Structure of the Solar Nebula, Growth and Decay of Magnetic Fields and Effects of Magnetic and Turbulent Viscosities on the Nebula

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First, distributions of surface densities of dust materials and gases in a preplanetary solar nebula, which give a good fit to the distribution of the planetary mass, are presented and the over-all structure of this nebula, which is in thermal and gravitational equilibrium, is studied.

Second, in order to see magnetic effect on the structure, electric conductivity of a gas ionized by cosmic rays and radioactivities contained in dust grains is estimated for each region of the nebula and, then, the growth and decay of seed magnetic fields, which are due to differential rotation of the nebula and to the Joule dissipation, respectively, are calculated. The results indicate that, in regions of the terrestrial planets, magnetic fields decay much faster than they grow and magnetic effects can be ignored, except for the outermost layers of very low density. This is not the case for regions of Uranus and Neptune where magnetic fields can be amplified to considerable extents.

Third, the transport of angular momentum due to magnetic and mechanical turbulent viscosities and the resultant redistribution of surface density in the nebula are investigated. The results show that the density redistribution occurs, in general, in a direction to attain a distribution of surface density which has nearly the same r-dependence as that obtained from the present distribution of the planetary mass. This redistribution seems to be possible if it occurs at a formation stage of the nebula where the presence of large viscosities is expected.

Finally, a comment is given on the initial condition of a collapsing interstellar cloud from which the solar nebula is formed at the end of the collapse.

§ 1. Introduction

The importance of magnetic effects on the origin of the solar system was pointed out and emphasized by many people in accordance with the development of plasma physics and cosmic plasma physics. Especially, Alfvèn[1,2] developed his own theory of planetary formation on a standpoint that magnetic effects can never be neglected as compared to the other effects such as due to gravity and gas pressure. On the other hand, Hoyle[3] pointed out that, at the time of formation of a preplanetary solar nebula, magnetic fields played an essential role in transferring angular momenta from the protosun to the surrounding solar nebula.

At present, it is still difficult to disentangle, quantitatively, magnetic effects from gravity effects since, contrary to gravity fields, magnetic fields grow and decay depending upon physical conditions in the evolving nebula such as the degree of ionization of matter and cross sections of collisions among electrons, ions and neutral atoms or molecules. A way to disentangle the effects will be to start from the construction of a very concrete model of the solar nebula where the magnetic effects are neglected at first. Such a model will be presented in § 2 of this paper. This model describes the structure of the solar nebula which has already settled into a nearly equilibrium state, after the end of gravitational collapse from a state of very low density in giant molecular clouds.

In this nebula the planetary system is supposed to have been formed in a very long period of time. Namely, according to current theories of planetary formation as reviewed recently by Wetherill[4] and Hayashi,[5] sedimentation of dust grains to the equatorial plane of the nebula forms a thin dust layer, gravitational fragmentation of this layer produces a great number of planetesimals of comet sizes and, finally, very gradual accumulation of these solid bodies leads to the formation of all the terrestrial planets and also the cores of all the giant planets. Among the giant planets, Jupiter and Saturn are supposed to have been formed as a result of further accretion of gases of the solar nebula onto the cores mentioned above.

For the above-mentioned model of the equilibrium nebula, physical conditions of the gas such as ionization degree and electric conductivity will be studied in § 3 and, then, the decay time of magnetic fields, if they exist somewhere in the nebula, will be estimated and compared with the Kepler period of rotation of the nebula. In § 4, one of magnetohydrodynamic equations which describes the growth of magnetic fields due to differential rotation and also their decay due to the Joule dissipation will be solved mathematically. The results of §§ 3 and 4 indicate that, in regions where the terrestrial planets are formed, the surface density of the disk is so high that cosmic rays cannot penetrate into the interiors and, because of resulting low electric conductivity, magnetic fields decay much faster than they grow by differential rotation. This is not the case, however, for surface regions of the nebula and also for the whole regions of Uranus and Neptune where the ionization degree is relatively high.

