Noisy Sine-Circle Map as a Model of Chaotic Phase Synchronization

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The sine-circle map driven by noise with bounded amplitude is shown to be a model exhibiting the characteristics of chaotic phase synchronization (CPS). Two different routes to CPS corresponding to the observation by Osipov et al. are also explained by the present model.

There is a wide class of chaotic oscillators which have the phase of oscillation as a well-defined dynamical variable, i.e., the phase variable can be taken as one of the phase space coordinates. In the coupled system of two such chaotic oscillators which are weakly nonidentical with each other, with a suitable coupling strength the transition to the chaotic phase synchronization[1)] (CPS) possibly takes place. In the CPS state, the phases of oscillation synchronize, but the amplitudes of the oscillators are chaotic each other. In the present paper, we propose a stochastic model exhibiting the CPS transition for a phenomenological understanding of CPS.

In the phase space, the attractor of weakly coupled system has a torus-like structure corresponding to the existence of the two phase variables. By taking a Poincaré section transversely across the faster phase variable of the torus-like attractor, the Poincaré map is introduced for the normalized phase difference θ_n and the vector r_n of its complemental variables as $\theta_{n+1} = F(\theta_n, r_n)$ and $r_{n+1} = G(\theta_n, r_n)$ where $F(\theta+1, r) = F(\theta, r)$ and $G(\theta+1, r) = G(\theta, r)$.[2)] By neglecting the time correlation of the chaotic component of r_n, it can be replaced by random noise, and then by assuming the simplest nonlinear form for $F(\theta_n, r_n)$ in θ_n, the following noisy sine-circle map is introduced as a model for CPS:

$$\theta_{n+1} = \theta_n - \frac{K}{2\pi}\sin 2\pi\theta_n + \varOmega + f_n \quad \mod 1, \tag{1}$$

where K and $\varOmega$ are control parameters and f_n is random noise that incorporates the modulation by chaotic amplitudes. Here we assume that f_n is noise uniformly distributed over an interval $[0, a]$ for simplicity. Note that the correlation between θ_n and

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r_n is also neglected and the noise term is restricted to additive one in Eq. (1). Equation (1) without noise term called the sine-circle map[3)] has been well-investigated as a typical model for the two-frequency systems. The only important point, here, is that f_n *is bounded*, which is required by the fact that the trajectory of the original dynamical system is confined in a bounded subset of the phase space. The boundedness of f_n brings about a clear transition from the phase desynchronized state to the phase synchronized state, which cannot be realized under unbounded noise such as Gaussian white noise.

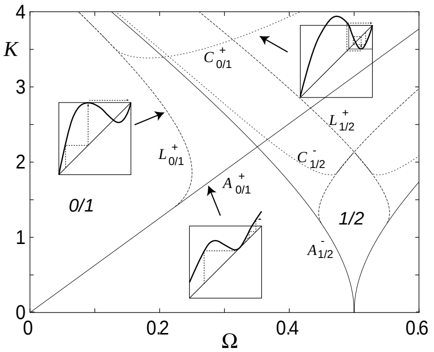

Fig. 1. The phase locking structure of the sine-circle map. The inserts illustrate the maps at the bifurcation points, where each dotted line illustrates the orbit with nonzero rotation number appearing after the bifurcation.

