# Business Cycle Based on Optimal DI Model

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We propose a dynamic model for the economic growth ($\Delta G$) and the diffusion index ($DI$) in order to explain the business cycle. This model is described by equations analogous to the optimal velocity model in traffic flow. In the model there exists a conserved quantity, which corresponds to the total energy in a dynamical system. We found that the business cycle with the period 5–8 years is favorably reproduced, since the model predicts a periodic motion in the conservative system.

## §1. Introduction

The gross domestic product (GDP) is one of the most important quantities in macroeconomics. The economic trend is visualized typically in terms of time evolution of GDP. It usually increases in the long term, but there exist fluctuations in the short term according to economic growth and decline (see Fig. 1).

In order to see the fundamental structure of such fluctuations more clearly, we propose to extract the economic growth $\Delta G(i) = G(i) - G(i-1)$, where we note $G(i)$ as the $i$-th year value of $GDP$.[2),3)] $\Delta G(i)$ commonly shows a kind of cycle repeating depression and prosperity as in Fig. 2, and it is called "business cycle".

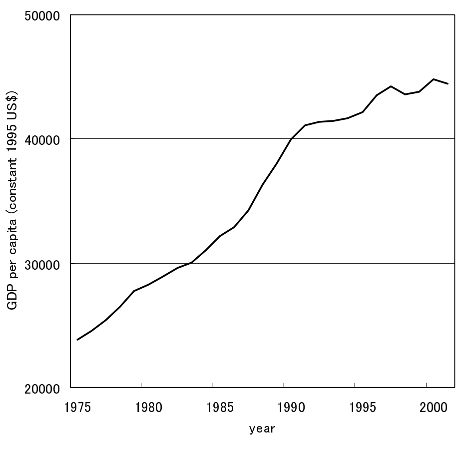


Fig. 1. Time dependence of $GDP$ per capita (constant 1995 US dollar) in Japan from 1975 to 2001.[1)]

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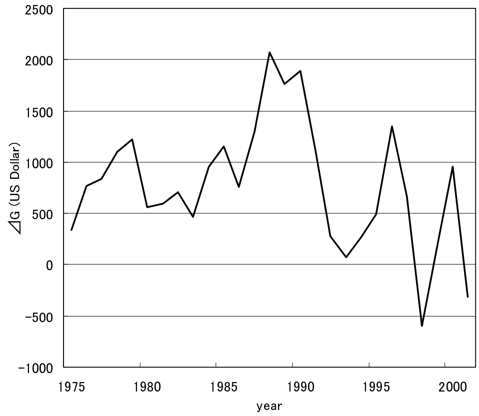


Fig. 2. Time dependence of $\varDelta G(i)$ in Japan ($i = 1975, 1976, \ldots, 2001$).

The origin of the fluctuations can be divided into external and internal sources. There are several models to explain the economic fluctuations caused by the internal origin.[4)–11)] Their main concern is to derive the fluctuations from the result of dynamics. For example, in non-linear models the economic fluctuations are explained in terms of a limit cycle. However, these studies were devoted to make a mathematical model, and did not succeed to explain the property of the real economic system very well. Therefore we propose a new model based on the equation of the motion in physics.

After we introduce a new variable, "diffusion index" ($DI$), we show the observed data of $DI$ and its relationship to $GDP$ in §2. In §3 we propose a model of the economic fluctuations and introduce a conserved quantity, namely the total energy of the system. Section 4 is devoted to investigate the properties of our model. We compare the calculated results of the energy and the period with the real data in §5. In §6, we summarize our results and make a short future prospect.

## §2. Observation

In this section, we introduce a new variable, "diffusion index" ($DI$) and show the observed data of $DI$, and its relationship to $GDP$.

The diffusion index is found in "Tankan", which is announced by Bank of Japan as the Short-term Economic Survey of Enterprises in Japan. "Tankan" is a report of the findings of the questionnaires on the business to about 10,000 private enterprises. $DI$ refers to a result of the question: "Is your business good?". From the number of "good / not good / bad" (denoted by $G$ / $N$ / $B$), $DI$ is defined by Bank of Japan as follows:

$$DI = 100\frac{G - B}{G + N + B}. \tag{2·1}$$

Therefore $DI$ represents whether business is thought good or bad in a society. The time dependence of $DI$ shows oscillating behavior (see Fig. 3).

