Kinetic Laws of Clustering Motions in N-Body Systems

Yoji Aizawa

Department of Applied Physics, Faculty of Science and Engineering
Waseda University, Tokyo 169-8555, Japan and
International Solvay Institute for Physics and Chemistry
CP231, ULB, Boulvard du Triomphe, B-1050 Brussels, Belgium

A kinetic theory of cluster formation is developed based on the recent numerical results of N-body systems. First, a stability criterion for equilibrium clusters is formulated in terms of residence time distributions. Secondly, the Negative-Weibull distribution is derived theoretically in a certain scaling hypothesis under a free particle approximation. Thirdly, the origin of the Universal Long Tail is explained from a general feature of nearly integrable hamiltonian dynamics. Lastly, the collapse of unstable cluster is briefly discussed.

§1. Introduction

Cluster formation is a universal phenomenon in many-body systems, although the detailed process of clustering motions has not yet been fully understood in the framework of hamiltonian system theories. Recent studies by computer simulations, however, elucidated some new aspects in cluster dynamics in relation to the internal structure of a cluster. [1)] It seems to be most interesting that there are two kinds of kinetic phases embedded in a cluster, i.e., one is the Negative-Weibull regime with the intermediate scale and another is the intrinsic long tail regime. The geometrical image for both phases is simply illustrated in Fig. 1, but the essential point is that each phase obeys the own kinetic law. In this paper, we will discuss the significance of those kinetic laws and the origin of the long time tails in cluster formation.

First, we summarize the general aspects of cluster formation discussed in the previous paper, [1)] where N particles are confined in a 2-dimensional box with periodic boundary conditions. A relatively large cluster is formed like a liquid droplet, and gaseous phase spreads outside the cluster. The cluster is sustained by exchanging member particles, and continues to survive for very long period (practically for infinite time in our simulations). The cluster seems to balance with the external gaseous phase. This is called "equilibrium cluster" for short in what follows, though it is not yet clear whether equilibrium clusters survive even when the box size goes to infinity.

The equilibrium cluster is generally described by the ergodic measure $\mu(\boldsymbol{x})$ defined in the one-particle phase space $\boldsymbol{x}$,

$$\mu(\boldsymbol{x}) = \langle r\rangle \mu_c(\boldsymbol{x}) + (1-\langle r\rangle)\mu_g(\boldsymbol{x}), \tag{1·1}$$

where $\langle r\rangle$ stands for the mean fraction of the phase space corresponding to the clustering motions; $\mu_c(\boldsymbol{x})$ and $\mu_g(\boldsymbol{x})$ are the normalized characteristic measures to describe the clustering phase and the gaseous one, respectively. Along an ergodic orbit, the particle is repeatedly trapped in either phase; the residence time in each phase is denoted by T_c (in cluster) or T_g (in gas). Therefore the fraction $\langle r \rangle$ is understood as the time average of $\frac{T_c}{T_g+T_c}$,

$$\langle r \rangle = \left\langle \frac{T_c}{T_g + T_c} \right\rangle . \qquad (1\cdot 2)$$

The behaviors in gaseous phase are well approximated by random motions when the particle-particle interaction is short-ranged. Therefore, the probability density for T_g, say $P_g(T_g)$, is poissonian,

$$P_g(T_g) = \langle T_g \rangle^{-1} \exp\left[\frac{-T_g}{\langle T_g \rangle}\right], \qquad (1\cdot 3)$$

which have been confirmed by simulations. Here and in what follows, $\langle \cdot \rangle$ stands for the mean value under the appropriate ensemble.

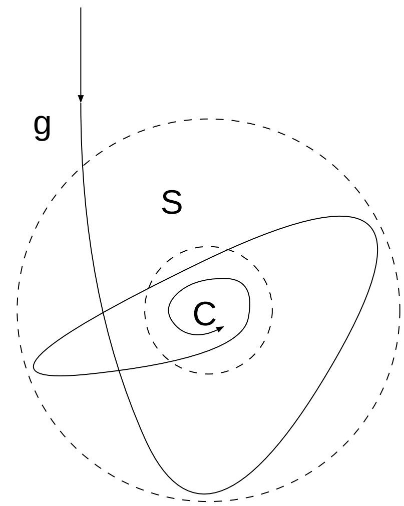

Fig. 1. Schematic picture of one particle orbit.