In § 5, study will be made of a possibility of the redistribution of gas density in the solar nebula, which is caused by angular momentum transport due to magnetic viscosity as well as mechanical turbulent viscosity. The result indicates that the density redistribution is possible at a formation stage of the nebula, where we can expect an existence of large viscosities. Finally, in § 6 a comment will be given on the initial condition of a rotating interstellar cloud from which both the protosun and the solar nebula under investigation are formed as a result of its collapse.

§ 2. Structure of an equilibrium solar nebula

We consider the equilibrium structure of a solar nebula which was at stages where most of dust grains had already sedimented onto the equatorial plane and was composed mainly of hydrogen molecules and helium atoms. The total mass of the nebula is of the order of $10^{-2}M_{\odot}$. We adopt a heliocentric cylindrical coordinate system (r, φ, z) where the equatorial plane is described by $z=0$. The nebula is disk-like and rotating around the sun with nearly Keplerian angular velocities since pressure gradient in the r-direction is very small compared with solar gravity. The half-thickness z_0 of the nebula is determined by a balance of pressure gradient and solar gravity in the z-direction and, in a case where the gas temperature depends on r only, it is given by

$$z_0(r) = \sqrt{2}\, c_s(r)/\Omega_K(r), \qquad (2\cdot 1)$$

where c_s is the sound velocity

$$c_s = \left(\frac{kT}{\mu m_H}\right)^{1/2} = 9.9\times 10^4 \left(\frac{2.34}{\mu}\frac{T}{280}\right)^{1/2} \text{cm s}^{-1}, \qquad (2\cdot 2)$$

($\mu=2.34$ being the mean molecular weight) and Ω_K is the Keplerian angular velocity, $(GM_{\odot}/r^3)^{1/2}$.

Since dust grains are highly depleted in the nebula at stages considered, the gaseous nebula is, almost everywhere, transparent to the solar visible radiation (but not to the solar UV) and the gas temperature as a function of r is given by

$$T = 280 r^{-1/2} (L/L_{\odot})^{1/4}\ {}^{\circ}\text{K}, \qquad (2\cdot 3)$$

where r is in astronomical units (a.u.) and L is the solar luminosity at stages considered. In the following we put $L=1L_{\odot}$ for simplicity.

The structure of the gaseous disk is completely determined if the surface density, ρ_s, is given as a function of r. Previously, Kusaka, Nakano and Hayashi[6] estimated the surface density under the assumption that dust materials (contained in the dust layer mentioned in § 1) accumulated into the terrestrial planets and the cores of the giant planets with minimum displacements in radial directions both inward and outward. The surface density obtained by them is now to be revised by taking into account recent developments of theories and observations on planetary formation and structure. As a result of looking for a smooth curve which gives a good fit to the present data, we have first for the surface density of dust materials

$$\rho_s(\text{rock}) = 7.1 r^{-1.5}\ \text{g cm}^{-2} \quad \text{for} \quad 0.35 < r < 2.7, \qquad (2\cdot 4)$$

$$\rho_s(\text{rock}+\text{ice}) = 30r^{-1.5}\ \text{g cm}^{-2} \quad \text{for} \quad 2.7<r<36, \tag{2·5}$$

where r is in a.u. and from these we have finally for the gas surface density

$$\rho_s(\text{gas}) = 1.7\times10^3 r^{-1.5}\ \text{g cm}^{-2} \quad \text{for} \quad 0.35<r<36. \tag{2·6}$$

The above distributions are shown in Fig. 1. The total mass of the nebula in the range, $r=0.35\sim36$, is $0.013M_{\odot}$ and this is to be considered as a minimum mass of the solar nebula required to form the present planets, in view of a possibility of mass flow across the inner and outer edges described above.

