A Manifestly Covariant Hamiltonian Formalism for Dynamical Geometry

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A frame independent, manifestly 4-covariant, reformulation of our covariant Hamiltonian formalism for dynamical geometry is presented. The validity of all the steps in the derivation of the Hamiltonian and the exact meaning of the quasi-local Hamiltonian boundary term expressions for the energy-momentum and angular momentum are thereby clarified.

§1. Introduction

Previously[1)–9)] we had developed a 4-covariant Hamiltonian formalism for dynamical geometry. All the principles were well explained in those works, however, due to using the components of the connection and a frame as the independent fields, the reference frame independent meaning of certain quantities and the validity of one or two steps in the argument leading to the quasi-local Hamiltonian boundary term expressions are not so apparent. Here we succinctly present a reformulation: a frame independent, manifestly covariant Hamiltonian formulation for dynamical geometry, which clarifies the meaning of the expressions.

§2. Our usual formulation

We find it convenient to work with fields which are differential forms, both because forms facilitate a "covariant" space-time split of derivatives and because of the boundary theorem,

$$\int_\Sigma \mathrm{d}\alpha = \oint_{\partial\Sigma} \alpha. \tag{2·1}$$

In our aforementioned work, for the dynamic geometric *potentials* we used the metric components, the *co-frame* one-form and the *connection* one-form *components*:

$$g_{\alpha\beta}, \qquad \vartheta^\alpha, \qquad \Gamma^\alpha{}_\beta. \tag{2·2}$$

The respective field strengths are the *non-metricity* one-form, the *torsion* 2-form and the *curvature* 2-form:

$$-Q_{\mu\nu} := Dg_{\mu\nu} := \mathrm{d}g_{\mu\nu} - \Gamma^\lambda{}_\mu g_{\lambda\nu} - \Gamma^\lambda{}_\nu g_{\mu\lambda}, \tag{2·3}$$

$$T^\alpha := D\vartheta^\alpha := \mathrm{d}\vartheta^\alpha + \Gamma^\alpha{}_\beta \wedge \vartheta^\beta, \tag{2·4}$$

$$R^\alpha{}_\beta := D\Gamma^\alpha{}_\beta := \mathrm{d}\Gamma^\alpha{}_\beta + \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta. \tag{2·5}$$

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This formulation facilitates regarding gravity as a gauge theory of local spacetime symmetry. The co-frame and connection one-forms are the gauge "vector potentials" for the "translations" and "rotations". Such a gauge approach to gravity enables a unified view of the physical interactions.[10),11)]

The Hamiltonian for evolving dynamic geometry within a spatial region Σ along the displacement vector field N was found to have the form

$$H(N,\Sigma) = \int_{\Sigma} \mathcal{H}(N), \tag{2·6}$$

where the 3-form integrand,

$$\mathcal{H}(N) = N^{\mu}\mathcal{H}_{\mu} + \tilde{D}_{\beta}N^{\alpha}\mathcal{H}_{\alpha}{}^{\beta} + \mathrm{d}\mathcal{B}(N), \tag{2·7}$$

included a total differential that leads to an integral over the boundary 2-surface $S = \partial\Sigma$. The value of the Hamiltonian, which is determined by the boundary term, gives the various quasi-local quantities (energy-momentum, angular momentum/center-of-mass). For each choice of boundary conditions there is a specific associated Hamiltonian boundary term. We identified several particular boundary term expressions, each is associated with certain boundary conditions (e.g., Dirichlet, Neumann), and we also found the associated quasi-local flux expressions. For Einstein's general relativity (GR) a particular boundary term was identified:

$$\mathcal{B}(N) = \Delta\Gamma^{\alpha}{}_{\beta} \wedge i_N\eta_{\alpha}{}^{\beta} + \bar{D}_{\beta}N^{\alpha}\Delta\eta_{\alpha}{}^{\beta}; \tag{2·8}$$

it is distinguished by directly giving the Bondi energy flux and by having an associated positive energy proof.[8),9)]

The virtue of this gauge approach naturally has a price: one must necessarily have dynamic equations for inherently gauge dependent, non-covariant objects: the metric, frame and connection components. Then covariance cannot be manifest. That the final resulting expressions obtained actually describe the desired proper physical covariant meaning is not manifestly obvious (e.g., the argument concerning Eq. (24) in Ref. 4) may leave some lingering doubts; also the formalism lacks clarity in exactly how to calculate $\Delta\Gamma^{\alpha}{}_{\beta}$).