In the cluster, on the other hand, motions are rather regular, and two different kinetic phases coexist as shown in Fig. 1, where the C-region stands for the core-part of the cluster and the S-region for the soft-dressing part surrounding the core. The residence time distribution for each phase is quite different from the other, namely the Negative-Weibull distribution in the dressing part and the Universal Long Tail distribution $P_{\rm UL}(T_c)$ in the core-part. As the result, the residence time distribution in the cluster $P_c(T_c)$ is given by the superposition of $P_{\rm NW}$ and $P_{\rm UL}$,

$$P_c(T_c) = p \cdot P_{\rm NW}(T_c;\alpha) + (1-p) \cdot P_{\rm UL}(T_c;\beta), \qquad (1\cdot 4)$$

where α and β are the characteristic exponents which represent their distributions, and p is the fraction of the soft-dressing part: $p \simeq 0.995$ in our simulations. The distribution function for each density, $P_{\rm NW}$ and $P_{\rm UL}$, are given by,

$$F_{\rm NW}(x;\alpha) = \int_0^x P_{\rm NW}(x';\alpha)dx' = \exp[-Ax^{-\alpha}] \qquad (1\cdot 5)$$

and

$$F_{\rm UL}(x;\beta) = \int_0^x P_{\rm UL}(x';\beta)dx' = \exp[-B(\ln(x_0+x))^{-\beta}]. \qquad (1\cdot 6)$$

Here A, B, α, β, and x_0 are positive finite parameters. The numerical results were, $\alpha \simeq 1.7$, $\beta \simeq 1.0$, and $x_0 \geq 1$. As the parameter x_0 only changes the linear scale of the variable x, so in the latter discussions we assume $x_0 = 1$ for the sake of simplicity. The remarkable features of $F_{\rm NW}(x;\alpha)$ and $F_{\rm UL}(x;\beta)$ are the divergence of q-th moment $\langle x^q \rangle$, e.g.,

$$\langle x \rangle_{\rm NW} = \text{finite}, \quad \langle x^q \rangle_{\rm NW} \longrightarrow \infty, \quad (q \geq \alpha) \qquad (1\cdot 7)$$

and

$$\langle x^q \rangle_{\mathrm{UL}} \longrightarrow \infty. \quad (q > 0) \tag{1·8}$$

These divergence play essential roles in the latter discussions of the stability conditions in §2. The theoretical background for F_{NW} and F_{UL} will be explained in §§3 and 4, respectively.

Denoting the invariant measures corresponding to the N-W phase and U-L phase by $\mu_{\mathrm{NW}}(\boldsymbol{x})$ and $\mu_{\mathrm{UL}}(\boldsymbol{x})$, respectively, Eq. (1·1) can be extended as

$$\mu(\boldsymbol{x}) = (1 - \langle r \rangle) \cdot \mu_g(\boldsymbol{x}) + \langle r \rangle p \cdot \mu_{\mathrm{NW}}(\boldsymbol{x}) + \langle r \rangle (1-p) \cdot \mu_{\mathrm{UL}}(\boldsymbol{x}). \tag{1·9}$$

Nonequilibrium aspects of cluster formation are known from the spontaneous organization of each kinetic phase. Under the markovian approximation of nucleation processes, the creation of the cluster phase μ_c obeys, [2)]

$$\langle r \rangle \cdot \mu_c(\boldsymbol{x}) \propto (t - t_0)^\gamma, \quad (\gamma < 1) \tag{1·10}$$

in the early stage. Here γ is a parameter determined by the order of the phase transition leading to cluster formation, and t_0 the onset time of nucleation.

In the last section (§5), the meaning of the parameter γ will be discussed in relation to the Negative-Weibull distribution.

§2. Stability of equilibrium clusters

Probability densities P_g and P_c play essential roles in describing the stability conditions of equilibrium clusters. Denote the successive residence times in gas and in cluster by T_g and T_c, as told in §1. Here we can assume that both quantities are independent stochastic variables without any correlation between them. This is theoretically justified from the chaotic effect due to the sensitive dependence of orbits under consideration. Actually, a lot of numerical simulations satisfy it very precisely.