As an order parameter of the CPS, the rotation number is introduced as

$$\omega \equiv \lim_{n\to\infty} \frac{\theta_n - \theta_0}{n} \qquad (2)$$

without taking mod 1 in Eq. (1). Zero and none zero ω corresponds to the phase synchronized and phase desynchronized states, respectively. Figure 1 shows the phase locking structure of the sine-circle map ($f_n \equiv 0$), where an attractor with $\omega = 0$ appears on the line $A^+_{0/1}$ ($\Omega = \frac{K}{2\pi}$) by the tangent bifurcation and this attractor is destroyed by crisis on the line $C^+_{0/1}$. On the line $L^+_{0/1}$, the attractor with $\omega = 0$ remains but the orbit with $\omega \neq 0$ appears by crisis. The inserts in Fig. 1 illustrate the maps on the bifurcation lines.[4)] On the Ω-K plane, the noise term in Eq. (1) acts as a parameter deviation $(\Omega + f_n, K)$ with $f_n \in [0, a]$. If (Ω, K) is inside the phase locking region with $\omega = 0$, then $\omega = 0$ is expected for the orbit of Eq. (1) with small enough value of a and $\omega \neq 0$ is expected for sufficiently large a. Thus, in the following, we take the maximum value a of f_n as a control parameter with fixed Ω and K.[*)]

Figure 2 shows the behavior of rotation number ω for $K = 1.4$ and several values of Ω as a function of a, where the critical point a_c of CPS satisfying $a_c = \frac{K}{2\pi} - \Omega$ implies this transition to CPS is associated with a tangent bifurcation. The critical behavior at the onset of CPS[2),5)] i.e., the normal scaling

$$\omega \sim \sqrt{a - a_*} \quad \text{or} \quad a - a_*, \qquad (3)$$

depending on the values of Ω and a_c, for $a > a_*$ with $a_* > a_c$ in the region relatively away from the critical point a_c and the anomalous scaling

$$\omega \sim \exp(-\mathrm{const}/\sqrt{a - a_c}) \qquad (4)$$

for a slightly above a_c is observed before the appearance of CPS state $\omega = 0$ for $a < a_c$. The two types of behavior in the normal scaling region are understood by

*) The value of K, Ω, or some their combination can also be used as a control parameter, but the critical behavior is not affected by the choice of the control parameter.

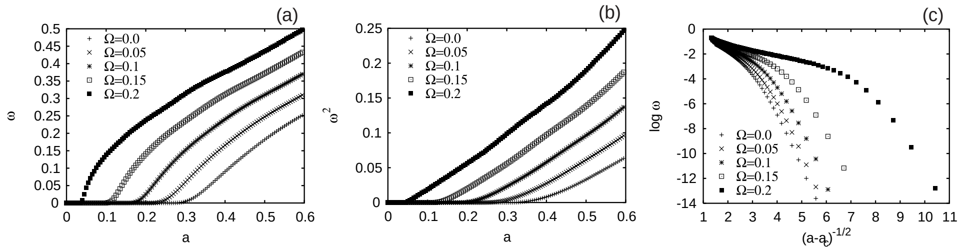

Fig. 2. The critical behavior of the rotation number ω as a function of a for $K = 1.4$ and $\Omega = 0, 0.05, 0.1, 0.15$, and 0.2, where $a_c = \frac{K}{2\pi} - \Omega$ is used. $\omega \sim \sqrt{a - a_*}$ and $\omega \sim a - a_*$ are suggested for $\Omega = 0.2$ and $\Omega = 0$, respectively.

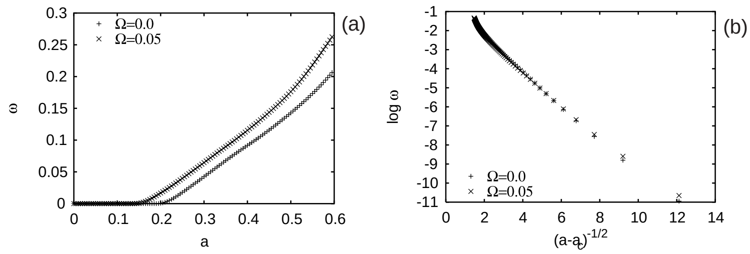

Fig. 3. The critical behavior of the rotation number around the onset of CPS due to crisis at $K = 3$ and $\Omega = 0$ and 0.05. Similar behavior as the case due to tangent bifurcation is observed.

considering the strength of the stochasticity of the chaotic modulation: If the stochasticity of the chaotic modulation is weak enough the tangent structure of the map is kept after the average over several time steps yielding the square root behavior. On the other hand, if the stochasticity of the chaotic modulation is strong enough the tangent structure of the map is averaged out and the parameter dependence becomes trivial. The anomalous scaling (4) can be theoretically explained in a similar manner as the unstable-unstable pair bifurcation[2),6)] despite the randomness of the modulation in the present system. Figure 2(c) is consistent with the theory.