Here a manifestly covariant version is presented. The benefit lies in clarifying the exact meaning of certain ideas, quantities, and expressions. A spinoff is that some mathematical techniques for manifestly covariant calculations are developed.

§3. Invariant foundations

We consider quite general geometries,[12)] with *a priori* independent *metric* g and *covariant derivative* ∇. The *torsion* tensor (a vector valued 2-form) is

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]\,, \tag{3·1}$$

and the *curvature* tensor is

$$R(Z,X,Y) := R_{XY}Z\,, \tag{3·2}$$

where the linear-operator-valued *curvature operator* 2-form is

$$R_{XY} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \,. \tag{3·3}$$

It is very convenient to generalize d, the exterior differential (which acts on *scalar* valued forms), to an *exterior covariant differential*[1),13)] which acts on *tensor valued forms* just by formally replacing $\mathrm{d} \to \nabla$. Thus, for a tensor-valued one-form

$$(\nabla\alpha)(X,Y) = i_X \nabla i_Y \alpha - i_Y \nabla i_X \alpha - i_{[X,Y]}\alpha \tag{3·4}$$

(here $i_X\beta := \beta(X,\dots)$ is the *interior product*).

For example for the the *identity* vector-valued one-form,

$$I(X) = X, \tag{3·5}$$

one finds the torsion:

$$\begin{aligned}(\nabla \wedge I)(X,Y) &:= i_X \nabla(i_Y I) - i_Y \nabla(i_X I) - i_{[X,Y]} I \\ &\equiv \nabla_X Y - \nabla_Y X - [X,Y] \equiv T(X,Y) \,.\end{aligned} \tag{3·6}$$

For the vector valued one-form ∇Z,

$$\begin{aligned}(\nabla \wedge \nabla Z)(X,Y) &:= i_X \nabla(i_Y \nabla Z) - i_Y \nabla(i_X \nabla Z) - i_{[X,Y]} \nabla Z \\ &\equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \equiv R(X,Y)Z \,,\end{aligned} \tag{3·7}$$

one gets the curvature. These results can be compactly expressed as

$$T \equiv \nabla \wedge I \,, \tag{3·8}$$

$$R(\,.\,,\,.\,)Z \equiv \nabla \wedge \nabla Z \,. \tag{3·9}$$

Thus ∇ gives a powerful and succinct notation.

§4. Invariant Lie derivatives

The *transpose connection* (it has the property $\tilde{T} = -T$) is defined by

$$\tilde{\nabla}_X Y := \nabla_Y X - \pounds_Y X \equiv \nabla_Y X - [Y,X] \equiv \nabla_X Y - T(X,Y) \,. \tag{4·1}$$

Using it one can express the Lie derivative in terms of the covariant connection. In particular for the metric,

$$\begin{aligned}(\pounds_Z g)(X,Y) &:= \pounds_Z\,(g(X,Y)) - g(\pounds_Z X, Y) - g(X, \pounds_Z Y) \equiv \nabla_Z\,(g(X,Y)) - \dots \\ &\equiv (\nabla_Z g)(X,Y) + g(\nabla_Z X - \pounds_Z X, Y) + g(X, \nabla_Z Y - \pounds_Z Y) \\ &\equiv (\nabla_Z g)(X,Y) + g(\tilde{\nabla}_X Z, Y) + g(X, \tilde{\nabla}_Y Z) \,.\end{aligned} \tag{4·2}$$

Concisely,

$$\pounds_Z g = \nabla_Z g + g(\tilde{\nabla} Z,\ ) + g(\ , \tilde{\nabla} Z) \,. \tag{4·3}$$

Now consider the Lie derivative of the *connection*:

$$\begin{aligned}(\pounds_Z\nabla)_X Y :&= \pounds_Z(\nabla_X Y) - \nabla_{\pounds_Z X}Y - \nabla_X \pounds_Z Y \\ &\equiv R(Z,X)Y + \nabla_X(\nabla_Z Y - \pounds_Z Y) - [\nabla_Z(\nabla_X Y) - \pounds_Z(\nabla_X Y)] \\ &\equiv R(Z,X)Y + \nabla_X \tilde{\nabla}_Y Z - \tilde{\nabla}_{\nabla_X Y} Z \\ &\equiv R(Z,X)Y + \langle \nabla\tilde{\nabla}Z | X, Y\rangle \,. \end{aligned} \tag{4·4}$$