The relative length r of collision time, for which a particle is trapped within the cluster, is given by,

$$r = \frac{T_c}{T_c + T_g} = \frac{1}{1+s}, \quad (0 \le r \le 1, 0 \le s) \tag{2·1}$$

where $s(= T_g/T_c)$ and r are also stochastic variables and their probability densities, $P_s(s)$ and $P_r(r)$, can be determined straightforwardly by use of $P_g(T_g)$ and $P_c(T_c)$;

$$P_s(s) = \iint dT_c dT_g P_c(T_c) P_g(T_g) \delta(s - T_g/T_c), \tag{2·2a}$$

$$P_r(r) = \int ds P_s(s)(1+s)^2 \delta\left(s - \frac{1-r}{r}\right). \tag{2·2b}$$

In the case of Eq. (1·3), i.e., a poissonian case for gaseous phase, Eq. (2·2a) is rewritten into a more transparent form,

$$P_s(s) = \frac{1}{\langle T_g \rangle} \int P_c(T_c) e^{-\frac{s}{\langle T_g \rangle} T_c} T_c dT_c$$

$$= -\frac{d}{ds}\hat{P}\left(\frac{s}{\langle T_g\rangle}\right)$$
$$= -\langle T_g\rangle^{-1}\,\frac{d}{d\lambda}\hat{P}_c(\lambda)\Big|_{\lambda=s/\langle T_g\rangle}, \tag{2·3}$$

where $\hat{P}_c(\lambda)$ is the Laplace transformation of $P_c(T_c)$. In other words, the distribution of variable s, say $F_s(s)$, is simply given by,

$$F_s(s) \equiv \int_0^x P_c(T_c)dT_c,$$
$$= 1 - \hat{P}_c\left(\frac{s}{\langle T_g\rangle}\right)$$
$$= 1 - \frac{s}{\langle T_g\rangle}\hat{F}_c\left(\frac{s}{\langle T_g\rangle}\right). \tag{2·4}$$

In the present paper, we use the capital letters F and $\hat{F}$ for a distribution function and for its Laplace transformation, respectively, e.g.,

$$F_c(x) \equiv \int_0^x P_c(T_c)dT_c,$$
$$\hat{F}_c(\lambda) \equiv \int_0^\infty e^{-\lambda x}F_c(x)dx. \tag{2·5}$$

$P_r(r)$ can be derived,

$$P_r(r) = \frac{-1}{\langle T_g\rangle}\frac{1}{r^2}\left\{\hat{F}_c\left(\frac{s}{\langle T_g\rangle}\right) + s\frac{d}{ds}\hat{F}_c\left(\frac{s}{\langle T_g\rangle}\right)\right\}\Big|_{s=(1-r)/r}. \tag{2·6}$$

Several examples of $P_r(r)$ and $P_s(s)$ are demonstrated in Appendix A.

Here let us formulate a naive criterion which characterizes the perpetual stability of clustering motions in terms of $P_r(r)$, i.e., the followings are two significant conditions for the criterion,

(i) $P_r(r) \longrightarrow 0 \quad (r \to 0)$, and
(ii) $P_r(r)$ is a unimodal function of r.

The first condition (i) ensures the persistency of the cluster under consideration, namely, as is shown in Case I in Appendix A, the cluster formation occurs in a completely accidental manner when this condition fails. The second condition (ii) persists the uniqueness of the equilibrium kinetic state in a cluster.

Let us see the examples in Appendix A; the poissonian case does not satisfy both conditions mentioned above, where cluster formation occurs quite randomly, and the creation as well as the annihilation of clusters are ruled over by chance. The second case (Γ-distribution) in Appendix A satisfies the stability conditions for $1 < \alpha$, but not for $\alpha \le 1$ since short residence times contribute dominantly. The last case (Lévy distribution) always satisfies both conditions, but the aspect is quite different from the previous two cases, because of the divergence of moments ($\langle T_c^q\rangle \to \infty,\ q \ge \alpha$).

$P_r(r)$ has only divergent peak at $r = 1$, which implies that all particles are condensed in the cluster and no particles in gaseous phase.

The situation obtained by numerical simulations in the previous paper [1] is a little bit different from the examples mentioned in Appendix A, though the overall aspects are similar to the case of Lévy distribution owing to the divergence of moments (see Eqs. (1·7) and (1·8)). Here let us analyze the stability of two cases mentioned in Eqs. (1·5) and (1·6), i.e., (1) $F_c(x) = F_{\rm NW}(x)$ and (2) $F_c(x) = F_{\rm UL}(x)$. As shown in Appendix B, the scaling forms are the same in either case,

$$\hat{F}_c(\lambda) \simeq O\left(\lambda^{\frac{-(\mu+2)}{2(\mu+1)}} e^{-a_\mu \lambda^{\frac{\mu}{\mu+1}}}\right), \quad (\lambda \to \infty) \tag{2·7}$$

where $a_\mu = \mu^{\frac{1}{\mu+1}} + \mu^{\frac{-\mu}{\mu+1}}$ ($\mu = \alpha$ or β). Therefore, the first condition (i) is satisfied;

$$P_r(r)|_{r\to 0} \simeq \lambda^{\frac{\mu}{2(\mu+1)}} e^{-a_\mu \lambda^{\frac{\mu}{\mu+1}}}\Big|_{\lambda\to\infty} \longrightarrow 0. \tag{2·8}$$

