)$ and $(\cos y, \sin y)$, respectively. If the counterclockwise way from $P$ to $Q$ on the circle is shorter than the clockwise way, we write $x<y$. Otherwise, we write $x>y$.

Let $x_1, x_2, \cdots, x_{2r+1}$ satisfy the conditions $0\leqq x_i<2\pi$, $x_{i+1}, x_{i+2}, \cdots, x_{i+r}<x_{i+r+1}$, for $i=1, 2, \cdots, 2r+1$. We denote the above conditions on $x_i$, $i=1, 2, \cdots, 2r+1$, by $E_r$.

Consider the integral

$$I_r=\int_{E_r}\cdots\int P(x_1,t)P(x_2,t)\cdots P(x_{2r+1},t)dx_1dx_2\cdots dx_{2r+1}\,.$$

THEOREM

Let $P(x,t)$ be the solution to Eq. (2). $I_r$ is the conserved quantities for $r=0,1,2,\cdots$.

*Proof*

We have

$$\frac{d}{dt}I_r$$

$$=(2r+1)\Big(\int_{\substack{E_r\\ y<x_{2r+1}}}\cdots\int P(x_1,t)P(x_2,t)\cdots P(x_{2r+1},t)P(y,t)dx_1dx_2\cdots dx_{2r+1}dy$$

$$-\int_{\substack{E_r\\ x_{2r+1}<y}}\cdots\int P(x_1,t)P(x_2,t)\cdots P(x_{2r+1},t)P(y,t)dx_1dx_2\cdots dx_{2r+1}dy\Big)\,.$$

Let us put

$$I_{r,1}=\int_{\substack{E_r\\ x_r+\pi<y\\ y<x_{2r+1}\\ x_{2r+1}<x_{r+1}+\pi}}\cdots\int P(x_1,t)P(x_2,t)\cdots P(x_{2r+1},t)P(y,t)dx_1dx_2\cdots dx_{2r+1}dy\,,$$

$$I_{r,2}=\int_{\substack{E_r\\ x_{2r+1}+\pi<y\\ y<x_r+\pi}}\cdots\int P(x_1,t)P(x_2,t)\cdots P(x_{2r+1},t)P(y,t)dx_1dx_2\cdots dx_{2r+1}dy\,,$$

$$I_{r,3}=\int_{\substack{E_r\\ x_r+\pi<x_{2r+1}\\ x_{2r+1}<y\\ y<x_{r+1}+\pi}}\cdots\int P(x_1,t)P(x_2,t)\cdots P(x_{2r+1},t)P(y,t)dx_1dx_2\cdots dx_{2r+1}dy\,,$$

and

$$I_{r,4}=\int_{\substack{E_r\\ x_{r+1}+\pi<y\\ y<x_{2r+1}+\pi}}\cdots\int P(x_1,t)P(x_2,t)\cdots P(x_{2r+1},t)P(y,t)dx_1dx_2\cdots dx_{2r+1}dy\,.$$

We have

$$\frac{d}{dt}I_r=(2r+1)(I_{r,1}+I_{r,2}-I_{r,3}-I_{r,4})\,.$$

The transposition of the variables $x_{2r+1}$ and $y$ changes a system of conditions $E_r$, $x_r+\pi<y$, $y<x_{2r+1}$ and $x_{2r+1}<x_{r+1}+\pi$ to a system of conditions $E_r$, $x_r+\pi<x_{2r+1}$, $x_{2r+1}<y$ and $y<x_{r+1}+\pi$, which will show $I_{r,1}=I_{r,3}$. A cyclic change of variables shows $I_{r,2}=I_{r,4}$. Hence we have

$$I_{r,1}=I_{r,3}$$

and

$$I_{r,2}=I_{r,4}\,.$$

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