Concisely,

$$\pounds_Z \nabla \equiv i_Z R + \nabla\tilde{\nabla}Z \,. \tag{4·5}$$

For a tensor-valued form S one finds a result of the form

$$\pounds_Z S = \nabla_Z S + \text{certain terms linear in both } S \text{ and } \tilde{\nabla}Z \,. \tag{4·6}$$

§5. First-order Lagrangian 4-form

We find it convenient to use a *first-order* Lagrangian approach. For example, for the Maxwell field,

$$\mathcal{L} = \mathrm{d}A \wedge H + \frac{1}{2} * H \wedge H \,, \tag{5·1}$$

has the variation

$$\delta\mathcal{L} \equiv \mathrm{d}\delta A \wedge H + (\mathrm{d}A + *H) \wedge \delta H \tag{5·2}$$

$$\equiv \mathrm{d}(\delta A \wedge H) + \delta A \wedge \mathrm{d}H + (\mathrm{d}A + *H) \wedge \delta H \tag{5·3}$$

$$\equiv \mathrm{d}(\delta A \wedge H) + \delta A \wedge \frac{\delta\mathcal{L}}{\delta A} + \frac{\delta\mathcal{L}}{\delta H} \wedge \delta H \,. \tag{5·4}$$

Hamilton's principle, requiring the action (the integral of the Lagrangian) to have an extreme with A fixed on the boundary, gives the pair of *first order* field equations:

$$0 = \frac{\delta\mathcal{L}}{\delta A} = \mathrm{d}H \,, \qquad 0 = \frac{\delta\mathcal{L}}{\delta H} = \mathrm{d}A + *H \,, \tag{5·5}$$

which are equivalent to the vacuum Maxwell equations. These equations can be split into their dynamic and initial value constraint pieces by "spatial pullback" and "contracting" with a timelike vector field N. These same dynamic and constraint equations can be derived from a Hamiltonian 3-form simply constructed via

$$i_N\mathcal{L} \equiv i_N \mathrm{d}A \wedge H + \mathrm{d}A \wedge i_N H + i_N(*H \wedge H) \tag{5·6}$$

$$\equiv \pounds_N A \wedge H - \mathrm{d}i_N A \wedge H + \mathrm{d}A \wedge i_N H + i_N(*H \wedge H) \tag{5·7}$$

$$\equiv \pounds_N A \wedge H - \mathcal{H} \,. \tag{5·8}$$

§6. Invariant first-order Lagrangian 4-form for dynamic geometry

A first-order Lagrangian for a dynamic geometry can always be put in the form

$$\mathcal{L} = \nabla g \wedge \pi + T \wedge \tau + R \wedge \rho - \Lambda(g, \pi, \rho, \tau, \lambda) \,. \tag{6·1}$$

Here $\mathcal{L}$ is a scalar valued 4-form, ∇g is a type $\binom{0}{2}$ one-form and π is a type $\binom{2}{0}$ 3-form. Since T is a vector valued 2-form, τ must be a co-vector valued 2-form. R is a linear-operator-valued 2-form, and likewise ρ. λ stands for some possible Lagrange multiplier fields. Appropriate contractions are implied.

We need the variations of certain quantities. First note that

$$\delta(\nabla_Z g)(X,Y) = (\nabla_Z \delta g)(X,Y) - g(\delta\nabla_Z X, Y) - g(X, \delta\nabla_Z Y)\,. \tag{6·2}$$

This can be concisely written as

$$\delta\nabla g = \nabla\delta g - g(\delta\nabla\ ,\ ) - g(\ ,\delta\nabla\ )\,. \tag{6·3}$$

Consider two connections: $\nabla' = \nabla + K$. Then

$$R'(X,Y)Z = R(X,Y)Z + [K_X, K_Y]Z + \langle \nabla \wedge K | (X,Y)\rangle Z\,. \tag{6·4}$$