On the other hand, the second relation (ii) is directly proved from Eq. (2·6);

$$\frac{dP_r(r)}{dr} = \langle T_g\rangle^2 \frac{d^2\hat{P}_c(\lambda)}{d\lambda^2} + 2\langle T_g\rangle \frac{d\hat{P}_c(\lambda)}{d\lambda}, \tag{2·9}$$

where $\hat{P}_c(\lambda)$ is a completely monotonic function because of the Laplace transformation. [3] Therefore the derivative of $\hat{P}_c(\lambda)$ is also monotonic, namely the r.h.s of Eq. (2·9) has at most one zero-point at $\lambda = \lambda_s$, and $P_r(r)$ is extremum at $r = r_s = \frac{1}{1+\lambda_s\langle T_g\rangle}$ if zero-point exists. When the zero-point does not exist, the maximum of $P_r(r)$ appears at $r = 0$ or $r = 1$. This proves the unimodal nature of $P_r(r)$; actually in either case of $P_{\rm NW}(x)$ or $P_{\rm UL}(x)$, the maximum of $P_r(r)$ appears at $r = 1$.

The analysis mentioned here demonstrates that the perpetual stability is always satisfied in the kinetic law of Eq. (1·4); in physics viewpoints the cluster of our case is necessarily formed in the system, and also it is never annihilated spontaneously even though large fluctuations ($\langle T_c^2\rangle \to \infty$) occur.

§3. Derivation of the Negative-Weibull distribution

The motions in the soft-dressing part of a cluster are analyzed under a kind of free particle approximation; the number density in the cluster is low and homogeneous, and that the effect of the mean field potential which confines clustering particles will be added later. Though this approximation includes a few contradictory points, essential aspects in the kinetic phase of the soft-dressing part can be described very well, because the soft-dressing part is an interfacial layer between the gaseous part consisting of perfectly free particles and the core part of strongly condensed particles (see Fig. 1). Our approximation is characterized by two statistical distributions: one is the poissonian distribution for free path $P_l(l)$ (Appendix C) and another is the gaussian distribution for velocities $P_v(v)$, i.e.,

$$P_l(l) = ae^{-al}, \tag{3·1}$$

$$P_v(v) = 4\pi \left(\frac{b}{\pi}\right)^{\frac{3}{2}} v^2 e^{-bv^2}, \quad (d=3) \tag{3·2a}$$

$$= 2\pi \left(\frac{b}{\pi}\right) v e^{-bv^2}, \quad (d=2) \tag{3·2b}$$

where a and b are positive parameters. The free time distribution $P_\tau(\tau)$ is obtained by,

$$P_\tau(\tau) = \int\int \delta\left(\tau - \frac{l}{v}\right) P_l(l) P_v(v) dv dl \tag{3·3}$$

$$= \frac{dF_\tau(\tau)}{d\tau}. \tag{3·4}$$

The explicit form of $F_\tau(\tau)$ can be demonstrated in terms of an error function erfc$(\frac{a\tau}{2\sqrt{b}})$, but the essential point of our concern is the scaling form in the limit of $\tau \to \infty$. By use of the Tauberian theorem for Laplace transformations, [3] the long tail part is derived as,

$$1 - F_\tau(\tau) \simeq O(\tau^{-d}), \tag{3·5}$$

where the exponent of the long tail is the same as the spacial dimension d. In other words, the long tail of the probability density $P_\tau(\tau)$ becomes,

$$P_\tau(\tau) \simeq O(\tau^{-d-1}), \tag{3·6}$$

so that the q-th moment $\langle \tau^q \rangle (q \geq d)$ is divergent.