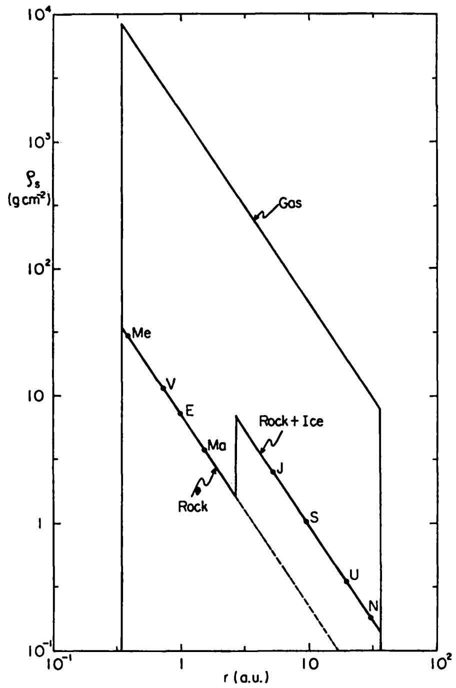

Fig. 1. Surface densities of rocky, icy and gaseous materials in the solar nebula as a function of the distance from the sun.

The above formulae have been obtained in the following way. First, we assume the minimum displacements of dust materials as mentioned above. We adopt Cameron's values[7] for the original chemical composition of the nebula, where the abundance (by mass) of all rocky and metallic materials is 0.0043 and that of icy materials (H_2O, CH_4 and NH_3) is 0.0137. The condensation temperature of ice is simply taken as 170°K. Second, the mass of rocky and icy materials contained in the cores of all the giant planets is chosen as $15M_E$[8)~10)] (M_E being the Earth mass). Third, it has been assumed that almost all of the dust materials existent in the region, $r=1.55\sim7.00$ a.u., were accumulated into Jupiter's core. The reason for this is that the surface density of dusts near Jupiter's orbit is much greater than that in the asteroid belt because of the ice condensation (see Fig. 1) and, consequently, the growth of Jupiter was much faster. Thus, for the original distribution of gas density, there was no gap in the present asteroid region.

If magnetic effects are negligible, the equilibrium structure of the gaseous nebula is uniquely determined from Eqs. (2·1), (2·3) and (2·6). The distribution of gas density as a function of r and z is expressed in the form

$$\rho(r,z) = \rho_0 r^{-2.75} \exp\{-z^2/z_0^2(r)\} \tag{2·7}$$

with

$$\rho_0 = 1.4\times10^{-9}\ \text{g cm}^{-3}, \tag{2·8}$$

$$z_0(r) = 0.0472 r^{5/4}, \tag{2·9}$$

where $z_0(r)$ is the half-thickness given by Eq. (2·1) and r, z and z_0 are all in a.u. Equi-density contours in the r-z plane are illustrated by Fig. 2 where the gas density is measured in units of ρ_0 given by Eq. (2·8). The dashed lines ($p = 10^4 p_{SW}$) represent outer layers where the gas pressure is equal to the dynamical pressure of the solar wind with an intensity 10^4 times greater than the present value (reflecting greater coronal activities in a T Tauri stage of the sun). The hatches (in the outer regions) denote surface regions of very low density where the gas is ionized by the irradiation of the solar UV.

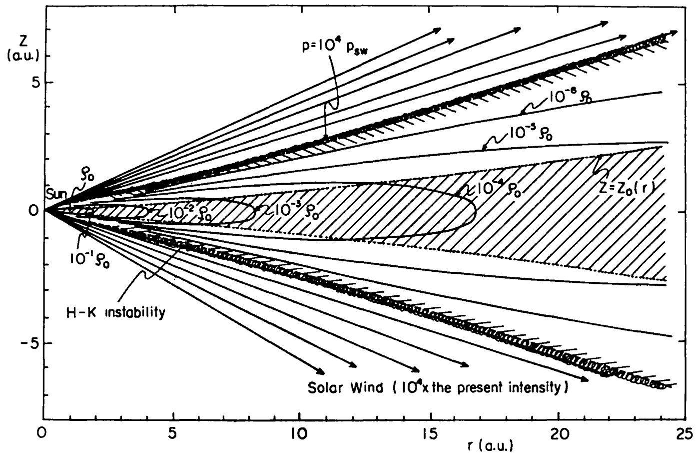

Fig. 2. Structure of the solar nebula and equi-density contours.