Briefly, from (3·3,3·9) with $K = \delta\nabla$,

$$\delta R = \delta(\nabla\wedge\nabla) = \nabla\wedge\delta\nabla\,. \tag{6·5}$$

For the torsion from the definition (3·1) and (3·6,3·8)

$$\delta T(X,Y) = \delta\nabla_X Y - \delta\nabla_Y X\,; \tag{6·6}$$

succinctly,

$$\delta T = \delta(\nabla\wedge I) = \delta\nabla\wedge I\,. \tag{6·7}$$

Now compute the variation of the Lagrangian 4-form:

$$\begin{aligned}
\delta\mathcal{L} &\equiv \delta(\nabla g)\wedge\pi + \nabla g\wedge\delta\pi + \delta T\wedge\tau + T\wedge\delta\tau + \delta R\wedge\rho + R\wedge\delta\rho - \delta\Lambda\\
&\equiv [\nabla\delta g - g(\delta\nabla\ ,\ ) - g(\ ,\delta\nabla\ )]\wedge\pi + \nabla g\wedge\delta\pi\\
&\quad +(\delta\nabla\wedge I)\wedge\tau + T\wedge\delta\tau + (\nabla\wedge\delta\nabla)\wedge\rho + R\wedge\delta\rho - \delta\Lambda &(6·8)\\
&\equiv \mathrm{d}\,[\delta g\pi + \delta\nabla\wedge\rho] - \delta g\nabla\pi - [g(\delta\nabla\ ,\ ) + g(\ ,\delta\nabla\ )]\wedge\pi + \nabla g\wedge\delta\pi\\
&\quad +(\delta\nabla\wedge I)\wedge\tau + T\wedge\delta\tau + \delta\nabla\wedge\nabla\rho + R\wedge\delta\rho - \delta\Lambda &(6·9)\\
&\equiv \mathrm{d}\,[\delta g\pi + \delta\nabla\wedge\rho]\\
&\quad +\delta g\frac{\delta\mathcal{L}}{\delta g} + \frac{\delta\mathcal{L}}{\delta\pi}\delta\pi + \frac{\delta\mathcal{L}}{\delta\tau}\wedge\delta\tau + \delta\nabla\wedge\frac{\delta\mathcal{L}}{\delta\nabla} + \frac{\delta\mathcal{L}}{\delta\rho}\wedge\delta\rho + \delta\lambda\frac{\delta\mathcal{L}}{\delta\lambda}\,. &(6·10)
\end{aligned}$$

6.1. *Field equations*

The variational derivatives—the vanishing of which, according to Hamilton's principle, are the vacuum field equations—are

$$\frac{\delta\mathcal{L}}{\delta g} := -\nabla\pi - \frac{\partial\Lambda}{\partial g}\,, \tag{6·11}$$

$$\frac{\delta\mathcal{L}}{\delta\pi} := \nabla g - \frac{\partial\Lambda}{\partial\pi}\,, \tag{6·12}$$

$$\frac{\delta\mathcal{L}}{\delta\tau} := T - \frac{\partial\Lambda}{\partial\tau}\,, \tag{6·13}$$

$$\frac{\delta\mathcal{L}}{\delta\nabla} := \nabla\rho + I\wedge\tau - g\pi - g\pi\,, \tag{6·14}$$
$$\frac{\delta\mathcal{L}}{\delta\rho} := R - \frac{\partial\varLambda}{\partial\rho}\,, \tag{6·15}$$
$$\frac{\delta\mathcal{L}}{\delta\lambda} := -\frac{\partial\varLambda}{\partial\lambda}\,. \tag{6·16}$$

6.2. *Diffeomorphism invariance*

Diffeomorphism invariance requires that the Lagrangian depend on position only through the fields, consequently the general variational relation (6·10) must become an identity under the substitution $\delta \to \pounds_Z$:

$$\mathrm{d}i_Z\mathcal{L} \equiv \pounds_Z\mathcal{L} \equiv \mathrm{d}[\pounds_Z g\pi + \pounds_Z\nabla\wedge\rho] + [\text{terms proportional to field eqns}]. \tag{6·17}$$