The Negative-Weibull distribution is obtained straightforwardly from Eq. (3·5). Here we assume that a lot of collisions are necessary for the particle to escape from the cluster. Denote the free times for waiting the collisions with the remaining M particles in the cluster by $\{\tau_1, \tau_2, \tau_3, \cdots, \tau_M\}$, where M stands for the number of collisions necessary for the escape of the particle under consideration. The τ_i are stochastic variables with the independent identical distribution given by Eq. (3·5). Therefore, the residence time of the particle is given by the maximum value τ^* of $\{\tau_1, \tau_2, \cdots, \tau_M\}$, i.e.,

$$\tau^* = \max\{\tau_1, \tau_2, \cdots, \tau_M\}, \tag{3·7}$$

and the distribution function for the residence time $F_c(T_c)$ is obtained by,

$$F_c(T_c) = \text{Prob}\{\tau^* \leq T_c\}$$

$$= [F_\tau(T_c)]^M. \tag{3·8}$$

By use of Eq. (3·5) and the strong laws of maximum value, [3), 7), 8)] $F_c(T_c)$ is derived in the limit of large M; $\ln F_c(T_c) = M \ln F_\tau(\tau_c) \simeq M(1 - F_\tau(T_c))$, i.e.,

$$F_c(T_c) \simeq \exp\{C_M T_c^{-d}\}, \tag{3·9}$$

where C_M is a constant proportional to M. Equation (3·9) is nothing but the Negative-Weibull distribution with the index $\alpha = d$. To compare this result with Eq. (1·5) obtained by our simulations, the value of the index $\alpha(\approx 1.7)$ demonstrates non-negligible deviation from the theoretical estimation of Eq. (3·9), though the typical regime adjustable by the Negative-Weibull distribution clearly exists.

The reason for the deviation is obvious, because the low density approximation based on Eqs. (3·1)-(3·3) is too simple to describe the correlated motions in the cluster with relatively high density. Particularly, we neglected the effect of the mean field potential generated by the cluster itself. Here we give a minimum modification of Eq. (3·3) to recover the consistency between Eq. (3·9) and Eq. (1·5). An essential idea is that the orbit of a particle is not straight line even in the free time, but a curved line which can be described as a geodesic line in the Riemannian space with positive curvature.[1] In other words, the quantity defined by (l/v), which describes the free time in Eucledian space, does not assign the physical time τ along the geodesics, e.g., if we assume the following scaling,

$$\tau = (l/v)^{\sigma}, \tag{3·10}$$

$\delta(\tau - (l/v)^{\sigma})$ takes the place of $\delta(\tau - l/v)$ in Eq. (3·3), and the long tail of $P_\tau(\tau)$ becomes,

$$P_\tau(\tau) \simeq \tau^{-1-d/\sigma}, \tag{3·11}$$

where the index σ stands for the effect of positive curvature within the cluster. Therefore, after the same calculations mentioned before, the residence time distribution $F_c(T_c)$ is rewritten into the Negative-Weibull distribution, i.e.,

$$F_c(T_c) = F_{\rm NW}(T_c, d/\sigma) \tag{3·12}$$

with the index $\alpha = d/\sigma$. The detailed mechanism leading to the scaling assumption of Eq. (3·10) is still unclear, though our numerical result is reproduced by, $\sigma = d/\alpha \approx 1.176$ ($\alpha \approx 1.7$ at $d = 2$).

§4. Origin of the universal long time tail

The universal long time tail distribution given by Eq. (1·6) is strongly correlated to the stagnant motions in nearly integrable hamiltonian systems.[4] Here we see the theoretical background and give a conjecture concerning three-dimensional cluster formation.

According to Nekhoroshev theorem for the hamiltonian systems with n-degrees of freedom;[5]

$$H = H_0(p, q) + \epsilon H_1(p, q), \tag{4·1}$$

the pausing time T_ϵ (so-called Nekhoroshev bound) of an orbit in the region $|P(t) - P(0)| < \epsilon^a$ obeys,

$$T_\epsilon \simeq \epsilon^{-\chi} \exp\,[\epsilon^{-b}], \quad (\chi = 1) \tag{4·2}$$

where a and b are positive parameters determined by the steepness condition of the integrable unperturbed hamiltonian H_0.