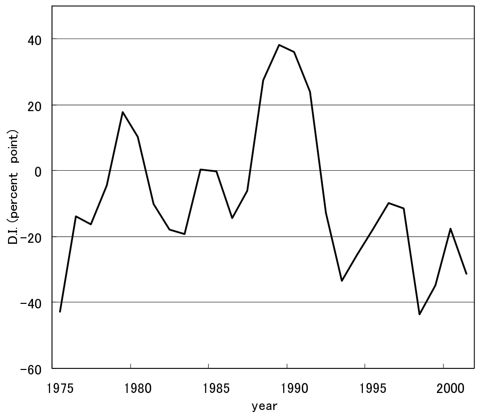


Fig. 3. Time dependence of $DI(i)$ in Japan ($i = 1975, 1976, \ldots, 2001$).

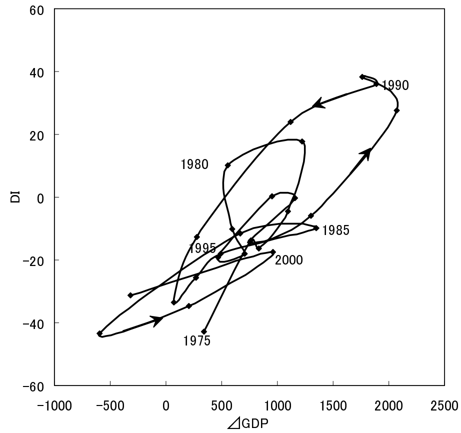


Fig. 4. Each dot represents a data point ($\Delta G(i)$, $DI(i)$), and solid curve represents a trajectory connecting the series of data points with spline curve ($i = 1975, 1976, \ldots, 2001$).

Originally the two variables, $GDP$ and $DI$ is mutually independent, but it is found that $\Delta G(i)$ and $DI(i)$ are strongly correlated from the observed data. The year-dependence of $\Delta G(i)$ and $DI(i)$ are shown in Figs. 2 and 3, which clearly indicates that $\Delta G(i)$ and $DI(i)$ behave almost similarly. Indeed the correlation coefficient is found to be 0.85.

A remarkable fact is that a point ($\Delta G(i)$, $DI(i)$) moves almost counterclockwise (see Fig. 4) in the space ($\Delta G$, $DI$). Its trajectory resembles that of a vehicle in a congested traffic flow. Figure 5 shows an example of trajectory of a vehicle motion in a phase space ($\Delta x$, $v$), where $\Delta x$ is headway, which is a distance between two vehicles, and $v$ is velocity of a vehicle. This trajectory is obtained by a numerical simulation based on the optimal velocity (OV) model.[12] The similarity of trajectories suggests that the dynamics of $\Delta G(t)$ and $DI(t)$ can be described by a similar equation to that of traffic flow.

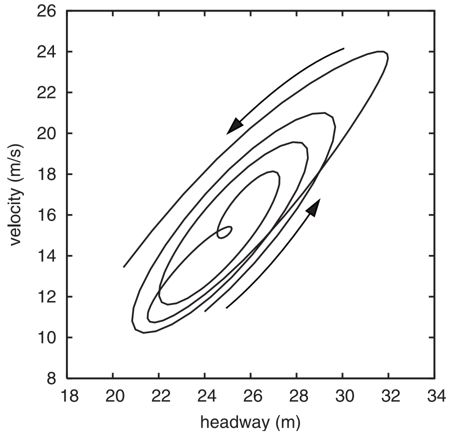


Fig. 5. An example of trajectory of a vehicle motion in a phase space of headway $\Delta x$ and velocity $v$. A motion of a vehicle in the phase space is counterclockwise. This trajectory is obtained by a numerical simulation based on the optimal velocity (OV) model.[12)]

## §3. The optimal DI model

We present a model[2),3)] to explain the behavior of economic fluctuations. The model is expressed by non-linear equations in terms of $GDP$ and $DI$, which is analogous to the OV model of traffic flow.