This naturally identifies (unique up to a total differential) the *translation current*:

$$\begin{aligned}\mathcal{H}[Z] &:= \pounds_Z g\pi + \pounds_Z\nabla\wedge\rho - i_Z\mathcal{L} &(6·18)\\ &\equiv [\nabla_Z g + g(\tilde{\nabla}Z,\ ) + g(\ ,\tilde{\nabla}Z)]\pi\\ &\quad +(i_Z R + \nabla\tilde{\nabla}Z)\wedge\rho - i_Z[\nabla g\wedge\pi + T\wedge\tau + R\wedge\rho - \varLambda]\\ &\equiv i_Z\varLambda + \nabla g\wedge i_Z\pi - T\wedge i_Z\tau - R\wedge i_Z\rho\\ &\quad +g(\tilde{\nabla}Z,\ )\pi + g(\ ,\tilde{\nabla}Z)\pi + (\nabla Z - \tilde{\nabla}Z)\wedge\tau + \nabla\tilde{\nabla}Z\wedge\rho &(6·19)\\ &\equiv i_Z\varLambda + \nabla g\wedge i_Z\pi - T\wedge i_Z\tau - R\wedge i_Z\rho - i_Z I\nabla\tau + \mathrm{d}[i_Z I\tau + \tilde{\nabla}Z\rho]\\ &\quad +g(\tilde{\nabla}Z,\ )\pi + g(\ ,\tilde{\nabla}Z)\pi - \tilde{\nabla}Z\wedge\tau - \tilde{\nabla}Z\nabla\rho\,, &(6·20)\end{aligned}$$

which satisfies the identity

$$-\mathrm{d}\mathcal{H}[Z] \equiv$$
$$\pounds_Z g\frac{\delta\mathcal{L}}{\delta g} + \frac{\delta\mathcal{L}}{\delta\pi}\pounds_Z\pi + \frac{\delta\mathcal{L}}{\delta\tau}\wedge\pounds_Z\tau + \pounds_Z\nabla\wedge\frac{\delta\mathcal{L}}{\delta\nabla} + \frac{\delta\mathcal{L}}{\delta\rho}\wedge\pounds_Z\rho + \pounds_Z\lambda\frac{\delta\mathcal{L}}{\delta\lambda}\,. \tag{6·21}$$

The rhs of this identity is proportional to variational derivatives, which are the (vacuum) field equations. Consequently, for each choice of Z, $\mathcal{H}[Z]$ is a (Noether 1st theorem) *conserved current* "on shell".

The "conserved" translational current (6·20) has the form

$$\mathcal{H}[Z] = \langle Z|h\rangle + \langle\tilde{\nabla}Z|k\rangle + \mathrm{d}\mathcal{B}[Z], \tag{6·22}$$

where $\langle Z|h\rangle$ and $\langle\tilde{\nabla}Z|k\rangle$ are *linear* in the indicated arguments. Explicitly

$$\begin{aligned}\langle Z|h\rangle &:= i_Z\varLambda + \nabla g\wedge i_Z\pi - T\wedge i_Z\tau - R\wedge i_Z\rho - i_Z I\nabla\tau\,, &(6·23)\\ \langle\tilde{\nabla}Z|k\rangle &:= g(\tilde{\nabla}Z,\ )\pi + g(\ ,\tilde{\nabla}Z)\pi - \tilde{\nabla}Z\wedge\tau - \tilde{\nabla}Z\nabla\rho &(6·24)\\ &\equiv \langle\tilde{\nabla}Z|g\pi + g\pi - I\wedge\tau - \nabla\rho\rangle\,. &(6·25)\end{aligned}$$

For the theories considered here we also have *local* diffeomorphism invariance. Thus (6·21) is an identity for all Z, and consequently the coefficients of Z, $\tilde{\nabla}Z$ and $\nabla\tilde{\nabla}Z$ on the lhs are identical to those on the rhs "off shell". This leads to certain differential identities (Noether 2nd theorem). In view of (6·22), the left-hand side of the identity takes the form

$$-\langle \nabla Z | h \rangle - \langle Z | \nabla h \rangle - \langle \nabla \tilde{\nabla} Z | k \rangle - \langle \tilde{\nabla} Z | \nabla k \rangle \,. \tag{6·26}$$

The corresponding terms on the rhs are linear combinations of the field equations. (The additional necessary Lie derivatives to find the explicit form can easily be computed in accordance with the techniques discussed earlier in §4.) Having identified the general form of this expression, we can conclude that h and k are proportional to field equations and hence *vanish* "on shell".

