Directed Network as a Chaotic Piece-Wise Linear One-Dimensional Map and Its Large-Deviation Properties

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A directed network such as the WWW can be represented by a transition matrix. Comparing this matrix to a Frobenius-Perron matrix of a chaotic piecewise-linear one-dimensional map whose domain can be divided into Markov subintervals, we are able to relate the network structure itself to chaotic dynamics. Just like various large-deviation properties of local expansion rates (finite-time Lyapunov exponents) related to chaotic dynamics, we can also discuss those properties of network structure. Recurrence time statistics and their relationships to double time correlation functions and to power spectra are also considered.

§1. Introduction

One of the most remarkable points about deterministic chaos is the duality consisting of irregular dynamics and fractal structure of the attractor in the phase space. Amplifying this relation between dynamics and geometry, we will try to construct dynamics corresponding to a directed network structure such as the WWW. A directed network or graph can be represented by a transition matrix. On the other hand, temporal evolution of a chaotic piecewise-linear one-dimensional map with Markov partition can be governed by a Frobenius-Perron matrix. Both transition matrices and Frobenius-Perron matrices belong to a class of transition matrices sharing the same mathematical properties. The maximum eigenvalue is equal to unity. The corresponding eigenvector is always a real vector, and evaluates the probability density of visiting a subinterval of the map or a site of the network, which is commercially valuable information in the field of the WWW.[1)] Relating these two matrices to each other, we are able to represent the structure of the directed network as a dynamical system. Once we relate the directed network to chaotic dynamics, several approaches to deterministic chaos can be also applied to graph theory.

In chaotic dynamical systems, the local expansion rates which evaluate an orbital instability fluctuate largely in time, reflecting a complex structure in the phase space. Its average is called the Lyapunov exponent, whose positive sign is a practical criterion of chaos. There exist numerous investigations based on large-deviation statistics in which one considers distributions of coarse-grained expansion rates (finite-time Lyapunov exponent) in order to extract large deviations caused by non-hyperbolicities or long correlations in the vicinity of bifurcation points.[2)]

In general, statistical structure functions consisting of weighted averages, variances, and these partition functions as well as fluctuation spectra of coarse-grained dynamic variables can be obtained by processing the time series numerically. In the case of the piecewise-linear map with Markov partition, we can obtain these structure functions analytically. This is one of the reasons why we correspond a directed network to a piecewise-linear map. We herein try to apply an approach based on large-deviation statistics in the research field of chaotic dynamical systems to network analyses. What is the Lyapunov exponent of the network? What becomes of fluctuations of the network Lyapunov exponent or other coarse-grained variables?

In §2, a one-dimensional piecewise-linear map corresponding to directed network is derived. In §3, large-deviation statistics of temporal fluctuation are reviewed. Based on these statistics, statistical structure functions and generalized spectral densities of directed network are obtained in §4. In §5, recurrence statistics are discussed. The final section is devoted to concluding remarks.

§2. One-dimensional map corresponding to directed network

We will consider the very simple example shown in Fig. 1. There exist two kinds of loops. One is a loop between node 1 and 2, the other a loop or a triangle 1, 3, 2. Let us define this adjacency matrix A, where A_{ji} is equal to unity if the node i is linked to j. If not, A_{ji} is equal to zero. Transition matrix H can be derived straightforwardly from the adjacency matrix. The element H_{ij} is equal to A_{ij} divided by the number of nonzero elements of column j. The transition matrix of the simple triangular graph mentioned before is explicitly given by the 3 by 3 matrix as

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{pmatrix}. \tag{2·1}$$

The maximum eigenvalue is always equal to unity. The corresponding eigenvector measures site importance in the context of the web network, which is commercially used as mentioned earlier.

In the case of chaotic dynamics caused by a one-dimensional map f, the trajectory is given by iteration. Its distribution at time n, $\rho_n(x)$, is given by the average of this delta function $\langle\delta(x_n - x)\rangle$, where $\langle\cdots\rangle$ denotes the average with respect to initial points x_0. The temporal evolution of ρ is given by the following relation $\rho_{n+1}(x) = \int_0^1 \delta(f(y) - x)\rho_n(y)\,dy \equiv \mathcal{H}\rho_n(x)$. This operator $\mathcal{H}$, called the Frobenius-Perron operator, is explicitly given as

$$\mathcal{H}G(x) = \sum_j \frac{G(y_j)}{|f'(y_j)|}, \tag{2·2}$$

where the sum is taken over all solutions $y_j(x)$ satisfying $f(y_j) = x$.