The characteristic feature of the OV model is to introduce "optimal velocity" which is the most desirable velocity for a vehicle with headway $\Delta x$. The equation of motion is determined by the rule that a driver accelerates or decelerates in such a way that he maintains this optimal velocity. Namely, if the velocity is smaller than the optimal velocity, he accelerates his car. On the other hand, he decelerates if the velocity is larger than the optimal velocity. Therefore the optimal velocity function gives the border between the acceleration region and the deceleration region.

In the economy, $DI$ represents the tendency of the presidents of companies to accelerate or decelerate their production rates. However $DI$ does not always increase even if the economic growth $\Delta G$ is positive. Sometimes $DI$ decreases even if $\Delta G$ is positive, and $DI$ increases if $\Delta G$ is negative. $DI$ seems to increase or decrease depending on the value of ($\Delta G$, $DI$). There seems to exist the "optimal" $DI$ value, and $DI$ decreases when $DI$ is larger than the value. On the other hand, when $DI$ is smaller than the optimal $DI$ value, $DI$ increases.

We can observe these phenomena in Fig. 6 which is the same scatter diagram as Fig. 4. We assign "+" or "−" to each point ($\Delta G(i)$, $DI(i)$) according to the signature of $\Delta DI(i) = DI(i+1) - DI(i)$ in Fig. 6. We call the "deceleration" region above the solid curve in Fig. 6, because many of $DI$s decrease in this region, and we call the "acceleration" region below the solid curve.

We here introduce an "optimal" $DI$ function. The optimal $DI$ curve is defined as the boundary of two regions, acceleration and deceleration regions of $DI$. From this figure, $ODI$ function can be read off.

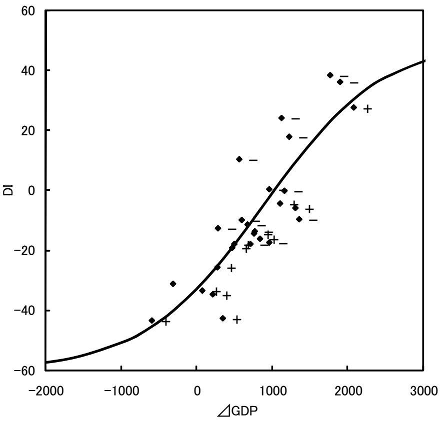


Fig. 6. Optimal *DI*, acceleration region, and deceleration region. Each dot represents a data point ($\Delta G(i)$, $DI(i)$) and a mark "+" or "−" means $\Delta DI(i)$ is positive or negative. Solid line represents the optimal $DI$ function that we choose.

We suppose the optimal $DI$ function in the following form:

$$ODI(\Delta G) = A + B\tanh(C(\Delta G - D)), \tag{3·1}$$

where $A$, $B$, $C$, and $D$ are constant parameters. We choose the values of the parameters as $A = -5$, $B = 55.3$, $C = 6.28 \times 10^4$, and $D = 880$. The $ODI$ function with these parameters is shown in Fig. 6.

When we consider a dynamical model of $\Delta G$ and $DI$, it is natural to introduce continuous variables $\Delta G(t)$ and $DI(t)$. Practically the data points of $\Delta G(t)$ and $DI(t)$ are given annually, and in such a case $\Delta G(t)$ and $DI(t)$ are expressed as $\Delta G(i)$ and $DI(i)$ with $i$ =1975, 1976, ..., and 2001.

The dynamical equation is determined in such a way that companies try to maintain the optimal $DI$. Using the optimal $DI$ function, a dynamical equation is

$$\frac{dDI(t)}{dt} = a(ODI(\Delta G(t)) - DI(t)), \tag{3·2}$$

where $a$ is a constant, and $\Delta G(t) = G(t) - G(t-1)$ is an economic growth of past one year.

In addition, we suppose that the derivative of $GDP$ is approximately proportional to $DI$,

$$\frac{dG(t)}{dt} = bDI(t) + c, \tag{3·3}$$

where $b$ and $c$ are constants. This is because from Figs. 1 and 3 we observe that $DI$ is large, if the inclination of $GDP$ is steep, and vice versa.