Consequently the value of the Noether current is entirely determined by the boundary term. However, this value is highly ambiguous, for the boundary term $\mathcal{B}$ may be modified in any way without affecting the conservation property. The Hamiltonian formalism includes the remedy for this ambiguity.

§7. The Hamiltonian

In §6.1 we have noted the 4-covariant first-order field equations. Each dynamic first-order equation can be split into a pair of equations, a constraint equation (the pullback to any spatial hypersurface) and an evolution equation, obtained by taking the interior product with any non-spatial vector. Formally, these pairs of equations can be obtained from a Hamiltonian constructed by evaluating the Lagrangian 4-form on a vector field. Thus

$$i_N \mathcal{L} =: \pounds_N g \, \pi + \pounds_N \nabla \wedge \rho - \mathcal{H}(N) \tag{7·1}$$

defines the Hamiltonian 3-form for dynamic geometry. A comparison with (6·18) shows that the Hamiltonian is just the Noether translational current, which has already featured in our discussion above. Let us formally vary this relation to get

$$\delta i_N \mathcal{L} = \delta \pounds_N g \pi + \pounds_N g \delta\pi + \delta \pounds_N \nabla \wedge \rho + \pounds_N \nabla \wedge \delta\rho - \delta \mathcal{H}[N]. \tag{7·2}$$

On the other hand consider that the contraction of $\delta\mathcal{L}$ (6·10) has the form

$$i_N \delta\mathcal{L} \equiv i_N \mathrm{d}\,[\delta g \pi + \delta\nabla \wedge \rho] + i_N\,[\dots] \tag{7·3}$$

$$\equiv \pounds_N\,[\delta g \pi + \delta\nabla \wedge \rho] - \mathrm{d} i_N\,[\delta g \pi + \delta\nabla \wedge \rho] + i_N\,[\dots] \tag{7·4}$$

$$\begin{aligned} &\equiv \delta \pounds_N g \pi + \delta \pounds_N \nabla \wedge \rho + \delta g \pounds_N \pi + \delta\nabla \wedge \pounds_N \rho \\ &\quad - \mathrm{d} i_N\,[\delta g \pi + \delta\nabla \wedge \rho] + i_N\,[\dots]\;, \end{aligned} \tag{7·5}$$

where the bracket contains terms which vanish "on shell". Since N is not varied, one can equate the two expressions and obtain a **key identity**:[8)]

$$\begin{aligned} &\delta\mathcal{H}[N] - \mathrm{d} i_N\,[\delta g \pi + \delta\nabla \wedge \rho] + i_N\,[\dots] \\ &\qquad \equiv \pounds_N g \delta\pi - \delta g \pounds_N \pi + \pounds_N \nabla \wedge \delta\rho - \delta\nabla \wedge \pounds_N \rho. \end{aligned} \tag{7·6}$$

Thus, assuming the 4-covariant field equations are satisfied, the Hamiltonian will exactly generate the desired equations of evolution for the canonical variables when the total differential term vanishes on the boundary of the region. Asymptotically one would like to have this term vanish automatically with standard fall-offs.

However the boundary term in the variation of the present Hamiltonian does not have such nice fall off properties. (In particular for GR the term $\delta\nabla\wedge\rho\sim O(1/r^2)$ gives a non-vanishing value.[14]) To make an improvement we may exploit our freedom to adjust the boundary term in the Hamiltonian. This is most easily done by simply dropping the present boundary term and then varying the (*unique*) pure 3-form Hamiltonian. Then one can examine the boundary term in the variation. As can be seen from (6·20), it is

$$\delta\mathcal{H}(Z)=\cdots+\mathrm{d}\left[\delta g i_Z\pi-i_Z I\delta\tau-\delta\nabla\wedge i_Z\rho-\tilde{\nabla}Z\delta\rho\right]. \tag{7·7}$$