In the case of a piecewise-linear map with Markov partition, invariant density is constant for each interval. Taking these 3 functions as a basis, we can represent the Frobenius-Perron operator as this 3 by 3 matrix. This is nothing but the transition matrix of the directed graph consisting of 3 nodes mentioned before. The map f can be chosen as

$$f(x) = \begin{cases} 2x + 1/3, & (0 \le x < 1/3) \\ x - 1/3, & (1/3 \le x \le 1) \end{cases} \tag{2·3}$$

and is shown in Fig. 2.

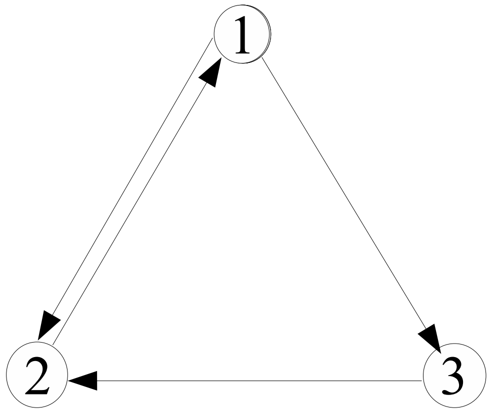

Fig. 1. Example.

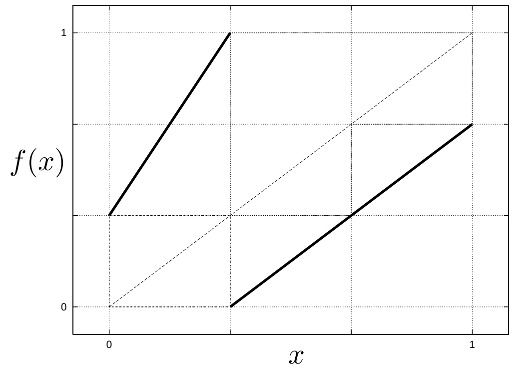

Fig. 2. One-dimensional map corresponding to the directed network shown in Fig. 1.

The Lyapunov exponent of the one-dimensional map is an average of the logarithm of the slope of the map with respect to its invariant density. Comparing the directed network, the map f, the matrix H, we find that the number of arrows originating from node i is equal to the slope of the interval I_i. Thus the Lyapunov exponent of the network is found to be an average of the logarithm of the number of arrows originating from each node. This exponent quantifies the complexity of link relations. Degree distribution is often used to characterize link relations. The network Lyapunov exponent and its fluctuation are also supposed to be useful in the context of the network. In this way, we can relate the network structure itself to a chaotic dynamical system, and we try to characterize the network based on an approach to deterministic chaos, namely large-deviation statistics, in other words, thermodynamical formalism.