Instead of the above differential equations, we propose difference equations as follows, because $\Delta G(t)$ is defined only at the points $t = i$ and we cannot define continuous $\Delta G(t)$ in a realistic way at present.

$$DI(i+1) - DI(i) = a(ODI(\Delta G(i)) - DI(i)), \tag{3·4}$$

$$\Delta_2 G(i+1) = bDI(i+1) + c, \tag{3·5}$$

where $\Delta G(i+1) = G(i+1) - G(i)$, and $\Delta_2 G(i+1) = (\Delta G(i+1) + \Delta G(i))/2$. Equation (3·5) is a discretized equation of Eq. (3·3), and is phenomenologically obtained because these two variables have very strong correlation, where the correlation coefficient is 0.94. (The correlation coefficient between $\Delta G$ and $DI$ is 0.85.) We consider that $\Delta_2 G$ is linearly dependent on $DI$. We obtain $b = 23.6$ and $c = 969$ by the linear fit of the observed data.

From Eq. (3·4) and (3·5), we obtain the following equation,

$$\Delta G(i+1) + a\Delta G(i) + (a-1)\Delta G(i-1) = 2ab\,ODI(\Delta G(i)) + 2ac. \qquad (3·6)$$

Here we note that the economic system in the above model is expressed in terms of a single dynamical variable $\Delta G(i)$. Hereafter we denote $x(i)$ instead of $\Delta G(i)$.

By solving Eq. (3·6) numerically, we find that the behavior of $x(i)$ depends on the parameter $a$.

(1) $a < 2$: The behavior of $x(i)$ is similar to that of the original $ODI$ model,[2] namely $x(i)$ tends to a fixed point irrelevant to the initial condition.
(2) $a = 2$: This is a special case and we find that $x(i)$ shows a periodic behavior.
(3) $a > 2$: Irrelevant to the initial condition, $x(i)$ tends to infinity.

Now that we have found that only the case (2) reproduce cyclic motion of economic system, the case (2) may be a candidate for reproducing a realistic feature of business cycle. So here we concentrate ourselves to the case $a = 2$ and rewrite Eq. (3·6) as follows:

$$x(i+1) - 2x(i) + x(i-1) = 4[\, b\,ODI(x(i)) - x(i) + c\,]. \qquad (3·7)$$

The left-hand side of Eq. (3·7) can be regarded as the second derivative of the continuous function $x(t)$. So we replace the difference equation (3·7) by the differential equation

$$\frac{d^2x(t)}{dt^2} = 4[\, b\,ODI(x(t)) - x(t) + c\,] \qquad (3·8)$$

as a fundamental equation.

This equation can be regarded as an equation of motion, and the right-hand side is a force, which is written in terms of $x(t)$ only. This indicates that our model represents a conservative system. Then we can define the conserved quantity "total energy"

$$E = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x), \qquad (3·9)$$

$$V(x) = -\int 4[\, b\,ODI(x) - x + c\,]\, dx, \qquad (3·10)$$

where $V(x)$ is the potential energy of this system.

## §4. Properties of the model

In this section we investigate the properties of our model. The dynamical variable of the model is $x(t)$ only, in terms of which the system is expressed by the following Hamiltonian,

$$H = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x), \tag{4·1}$$

$$V(x) = -4\beta \log(\cosh(C(x-D)) + 2(x-\gamma)^2 + \text{const}, \tag{4·2}$$

where we have used the *ODI* function, and redefine the parameters $\beta \equiv bB/C$ and $\gamma \equiv bA + c$. Hereafter we take the unit of $x(t)$ as $10^3$ dollars; $x(t) = \Delta G(t) \times 10^{-3}$ and the constant in Eq. (4·2) is set to zero for convenience. The parameters $A$, $B$, $C$, $D$, $b$ and $c$ are given in Eqs. (3·1) and (3·5), and the values of parameters of the above equations are converted to

$$\beta = 2.08,\ \gamma = 0.851,\ C = 0.628,\ D = 0.880. \tag{4·3}$$

Figure 7 shows the shape of the potential energy. The total energy, which is conserved quantity, is a sum of this potential energy and kinetic energy. However in realistic situation irregular changes of economy such as the Depression in 1929 happen to occur. In our model, such irregular changes correspond to the external forces. We shall discuss this point in §5. The total energy jumps to a different value due to such effects, and accordingly the total energy evaluated from the observed data is not really constant. For example, the dashed line (a) in Fig. 7 represents the total energy evaluated from the data in 1974 (high energy case), and (b) represents that in 1979 (relatively low energy case). From this figure, we find that $x(t)$ oscillates between $-0.8$ and 2.5 in the case (a), and oscillates between 0.2 and 1.3 in the case (b).