If magnetic fields are present, the equilibrium state is determined from the magnetohydrodynamic equation

$$\rho \frac{d\boldsymbol{v}}{dt} = \rho \operatorname{grad}\left(\frac{GM_\odot}{r}\right) - \operatorname{grad} p_g + \frac{1}{c} \boldsymbol{j} \times \boldsymbol{H}, \tag{2·10}$$

where $p_g = \rho c_s^2$ is the gas pressure and, as is well-known, the magnetic force is written in the form

$$\frac{1}{c} \boldsymbol{j} \times \boldsymbol{H} = -\operatorname{grad}\left(\frac{H^2}{8\pi} - \frac{\boldsymbol{H} \cdot \boldsymbol{H}}{4\pi}\right). \tag{2·11}$$

The magnitude of a magnetic field for which the magnetic pressure, $p_m = H^2/8\pi$, is equal to the gas pressure at the equator is denoted by H_1 and that for which p_m is equal to the gravity term, $GM_\odot \rho / r$, is denoted by H_2. The values of H_1 and H_2 for different regions are shown in Table I. If magnetic fields greater than H_1 exist the density distribution in the z-direction is greatly affected and if $H > H_2$ a considerable deviation from the Keplerian law of rota-

Table I. Magnitudes of the magnetic fields H_1 and H_2 in Gauss.

Region	Mercury	Earth	Jupiter	Neptune
r (a.u.)	0.39	1.0	5.2	30
H_1	85	18	1.2	0.072
H_2	3200	550	25	0.92

tion is expected. Now, the magnitude of magnetic fields which are expected to be present in the solar nebula will be estimated in the following two sections.

§ 3. Decay time of magnetic fields

It is well-known that magnetic fields in a uniformly ionized gas grow and decay according to the equation (see a book of Cowling[11] for derivation)

$$\frac{\partial \boldsymbol{H}}{\partial t}=\mathrm{rot}\,(\boldsymbol{v}\times\boldsymbol{H})+\frac{c^2}{4\pi\sigma_e}\left[\nabla^2\boldsymbol{H}+\frac{(\omega\tau)_e(\omega\tau)_i}{H^2}\mathrm{rot}\{(\mathrm{rot}\,\boldsymbol{H}\times\boldsymbol{H})\times\boldsymbol{H}\}\right], \quad (3\cdot1)$$

where $\boldsymbol{v}$ is the gas velocity, σ_e is the electric conductivity, ω $(=eH/mc)$ is the cyclotron frequency, τ is the mean collision time for electrons and ions (denoted by the subscripts e and i, respectively) with neutral atoms and molecules. The Gauss unit is used throughout this paper. In the above equation, terms depending upon pressure gradients of electrons, ions and atoms have been omitted.

Here a brief comment will be given on the derivation of the above equation. An essential point is that, starting from equations for the three components, electrons, ions and atoms (denoted by the subscript a), we have for the relative motion between ions and atoms an equation written in the form

$$d\boldsymbol{v}_i/dt-d\boldsymbol{v}_a/dt=A-(\boldsymbol{v}_i-\boldsymbol{v}_a)/\tau_{ia}\,, \quad (3\cdot2)$$

where $d/dt=\partial/\partial t+(\boldsymbol{v}\cdot\mathrm{grad})$ is the Lagrangian derivative, A denotes a somewhat complicated term depending on $\boldsymbol{j}$, $\boldsymbol{j}\times\boldsymbol{H}$ and pressure gradients but not on $\boldsymbol{v}_i$ nor $\boldsymbol{v}_a$, and τ_{ia} is the collision time of ions with respect to atoms. In the case of a partially ionized gas where τ_{ia} is very small, both sides of Eq. $(3\cdot2)$ tend to zero rapidly even if they are not initially, owing to a rapid relaxation of the relative velocity, $\boldsymbol{v}_i-\boldsymbol{v}_a$.