§3. Large-deviation statistics

Let us briefly describe large deviation statistics following the series of studies by Fujisaka and his coworkers.[3),4)] Consider a stationary time series u. The average over time interval T is given by this formula, $\overline{u}_T(t) = \frac{1}{T}\int_t^{t+T} u\{s\}\,ds$, which distributes when T is finite. When T is much larger than the correlation time of u, the distribution of coarse-grained u is assumed to be an exponential form $P_T(u) \propto e^{-S(u)T}$. Here we can introduce fluctuation $S(u)$ as $S(u) = -\lim_{T\to\infty}\frac{1}{T}\log P_T(u)$. When T is comparable to the correlation time, correlation cannot be ignored, so non-exponential or non-extensive statistics will be a problem, but here we do not discuss this point any further. Let q be a real parameter. We introduce the generating function M_q of T by this definition: $M_q(T) \equiv \langle e^{qT\overline{u}_T}\rangle = \int_{-\infty}^{\infty} P_T(u)e^{qTu}\,du$. We can also here assume the exponential distribution and introduce characteristic function $\phi(q)$ as $\phi(q) = \lim_{T\to\infty}\frac{1}{T}\log M_q(T)$. The Legendre transform holds between fluctuation spectrum $S(u)$ and characteristic function $\phi(q)$, which is obtained from saddle-point calculations. $\frac{dS(u)}{du} = q, \quad \phi(q) = -S(u(q)) + qu(q)$. In this transform a derivative $d\phi/dq$ appears, and is a weighted average of $\overline{u}_T$, $u(q) = \frac{d\phi(q)}{dq} = \lim_{T\to\infty} \frac{\overline{u}_T e^{qT\overline{u}_T}}{M_q(T)}$, so we find that q is a kind of weight index. We can also introduce susceptibility $\chi(q) = \frac{du(q)}{dq}$ as a weighted variance. These statistical structure functions $S(u), \phi(q), u(q), \chi(q)$ constitute the framework of statistical thermodynamics of temporal fluctuation, which characterize the static properties of chaotic dynamics. In order to consider dynamic properties, we can introduce this generalized spectrum density as a weighted average of conventional spectrum density as $I_q(\omega) = \lim_{T\to\infty}\langle|\int_0^T[u\{t+s\}-u(q)]e^{-i\omega s}ds|^2 e^{qT\overline{u}_T}\rangle/(T\,M_q(T))$. In the same way, the generalized double time correlation function is $C_q(t) = \lim_{T\to\infty}\lim_{\tau\to\infty}\left\langle(u\{t+\tau\}-u(q))(u\{\tau\}-u(q))e^{qT\overline{u}_T}\right\rangle$ $/M_q(T)$. The relation between the two is given by the Wiener-Khintchine theorem: $C_q(t) = \int_{-\infty}^{\infty} I_q(\omega)e^{-i\omega t}d\omega/(2\pi)$ and $I_q(\omega) = \sum_{t=-\infty}^{\infty} C_q(t)e^{i\omega t}$.

Let us consider the case of a one-dimensional map. Let $u[x_n]$ be a unique function of x, which is governed by the map $x_{n+1} = f(x_n)$. The question is how to obtain statistical structure functions and generalized spectral densities of u. The answer is to solve the eigenvalue problems of a generalized Frobenius-Perron operator. As we mentioned before, the characteristic function $\phi(q)$ is given by the asymptotic form of the generating function $M_q(n)$ in the limit of $n\to\infty$ corresponding to the temporal coarse-grained quantity $\overline{u}_n = \frac{1}{n}\sum_{j=0}^{n-1} u[x_{j+m}]$, where we assume an exponential fast decay of time correlations of u. A generating function can be expressed in terms of invariant density as $M_q(n) = \int \rho_\infty(x)\exp\left[q\sum_{j=0}^{n-1}u[f^j(x)]\right]dx = \int \mathcal{H}_q^n \rho_\infty(x)dx$, where the generalized Frobenius-Perron operator $\mathcal{H}_q$ is defined and related to the original one as $\mathcal{H}_q G(x) = \mathcal{H}[e^{qu[x]}G(x)] = \sum_k \frac{e^{qu[x]}G(y_k)}{|f'(y_k)|}$ for an arbitrary function $G(x)$ ($\mathcal{H}_0 = \mathcal{H}$). To obtain the above equation, the following relation is repeatedly used: $\mathcal{H}\left\{G(x)\left[q\sum_{j=0}^{m}u[f^j(x)]\right]\right\} = (\mathcal{H}_q G(x))\left[q\sum_{j=0}^{m-1}u[f^j(x)]\right]$.

Let $\nu_q^{(0)}$ be the maximum eigenvalue of H_q. The characteristic function is given by its logarithm as $\phi(q) = \log\nu_q^{(0)}$. The weighted average $u(q)$ and the susceptibility $\chi(q)$ are given by the first and the second derivatives of $\phi(q)$.

The generalized spectral density is given by $I_q(\omega) = \int v_{(0)}(x)[u[x]-u(q)][J_q(\omega)+J_q(-\omega)-1][u[x]-u(q)]h^{(0)}(x)dx$, where $J_q(\omega) = 1/\left[1-(e^{i\omega}/\nu_q^{(0)})H_q\right]$, $v^{(0)}(x)$ and $h^{(0)}(x)$ are respectively the left and right eigenfunctions corresponding to the maximum eigenvalue $\nu_q^{(0)}$ of H_q. The generalized double time correlation function is given by $C_q(t) = \int v^{(0)}(x)[u[x]-u(q)][H_q/\nu_q^{(0)}]^t[u[x]-u(q)]h^{(0)}(x)dx$.