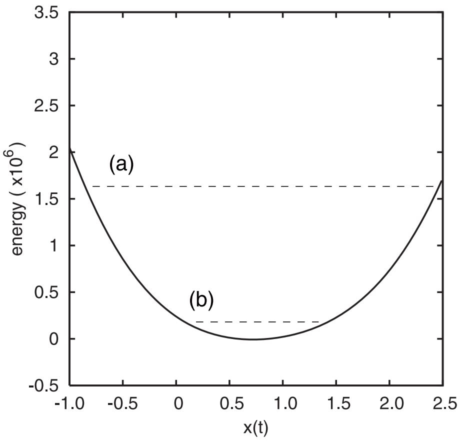


Fig. 7. Solid line represents the potential energy (Eq. (4·1)). Two dashed lines (a) and (b) represent the energy calculated from the real *GDP* data in 1974 and 1979 respectively.

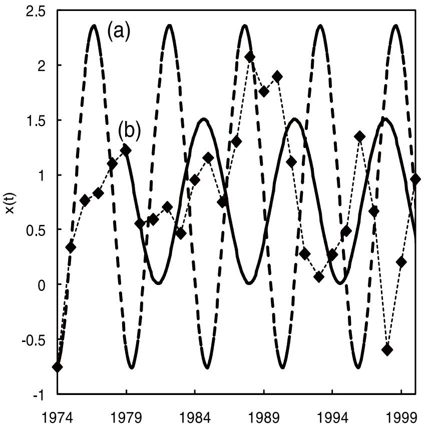


Fig. 8. Behaviors of $x(t)$ using the data in (a) 1974 (dashed line) and (b) 1979 (solid line) as the initial condition. As a reference, we add the observed data (dotted line with diamond marks).

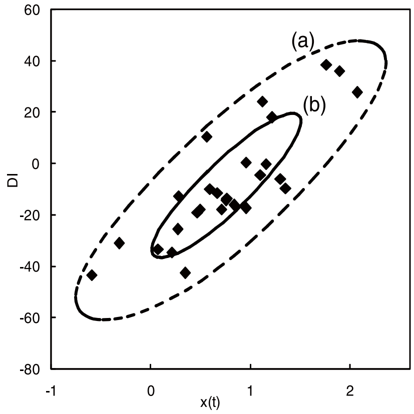


Fig. 9. Trajectories in the phase space $(x(t), DI(t))$. Dashed and solid curves represent trajectories starting from the data in (a) 1974 and (b) 1979. Diamond marks show the observed data.

In order to investigate the motion of $x(t)$, we solve Eq. (3·8) numerically in the above typical cases. Figure 8 shows the results of the simulation using the data in (a) 1974 and (b) 1979 as the initial condition. The total energy defined in Eq. (4·1) is (a) $E = 1.57$ in $10^6$ unit and (b) $E = 0.12$. The period of the motion is 5.5 and 7 years in the case (a) and (b), respectively. To guide the eye, we plot the real data points.

In order to understand the behavior of business cycle more clearly, we show the trajectories in the phase space $(x(t), DI(t))$ (Fig. 9). This phase space plays an essential role in the original $ODI$ model.[2)] Here $DI(t) = (1/2b)[x(t) + x(t-1) - 2c]$ (see Eq. (3·5)). As a reference, we add the observed data. The periodic motion of $x(t)$ with corresponding $DI(t)$ is visualized by close circles in Fig. 9. It is found that the higher the total energy is, the larger the circle becomes.