In a simple case where H_0 is given by positive definite quadratic forms of momentum p, the parameters a and b satisfy,[4)]

$$a = b. \tag{4·3}$$

It can be shown that Eq. (1·6) is consistent with Eq. (4·2), which describes the stagnant motions.[4),6)] Denote the depth of the stagnant layer by ϵ_0^a, then the invariant measure proportional to the phase volume for the stagnant motions can be given by $(\epsilon/\epsilon_0)^{a(n-1)}$, and the probability for the stagnant orbit to be trapped in the region $|P(t) - P(0)| < \epsilon^a$ satisfies,

$$\begin{aligned} 1 - (\epsilon/\epsilon_0)^{a(n-1)} &= F_{\mathrm{UL}}(T_\epsilon; \beta) \\ &= \exp\left[-B(\ln(1 + T_\epsilon))^{-\beta}\right]. \end{aligned} \tag{4·4}$$

Therefore, the scaling forms for two cases, (i) $\epsilon \to 0$ and (ii) $\epsilon \to \infty$ $(\epsilon_0 \gg \epsilon)$ are derived as follows,
(i) $\epsilon \to 0$ (i.e., $T_\epsilon \to \infty$)

$$\begin{aligned} [\ln(1 + T_\epsilon)]^{-\beta} &\simeq -\ln[1 - (\epsilon/\epsilon_0)^{a(n-1)}] \\ &= (\epsilon/\epsilon_0)^{a(n-1)}, \end{aligned}$$

namely,

$$T_\epsilon \simeq \exp\left[\epsilon^{-a(n-1)/\beta}\right]. \tag{4·5}$$

(ii) $\epsilon \to \epsilon_0$ (i.e., $T_\epsilon \to 0$)

$$\begin{aligned} [\ln(1 + T_\epsilon)]^{\beta} &\simeq \epsilon_0^{-a(n-1)} \ln[1 - (\epsilon/\epsilon)^{a(n-1)}]^{\epsilon_0^{a(n-1)}} \\ &= \epsilon_0^{-a(n-1)} \epsilon^{a(n-1)}, \end{aligned}$$

namely,

$$T_\epsilon \simeq O(\epsilon^{-a(n-1)/\beta}). \tag{4·6}$$

Equations (4·5) and (4·6) are the same as the Nekhoroshev bound, i.e., the indexes b and χ become,

$$\begin{aligned} b &= a(n-1)/\beta, \qquad \text{and} \\ \chi &= a(n-1)/\beta. \end{aligned} \tag{4·7}$$

Now we apply the scaling relations mentioned here to the case of clustering motions of our concerns, where the degrees of freedom n is equal the spacial dimension d since the cluster under consideration is almost axially symmetric ($d = 2$) or spherically symmetric ($d = 3$), and that the near integrability for one-particle motion is satisfied due to the conservation of angular momentum. Thus the index β satisfies, $\beta = d - 1 = \beta(d)$ if we assume the relation of Eq. (4·4). Therefore, the conjecture $\beta(3) \simeq 2$ is expected to hold in three-dimensional simulations.

§5. Discussion

In this paper we have discussed about new kinetic laws in cluster formation, especially the internal structure of an equilibrium cluster. However, another crucial problem is found in nonequilibrium aspects of clusters; clusters are not the passive entities which only adapt to the environment, but the active ones which also affect the environment through their own inherent internal structures.

The internal structure plays an important role in the destabilization or the collapse of the cluster. When the box confining the particles goes to infinity non-adiabatically, the equilibrium cluster turns to an unstable one. Then, the clustering particles begin to evaporate and finally the cluster disappears. The process recalls the inverse of nucleation. As shown in the previous paper,[1] small clusters do not have any core-parts which are described by the universal long tail distribution, so that the evaporation process can be determined only by the soft-dressing part described by the Negative Weibull distribution. In other words, the kinematic scaling exponent σ in Eq. (3·10) plays an essential role in determining the critical exponent γ in Eq. (1·10). The detailed mechanism will be discussed in the forthcoming paper together with the positive curvature effect to clustering particles.

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Appendix A
—— *Several Examples for $P_s(s)$ and $P_r(r)$* ——

Here we use a simple notation; $P_g(x) = ae^{-ax}$.

Case I $(P_c(y) = be^{-by}$; poissonian)

$$P_s(s) = ab(as+b)^{-2},$$
$$P_r(r) = ab\{a(1-r)+br\}^{-2}. \tag{A·1}$$

Case II $(P_c(y) = \Gamma(\alpha)x^{\alpha-1}e^{-x}$; Γ-distribution $(\alpha > 0))$

$$P_s(s) = a\alpha(1+as)^{-1-\alpha},$$
$$P_r(r) = a\alpha r^{\alpha-1}\{r+a(1-r)\}^{-1-\alpha}. \tag{A·2}$$

Case III $(P_c(y)$ is the Levy distribution $\quad (1 > \alpha > 0))$.