For larger values of K such as $K = 3$, a transition to CPS due to crisis is observed. Figure 3 shows the behavior of ω for $K = 3$ and $\Omega = 0$ and 0.05, where the critical point a_c is determined such that $(\Omega + a_c, K)$ is on the line $L^+_{0/1}$. The scaling behavior

$$\omega \sim a - a_* \tag{5}$$

for $a > a_*$ and

$$\omega \sim \exp(-\mathrm{const}/\sqrt{a - a_c}) \tag{6}$$

for a slightly above a_c is confirmed. The anomalous scaling can be explained by considering the escape time by crisis and the probability that the state of finite escape time continues for the time interval as long as its escape time. Note that the inverse square root in Eq. (6) stems form the scaling of the escape time of crisis and it may change to another power for higher dimensional crisis.[7)] This transition to CPS due to crisis may correspond to the second one of the three types of CPS found by Osipov et al.[8)] and the normal scaling $\omega \sim a - a_*$ coincides with their result.

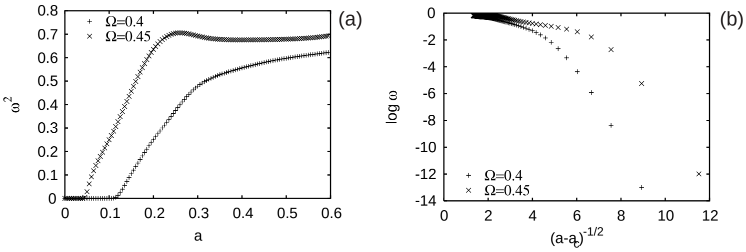

Fig. 4. The critical behavior of the rotation number around the onset of CPS due to tangent bifurcation at $K = 3$ and $\Omega = 0.4$ and 0.45 with a_c satisfying $a_c = \frac{K}{2\pi} - \Omega$.

At $K = 3$ a transition to CPS due to tangent bifurcation is also observed provided that the value of Ω is set close to $A^{+}_{0/1}$ such that the amplitude of modulation is small enough. Indeed, as shown in Fig. 4, similar behavior of the rotation number as the case of $K = 1.4$ is observed. Since the amplitude of modulation is kept small, only the scaling $\omega \sim \sqrt{a - a_*}$ is observed.

In summary, by introducing a noisy sine-circle map with bounded noise, two routes to CPS associated with tangent bifurcation and crisis, which may coincide the first two types among the three found by Osipov et al.,[8] are investigated. In the case of CPS by tangent bifurcation, the normal scaling of the rotation number is summarized as $\omega \sim \sqrt{a - a_*}$ or $a - a_*$, while only $\omega \sim a - a_*$ is observed in the case of crisis. In the anomalous scaling region, the same scaling law $\omega \sim \exp(-\mathrm{const}/\sqrt{a - a_c})$ holds for the both cases of tangent bifurcation and crisis. In applying this result for the case of crisis to coupled chaotic oscillators, the inverse square form in the exponent of the scaling law may need to be modified due to the dimensionality of crisis, which is not incorporated in the noisy 1d sine-circle map. The transition to CPS by crossing another crisis line $C^{+}_{0/1}$, which is not investigated in the present paper, may be a possible another route to CPS. The behavior of the phase diffusion coefficient is also a matter of the future investigation.

Finally, let us remark that the systems under bounded noise are worth to be investigated and useful for understanding nonlinear dynamics, and possibly show a variety of interesting phenomena, e.g., a bistable system driven by dichotomous noise[9] exhibits dynamical phase transition which can not be possible under unbounded noise such as Gaussian white noise.

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