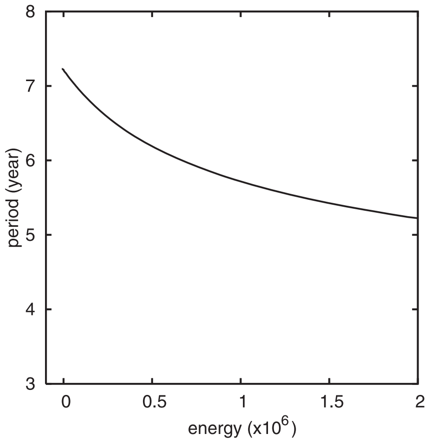


Fig. 10. The period of this system is shown. The period is a monotonically decreasing function of $E$, which tends to the value $\pi$.

Here let us investigate the energy dependence of the period of motion. We have seen in Fig. 7 that the global shape of the potential is approximately of the harmonic oscillator type. Indeed, if $x(t)$ is large, the second term in Eq. (4·2) dominates and then the potential becomes that of a harmonic oscillator, $V(x) \to 2x^2$. This is independent of the parameters $\beta$, $\gamma$, $C$ and $D$. Thus, for large $x(t)$, the equation of motion (3·8) reduces to the following simple form,

$$\frac{d^2x(t)}{dt^2} = -4x(t). \tag{4·4}$$

If the system is described by this equation, $x(t)$ moves periodically with the period $\pi$, independent of the total energy. The larger the total energy is, the wider the region becomes where the equation of motion of $x(t)$ is approximated by Eq. (4·4). In such case, the period is determined mainly by the behavior of large $x(t)$, even if the potential is deviated from the harmonic one for smaller $x(t)$. On the other hand, if the total energy is small, the effect from the potential deviated from the harmonic one becomes important. We have found by numerical calculation that this effect makes the period longer (see Fig. 8).

To see the energy dependence of the period, we calculated the period numerically. Figure 10 shows that the period is a monotonically decreasing function of the energy $E$, and tends to the value $\pi$. The reason of this result is as follows. The first term of Eq. (4·2) makes the potential flatter, and consequently the period becomes longer. Note that this result depends on the choice of parameters as we shall see in the next section.

## §5. Energy in the real system

Our model is a conservative system and the total energy $E$ is invariant. In order to see this, we calculate the total energy by using the real $GDP$ data. Figure 11 shows the plot of the energy calculated from the real data of each year.

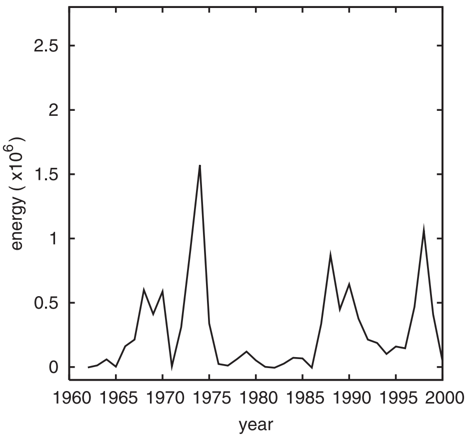


Fig. 11. Total energy $E$ calculated from the real $GDP$ data.

Table I. Big change of Japanese economy.

| years | economic event |
|---|---|
| 1965–70 | Izanagi boom |
| 1974 | first oil shock |
| 1979 | second oil shock |
| 1986–91 | bubble economy |
| 1997–98 | Heisei depression |

There are two distinct peaks in 1974 and 1998, and two broad hills around 1968 and 1990. For other years, the energy seems to be almost constant. The positions of these peaks and hills coincide with the years of big changes of Japanese economy. We list the years of the big changes of Japanese economy in Table I. The first peak corresponds to the so-called "first oil shock" and the second peak is "Heisei depression". The first and second broad hills correspond to "Izanagi boom" and "bubble economy" respectively. We find the sharp peaks correspond to depressions and the broad hills correspond to booms.

Basically the total energy is an invariant quantity and we assume its change arises from the external forces. Therefore the change of the total energy can be used as an index of the change of the economic situation.

As a reference, we also show the periods of business cycle for three cases in Fig. 12. The periods are obtained from the relation between the period and the total energy (Fig. 10) and the observed value of total energy (Fig. 11).