To compensate one could add the differential of one of the boundary terms

$$\mathcal{B}(Z)=-\varDelta g\left\{\begin{array}{c} i_Z\pi \\ i_Z\bar{\pi}\end{array}\right\}+i_Z I\varDelta\tau+\varDelta\nabla\wedge\left\{\begin{array}{c} i_Z\rho \\ i_Z\bar{\rho}\end{array}\right\}+\left\{\begin{array}{c}\tilde{\nabla}Z \\ \tilde{\bar{\nabla}}Z\end{array}\right\}\varDelta\rho\,, \tag{7·8}$$

where for any quantity $\varDelta\alpha:=\alpha-\bar{\alpha}$; the bar indicates reference values which are not varied. For each curly bracket here choose either the upper or lower expression. The boundary terms in the variation of the respective Hamiltonians are then

$$\mathcal{C}(Z)=\left\{\begin{array}{c}-\varDelta g i_Z\delta\pi \\ \delta g i_Z\varDelta\pi\end{array}\right\}+0+\left\{\begin{array}{c}\varDelta\nabla\wedge i_Z\delta\rho \\ -\delta\nabla\wedge i_Z\varDelta\rho\end{array}\right\}+\left\{\begin{array}{c} i_Z\delta\nabla\varDelta\rho \\ -i_Z\varDelta\nabla\delta\rho\end{array}\right\}, \tag{7·9}$$

which, with the standard spatial asymptotics[14)–16)] regarding parity and fall off:

$$\varDelta g,\ \varDelta\rho\sim O_1^+ + O_2^-\,, \tag{7·10}$$

$$\varDelta\pi,\ \varDelta\nabla\sim O_2^- + O_3^+\,, \tag{7·11}$$

vanish for all the indicated boundary term choices. At null infinity these expressions give a non-vanishing radiation flux.[8]

§8. Einstein application

The formalism described hereinbefore applies to quite general dynamic geometry gravity theories. Specifically, for the important special case of GR we may take

$$\varLambda=V\wedge(\rho-(2\kappa)^{-1}\eta)\,, \tag{8·1}$$

where η is a $\binom{1}{1}$ valued 2-form implicitly determined by the metric:

$$\langle\alpha|\eta(X,Y)Z\rangle:=\mathrm{Vol}_g\left[g^{-1}(\alpha,\cdot)\wedge X\wedge Y\wedge Z\right]\,, \tag{8·2}$$

and $\kappa=8\pi G/c^4$. The π variation (6·12) then enforces metric compatibility and the τ variation (6·13) leads to vanishing torsion; variation of the multiplier field V enforces the definition of ρ; the ρ variation (6·15) gives $V=R$. The connection variation (6·14) then leads to vanishing π and τ. Our preferred boundary term for GR (the one distinguished by directly giving the Bondi energy flux and by having a positivity proof) then reduces simply to

$$\mathcal{B}(Z)=(\nabla-\bar{\nabla})\wedge i_Z\rho+\tilde{\bar{\nabla}}Z(\rho-\bar{\rho})\,. \tag{8·3}$$

8.1. *Asymptotics*

The above expression gives good asymptotic values for the total energy-momentum, and also for the angular momentum/center-of-mass. The associated *energy flux* is

$$\pounds_Z\mathcal{H}(Z) = \mathrm{d}[(\nabla - \bar{\nabla}) \wedge i_Z \pounds_Z \rho - i_Z(\nabla - \bar{\nabla})\pounds_Z \rho] \,, \tag{8·4}$$

which gives good values in the null-infinity (Bondi) limit.

8.2. *Local density*

Let $\nabla = \bar{\nabla} + K$. We assume that $\bar{\nabla}$ is metric compatible, flat and symmetric, hence $\bar{R} = 0 = \bar{\nabla}\bar{\eta}$. Then

$$\mathcal{H} \simeq \mathrm{d}\mathcal{B}(Z) \equiv \mathrm{d}[K \wedge i_Z\rho + \bar{\bar{\tilde{\nabla}}} Z(\rho - \bar{\rho})] \tag{8·5}$$