The normal Frobenius-Perron operator $\mathcal{H}$ depends on the map f only. The generalized one $\mathcal{H}_q$ depends also on a dynamic variable u and determines statistical structure functions and generalized spectral densities of u. For example, in the case of local expansion rates $u[x] = \log|f'(x)|$ whose average is the Lyapunov exponent, the generalized operator is explicitly given by

$$\mathcal{H}_q G(x) = \sum_k \frac{G(y_k)}{|f'(y_k)|^{1-q}}. \tag{3·1}$$

In the case of the triangular network mentioned earlier, three subintervals constitute the Markov partition, such that $\mathcal{H}_q$ can be represented by a 3×3 matrix as

$$H_q = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{qu(I_1)} & 0 & 0 \\ 0 & e^{qu(I_2)} & 0 \\ 0 & 0 & e^{qu(I_3)} \end{pmatrix} \tag{3·2}$$

in the same way as $\mathcal{H}$.

§4. Statistical structure functions and generalized spectral densities of directed network

Let us analyze the triangular directed network based on the large-deviation statistics. As a dynamic variable we choose the local expansion rates (the logarithmic number of arrows originating from each nodes) $u(I_i) = \log(|f'(I_i)|)$, so that we have $u(I_1) = \log 2$, $u(I_2) = 0$, $u(I_3) = 0$.

In any choice of a dynamic variable u, right and left eigenvectors corresponding to eigenvalue 1 of the aforementioned Frobenius-Perron matrix H are determined. The left eigenvector is given by $(1/3, 1/3, 1/3)$, where it is so normalized that the sum of all elements is equal to unity. Note that the element is equal to the width of the subinterval of the Markov partition. The right eigenvector gives the probability density to visit each subinterval, and is equal to $(6/5, 6/5, 3/5)$, where it is so normalized that the inner product of the right and the left eigenvectors is equal to unity. The generalized Frobenius-Perron matrix H_q can be represented as

$$H_q = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{q\log 2} & 0 & 0 \\ 0 & e^{q\log 1} & 0 \\ 0 & 0 & e^{q\log 1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2^{q-1} & 0 & 1 \\ 2^{q-1} & 0 & 0 \end{pmatrix}. \tag{4·1}$$

The statistical structure functions $\phi(q)$, $u(q)$, $\chi(q)$, and $S(u)$ are obtained from the maximum eigenvalue of H_q and are shown in Figs. 3 – 6. The weighted average $u(q)$ has two asymptotes. The upper asymptote $u(+\infty) = (\log 2 + \log 1)/2 = (\log 2)/2$ and the lower one $u(-\infty) = (\log 2 + \log 1 + \log 1)/3 = (\log 2)/3$ correspond to period-2 and period-3 loops, respectively. This can be regarded as a kind of phase transition from period-2 loop phase to period-3. The mean logarithmic number of arrows originating from each node is given by $u(0)$, which is also obtained from the right eigenvalue (the probability density) corresponding to the eigenvalue 1 of $H = H_0$ as $\frac{1}{3} \times \frac{6}{5} \times \log 2 + \frac{1}{3} \times \frac{6}{5} \times \log 1 + \frac{1}{3} \times \frac{3}{5} \times \log 1 = \frac{2}{5}\log 2$. The average number of arrows originating from each node corresponds to $\exp(u(0))$. Its value is equal to $2^{2/5} \simeq 1.3$, which lies between 1 and 2 as expected. Period 2 and 3 trajectories are $(0,0) \to (0, 1/3+0) \to (1/3+0, 1/3+0) \to (1/3+0, 0) \to (0,0)$ and $(1/3-0, 1/3-0) \to (1/3-0, 1) \to (1,1) \to (2/3, 1) \to (2/3, 2/3) \to (2/3, 1/3-0) \to (1/3-0, 1/3-0)$, where $f(1/3-0) = 1$ and $f(1/3+0) = 0$. One can trace these trajectories along the grid lines shown in Fig. 2.

The fluctuation spectrum $S(u)$ (the line in Fig. 6) can be defined between two values given by the asymptotes of the weighted average $0.23 \simeq (\log 2)/3 \leq u \leq$

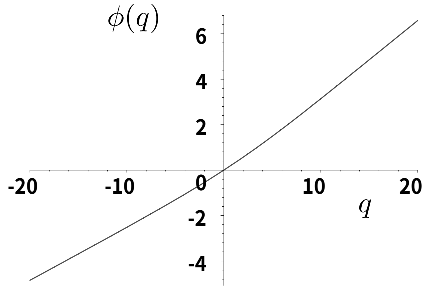

Fig. 3. Characteristic function $\phi(q)$.