The period is roughly 5–8 years. It is known that the period of the Juglar fixed investment cycle is 7–11 years, and our model could reproduce it.

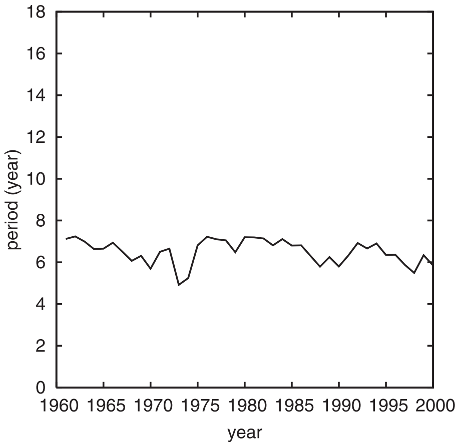


Fig. 12. The period of the motion of $x(t)$ calculated from the real $GDP$ data.

## §6. Concluding remarks

In this paper, we have introduced a model in order to reproduce the business cycle. We have concentrated on the case $a = 2$ in order to investigate the periodic behavior of the model as the first step of our investigation. In this case, the model describes a conservative system, having a Hamiltonian with a single dynamical variable $x(t)$. The economic system is described in terms of a point particle moving in the harmonic like potential. The dynamical variable $x(t)$ moves periodically and the total energy $E$ of this system is conserved.

There exist the business cycles in the economies of almost all modern countries in the world. It is a quite general feature of economic system, and has long been one of the most interesting questions how to explain such business cycles. Our model naturally explains the origin of the business cycle by the periodicity of $x(t)$. We also find the relation between the total energy and the period of business cycle. A constant period of business cycle is a result of the energy conservation. From our model the business cycle with the period 5–8 emerges in Japan. This cycle may be identified as the Juglar fixed investment cycle.

The observed data of $GDP$ in Japan shows that the total energy $E$ is not always constant. At several points $E$ takes very large values as seen in Fig. 11. Such sudden changes correspond to "Izanagi boom", "first oil shock", "bubble economy" and "Heisei depression". From the viewpoint of our model, these events can be interpreted as consequences of some external forces, which cannot be predicted from our dynamical model. We can use the total energy as a kind of the index of such economic events.

The behavior of the total energy calculated from the real $GDP$ data also has an interesting feature. The energy increases quickly by irregular events, but the effects are washed out immediately and the energy returns to the value before the events. This feature suggests that our model extracts only a conservative nature of the economic system. To construct a more realistic model the dissipation term may be necessary. The dissipation term emerges in the case $a < 2$. It is our future work to investigate the dynamics in this case.

It would be interesting task to investigate the behavior of total energy and the period of business cycle of the countries other than Japan. Also we note that in real world the economic system of each country is affected by other countries. It is also an interesting problem to investigate the interaction among many economic systems and their collective dynamics.

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### References

1) World Bank, *World Development Indicators 2003* (World Bank Office of the Publisher, 2003).
2) M. Taniguchi, M. Bando and A. Nakayama, J. Phys. Soc. Jpn. **76** (2007), 124003.
3) M. Taniguchi, M. Bando and A. Nakayama, J. Phys. Soc. Jpn. **77** (2008), 114001.
4) P. A. Samuelson, *Foundations of Economic Analysis* (Harvard University Press, Cambridge, Mass., 1947).
5) J. R. Hicks, *A Contribution to the Theory of the Trade Cycle* (Oxford University Press, Oxford, 1950).
6) K. Inada and H. Uzawa, *Economical Development and Fluctuations* [in Japanese] (Iwanami, 1972).
7) L. Arnord, *Business Cycle Theory* (Oxford University Press, Oxford, 2002).
8) T. Puu, *Nonlinear Economic Dynamics* (Springer-Verlag, New York, 2002).
9) P. A. Samuelson, Review of Economic Statistics **21** (1939), 75.
10) N. Kaldor, Economic Journal **50** (1940), 78.
11) R. M. Goodwin, Econometrica **19** (1951), 1.
12) M. Bando, K. Hasebe, A. Nakayama, A. Shibata and Y. Sugiyama, Phys. Rev. E **51** (1995), 1035.