The Laplace transformation $\hat{P}_c(\lambda)$ is given by $\hat{P}_c(\lambda) = e^{-\lambda^\alpha}$, where the variable y is defined for $0 \le y$, and the stable index α should obey $0 < \alpha < 1$. This example is quite different from previous ones, as the mean value $\langle y \rangle$ is divergent (i.e., $\langle T_c \rangle = \infty$);

$$P_s(s) = \alpha a^\alpha s^{\alpha-1} e^{-(as)^\alpha},$$
$$P_r(r) = \alpha a^\alpha r^{-(\alpha+1)}(1-r)^{\alpha-1} e^{-a(1-r)^\alpha r^{-\alpha}}. \tag{A·3}$$

Appendix B
—— *Asymptotic Behavior of $\hat{F}_c(\lambda)$ and $\hat{P}_c(\lambda)$* ——

The steepest descent method enables us to get the scaling form ($\lambda \to \infty$) of $F_{\rm NW}(\lambda)$ and $F_{\rm UL}(\lambda)$. By putting $\varphi(x) = \lambda x + Ax^{-\alpha}$,

$$\varphi(x) \simeq \varphi(x_s) + \frac{1}{2!}\varphi''(x_s)(x - x_s)^2,$$
$$\varphi'(x_s) = \lambda - A\alpha x_s^{-\alpha-1} = 0, \qquad \text{(B·1)}$$

the Laplace transformation $\hat{F}_{\rm NW}(\lambda)$ is rewritten,

$$\begin{aligned}\hat{F}_{\rm NW}(\lambda) &\simeq \int_0^\infty e^{-\varphi(x)}dx \\ &= e^{-\varphi(x_s)}\int_0^\infty e^{-\frac{\varphi''(x_s)}{2}(x-x_s)^2}dx \\ &= \sqrt{\frac{2\pi}{A\alpha(\alpha+1)}}(A\alpha)^{\frac{\alpha+2}{2(\alpha+1)}}\lambda^{\frac{-\alpha-2}{2(\alpha+1)}}e^{-\varphi(x_s)}. \end{aligned} \qquad \text{(B·2)}$$

The scaling form of $\hat{F}_{\rm UL}(\lambda)$ is also obtained by replacing by $\varphi(x) = \lambda x + B[\ln(1+x)]^{-\beta}$, but it will be attainable more easily if we notice $x_s \to 0\,(\lambda \to \infty)$ and that $\ln(1 + x_s) \simeq x_s$, namely

$$\hat{F}_{\rm UL}(\lambda) = \hat{F}_{\rm NW}(\lambda), \quad (\lambda \to \infty) \qquad \text{(B·3)}$$

by changing the parameter $(A, \alpha) \to (B, \beta)$.

Appendix C
—— *The Clausius Type of Free Paths* ——

Consider N particles confined in a d-dimensional box, where the number density is ρ. The probability $P(r)dr$, for a particle to collide with another one after running the distance $(r, r + dr)$, is given by,

$$\begin{aligned}P(r)dr &= \frac{4\pi\delta^2}{N}\rho dr, \quad \text{(3-d)} \\ &= \frac{2\delta}{N}\rho dr. \qquad \text{(2-d)}\end{aligned} \qquad \text{(C·1)}$$

Therefore, the probability for the representative particle to collide within the region $(r \leq l)$, say $Q(l)$, becomes $Q(l) = \int_0^l P(r)dr$, and the distribution for the free path $F_l(l)$ satisfies,

$$\begin{aligned}1 - F_l(l) &= [1 - Q(l)]^{N-1} \\ &= \exp\left[-\frac{l}{\langle l\rangle}\right], \quad (N \to \infty)\end{aligned} \qquad \text{(C·2)}$$

in other words, the probability density for the free path $P_l(l)$ becomes,

$$P_l(l) = \frac{dF_l(l)}{dl} = \frac{1}{\langle l\rangle}\exp\left[-\frac{l}{\langle l\rangle}\right], \qquad \text{(C·3)}$$

where δ is the effective radius of a particle and

$$\langle l \rangle = \frac{\rho}{\pi}(2\pi\delta)^{d-1}. \quad (\text{for } d = 2, 3) \qquad (\text{C}\cdot 4)$$

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