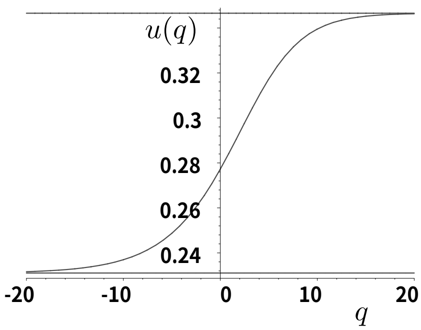

Fig. 4. Weighted average $u(q)$.

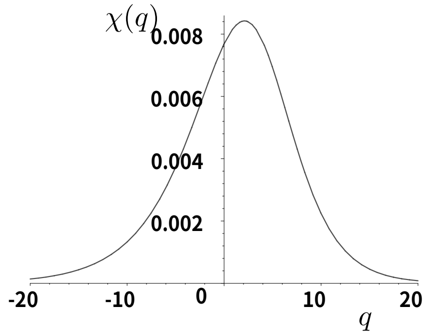

Fig. 5. Weighted variance $\chi(q)$.

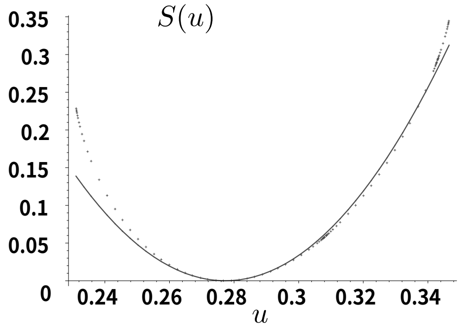

Fig. 6. Fluctuation spectrum $S(u)$ (line) and parabola indicating the central limit theorem (symbol).

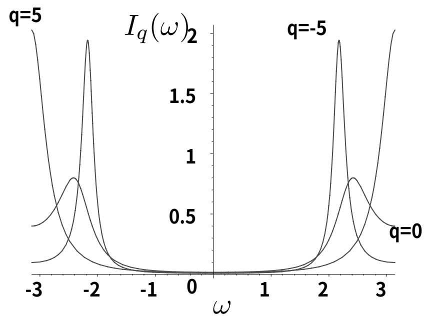

Fig. 7. Generalized spectral density $I_q(\omega)$ ($q = -5, 0, 5$).

$(\log 2)/2 \simeq 0.35$. Expanding the spectrum around the average up to the quadratic term, we have a parabola (the symbols in Fig. 6) indicating the central limit theorem. Large deviation statistics obviously do not coincide with the central limit theorem. The generalized spectral densities $I_q(\omega)$ for $q = -5, 0, 5$ are shown in Fig. 7. As q goes to $+\infty$, the density has a sharp peak at $\omega = \pi$, which corresponds to the period-2 loop. In the same way, As q goes to $-\infty$, the density has a sharp peak at $\omega = 2\pi/3$, which corresponds to the period-3 loop. There is nothing remarkable about the normal spectral density $I_0(\omega)$.

§5. Recurrence time statistics

One of the other choices of a dynamic variable is $u(I_i) = \delta_{il}$, which implies that the variable is equal to unity if the specific node l is visited. If not, it is zero. In this case, the recurrence time to visit a specific interval or node is the time span between the succeeding times at which $u = 1$ is satisfied.

Let $\psi_l(t)$ be the probability density function of the recurrence time of the node l. For nodes 1 and 2, only two and three time steps are allowed, so that we have $\psi_1(t) = \psi_2(t) = (\delta_{2\,t} + \delta_{3\,t})/2$. For node 3, three or a larger odd number of steps are allowed, which is caused by the bouncing motion between nodes 1 and 2, so that we have $\psi_3(t) = (1/2)^{(t-1)/2}$ $(t = 3, 5, 7, \cdots)$ and $\psi_3(t) = 0$ $(t = 0, 1, 2, 4, 6, \cdots)$. The conventional double time correlation time $C_0^{(l)}(t)$ for the dynamic variable $u(I_i) = \delta_{il}$ can be expressed in terms of $\psi_l(t)$ as $C_0^{(l)}(1) = \psi_l(1)$, $C_0^{(l)}(2) = \psi_l(2) + \psi_l(1)^2 = \psi_l(2) + C_0^{(l)}(1)\psi_l(1)$, $\cdots$,