$$\begin{aligned}
&\equiv \nabla K \wedge i_Z\rho - K \wedge \nabla i_Z\rho + \bar{\nabla}(\bar{\bar{\tilde{\nabla}}} Z) \wedge (\rho - \bar{\rho}) + \bar{\bar{\tilde{\nabla}}} Z \bar{\nabla}(\rho - \bar{\rho}) \\
&\equiv (R - \bar{R} + K \wedge K) \wedge i_Z\rho - K \wedge \nabla i_Z\rho \\
&\quad + (\pounds_Z\bar{\nabla} - i_Z\bar{R}) \wedge (\rho - \bar{\rho}) + \bar{\bar{\tilde{\nabla}}} Z \bar{\nabla}(\rho - \bar{\rho}) \\
&\equiv R \wedge i_Z\rho + K \wedge K \wedge i_Z\rho - K \wedge \nabla i_Z\rho \\
&\quad + \pounds_Z\bar{\nabla} \wedge (\rho - \bar{\rho}) + \bar{\bar{\tilde{\nabla}}} Z \bar{\nabla}\rho \\
&\equiv R \wedge i_Z\rho + K \wedge K \wedge i_Z\rho - K \wedge \nabla i_Z\rho \\
&\quad + \pounds_Z\bar{\nabla} \wedge (\rho - \bar{\rho}) + \bar{\bar{\tilde{\nabla}}} Z(\nabla\rho - K \wedge \rho + \rho \wedge K) \\
&\equiv R \wedge i_Z\rho + (\tilde{\nabla} Z - i_Z K)\nabla\rho + K \wedge K \wedge i_Z\rho - K \wedge \nabla i_Z\rho \\
&\quad + \pounds_Z\bar{\nabla} \wedge (\rho - \bar{\rho}) + \bar{\bar{\tilde{\nabla}}} Z(-K \wedge \rho + \rho \wedge K) \,,
\end{aligned} \tag{8·6}$$

where at a key step we used

$$\bar{\bar{\tilde{\nabla}}}_Y N \equiv \bar{\nabla}_N Y - \pounds_N Y \equiv \nabla_N Y - \pounds_N Y - K_N Y \equiv \tilde{\nabla}_Y N - (i_N K) Y. \tag{8·7}$$

Since torsion vanishes we have

$$\begin{aligned}
\mathcal{H} \simeq \mathrm{d}\mathcal{B}(Z) &\simeq R \wedge i_Z\rho + K \wedge K \wedge i_Z\rho - K \wedge i_{\nabla Z}\rho + \bar{\bar{\tilde{\nabla}}} Z(-K \wedge \rho + \rho \wedge K) \\
&\equiv R \wedge i_Z\rho + K \wedge K \wedge i_Z\rho - K \wedge i_{KZ}\rho \\
&\quad + \bar{\bar{\tilde{\nabla}}} Z(-K \wedge \rho + \rho \wedge K) - K \wedge i_{\bar{\nabla} Z}\rho \,.
\end{aligned} \tag{8·8}$$

Now, assuming that Z is a Killing field of the reference, we have

$$\mathcal{H}(Z) \simeq \mathrm{d}\mathcal{B}(Z) \simeq R \wedge i_Z\rho + K \wedge K \wedge i_Z\rho - K \wedge i_{KZ}\rho. \tag{8·9}$$

This is a conserved current density. It includes both an energy-momentum and an angular momentum density. The first term, using Einstein's equation, is just the source energy-momentum density. The remaining terms can then be interpreted as the energy-momentum density of gravity (measured wrt the chosen $\bar{g}$, $\bar{\nabla}$).

8.3. *Small region*

Consider the small region limit. One can choose the reference so that to lowest order K vanishes, hence within matter one gets just the source energy density—in accord with the *equivalence principle*. In vacuum the first term vanishes and the remainder is a *gravitational energy-momentum density*. Taylor series expanding it around a chosen point in *normal coordinates and frames* gives, to 2nd order with $K(x) \sim \frac{1}{2} i_x R$, after detailed calculation, *the Bel-Robinson tensor*—as desired.[17)]

§9. Summary

Here we presented a concise summary of a *Hamiltonian formulation* which is *manifestly covariant* and *reference frame independent* and which can be applied to general dynamic geometry gravity theories as well as Einstein's GR.

The Hamiltonian includes a boundary term which depends of the choice of boundary conditions, a choice of reference, and spacetime displacement. The value of the Hamiltonian, as determined by the boundary term, gives the quasi-local energy-momentum, angular mometum/center-of-mass, and energy flux. The expression has good limits asymptotically to spatial infinity and null infinity and to small regions.

This manifestly covariant formulation clarifies certain points. A spinoff is the development of some mathematical techniques.

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