$$C_0^{(l)}(m) = \sum_{k=0}^{m} C_0^{(l)}(m-k)\psi_l(k) + \delta_{m\,0}, \tag{5·1}$$

where the correlation function is normalized in such a way that $C_0^{(l)}(0) = 1$ is satisfied.[5] It can be straightforwardly shown that $C_0^{(1)}(t) = C_0^{(2)}(t) = C_0^{(3)}(t) = (1/2)^{t/2}\cos(3\pi\, t/4)$. It can be shown that $C_0^{(1)}(m) = \sum_{k=0}^{m} C_0^{(1)}(m-k)\psi_1(k) + \delta_{m\,0}, C_0^{(2)}(m) = \sum_{k=0}^{m} C_0^{(2)}(m-k)\psi_2(k) + \delta_{m\,0}$. For even m, we have $\sum_{k=0}^{m} C_0^{(3)}(m-k)\psi_3(k) + \delta_{m\,0} = C_0^{(3)}(m)$. For odd m, $\sum_{k=0}^{m} C_0^{(3)}(m-k)\psi_3(k) + \delta_{m\,0} = C_0^{(3)}(m) + 1/\sqrt{2}$. In this case, the convolution relation for a fixed time m does not exactly hold, since recurrence times longer than m always exist.

In the long time limit, we can pass on to continuous time. The Laplace transform of the convolution relation yields

$$\hat{C}_0(s) = \frac{1}{1 - \hat{\psi}(s)} \tag{5·2}$$

with $\hat{f}(s) = \int_0^\infty dt\, f(t)e^{-st}$, so that we have

$$I_0(\omega) = \frac{\hat{C}_0(i\omega) + \hat{C}_0(-i\omega)}{2} = \frac{1}{2}\left(\frac{1}{1 - \hat{\psi}(i\omega)} + \frac{1}{1 - \hat{\psi}(-i\omega)}\right). \tag{5·3}$$

If the recurrence time distribution is given by a power law, the correlation decay also obeys a power law. The relationship between the two power-law indices is obtained from the above formula. Such a power law will be observed in more complex networks such as a scale-free network. Furthermore, the average recurrence time $\bar{t} = \int_0^\infty t\,\psi(t)\,dt$ can be alternatively used as a measure of dynamically observed network size instead of average path length which is geometrically defined.

§6. Concluding remarks

We introduced a way of relating a directed network to chaotic dynamics. Relating the directed network to a chaotic dynamical system, and applying the formalism of statistical thermodynamics to it, we can extract important loops individually and elucidate the fluctuations of arrows originating from each node.

The reason why the author insists on deterministic chaos is that chaos has a rigid skeleton of unstable periodic orbits and it determines the distribution of the coarse-grained variables concerned. Stochastic processes such as random walk or Brownian motion have no skeleton of unstable periodic orbits. In the preceding section, we discussed the recurrence time, which is nothing but one of the periods of the unstable periodic orbits. The author will try to express large deviation statistics of directed networks in terms of these periodic orbits in the future.

In the case of chaotic dynamics, q of thermodynamical formalism is merely a weight index to process time series. This index q can be used to control traffic in the context of the web network.

Here we illustrated an idea which relates a directed network to chaotic dynamics using a very simple example consisting of three nodes. What becomes of more complicated networks? Dynamics corresponding to the small world are thought to need relatively short periodic orbits only. For a scale-free network, longer periodic orbits will take an important role, strongly correlated dynamics will appear, and recurrence time statistics will have interesting properties. Investigation in this direction is a future problem. Furthermore, random matrix theory of conventional and generalized transition matrices must play an important role.

Acknowledgements

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References

1) L. Page, S. Brin, R. Motwani and T. Winograd, The Page Rank Citation Ranking: Bringing Order to the Web (1998), `http://www-db.stanford.edu/ backrub/pageranksub.ps`
2) H. Mori and Y. Kuramoto, *Dissipative structures and chaos* (Springer, Berlin, 1998).
3) H. Fujisaka and M. Inoue, Prog. Theor. Phys. **77** (1987), 1334; Phys. Rev. A **39** (1989), 1376.
4) H. Fujisaka and H. Shibata, Prog. Theor. Phys. **85** (1991), 187.
5) H. G. Schuster, *Deterministic Chaos* (VCH, Weinheim, 1988), Chap. 4.