Using this condition and also the equation of motion for electrons, we can express the electric field $\boldsymbol{E}$ in the form

$$\boldsymbol{E}+\frac{1}{c}(\boldsymbol{v}+\boldsymbol{v}_D)\times\boldsymbol{H}=\left(\frac{\alpha_e\alpha_i}{\alpha_e+\alpha_i}+\beta\right)\frac{\boldsymbol{j}}{n_e e^2}\,, \quad (3\cdot3)$$

where terms depending upon pressure gradients have been omitted, $\boldsymbol{v}_D$ is the drift velocity depending upon both $\boldsymbol{j}$ and $\boldsymbol{j}\times\boldsymbol{H}$ and α_e, α_i and β are given by

$$\alpha_e = n_a m_e \langle\sigma v\rangle_{ea}\,, \tag{3·4}$$

$$\alpha_i = n_a m_i \langle\sigma v\rangle_{ia}\,, \tag{3·5}$$

$$\beta = n_e m_e \langle\sigma v\rangle_{ei}\,, \tag{3·6}$$

where n and m are the number density and the mass of each species, respectively, and $\langle\sigma v\rangle_{jk}$ denotes the Maxwellian mean of the product of the relative velocity v and the collision cross section σ of the species j with respect to the species k. From Eq. (3·3) and the Maxwell equation, we obtain Eq. (3·1) and, in this sense, Eq. (3·1) is valid for fully as well as partially ionized gases if the collisional effects are duly taken into account. The term $\nabla^2\boldsymbol{H}$ in Eq. (3·1) comes from the term on the right-hand side of Eq. (3·3) which represents the Joule loss, while the term proportional to $(\omega\tau)_e(\omega\tau)_i$ comes from a component of $\boldsymbol{v}_D$ which is proportional to $\boldsymbol{j}\times\boldsymbol{H}$.

Now, we consider a partially ionized gas of the solar nebula where the main neutral component is an H_2 molecule. For low-velocity collisions between electrons and H_2 molecules occurring at temperatures as given by Eq. (2·3), σv is almost independent of the relative velocity v and is given by

$$(\sigma v)_{eH_2} = \pi(\alpha e^2/m_e)^{1/2} = 4.4\times10^{-8}\,\mathrm{cm^3\,s^{-1}}\,, \tag{3·7}$$

where $\alpha(=7.9\times10^{-25}\,\mathrm{cm^3})$ is the polarizability of an H_2 molecule. For the collision between a proton and an H_2 molecule, the electron mass in Eq. (3·7) is to be replaced by the reduced mass $2m_H/3$ and we have

$$(\sigma v)_{pH_2} = 1.3\times10^{-9}\,\mathrm{cm^3\,s^{-1}}\,. \tag{3·8}$$

This value of σv is taken as representative for the ion-H_2 collision although ions of the other kinds may be more abundant than protons. Then, comparing Eqs. (3·4) with (3·5) we find $\alpha_i \gg \alpha_e$ and also we have $\alpha_e \gg \beta$ for a slightly ionized gas. The right-hand side of Eq. (3·3) is then given by $\alpha_e\boldsymbol{j}/n_e e^2$ and we have for the electric conductivity

$$\sigma_e = n_e e^2/n_{H_2} m_e \langle\sigma v\rangle_{eH_2}\,, \tag{3·9}$$

which can be written with Eq. (3·7) as

$$\sigma_e = 1.1\times10^{16}x \quad \text{(in the Gauss unit)}, \tag{3·10}$$

where $x = n_e/n_H$ (with $n_H = 2n_{H_2}$) is the ionization degree of a gas considered.

The ionization degree is determined by a balance between the ionization of H_2 molecules, which is due to cosmic rays and also radioactivities contained in dust grains floating in the gas, and the recombination of ions and electrons on the surfaces of these grains (radiative recombination being negligible in the solar nebula under consideration). The rate of ionization (per H atom) due to cosmic rays with energies above 1 GeV and with the present intensity is written in the form

$$\zeta_{\mathrm{CR}} = 1 \times 10^{-17} e^{-l/l_0} \quad \mathrm{s}^{-1}, \tag{3·11}$$

where l_0 is the range of cosmic rays in a gas of H_2, which is about 100 g cm^{-2} according to Umebayashi and Nakano,[12] and l $(=\int_z^\infty \rho dz)$ is the column density of mass traversed by cosmic rays to reach a point considered. The rate of ionization by radioactivities is due mainly to ^{40}K[12] and written in the form

$$\zeta_{\mathrm{RA}} = 4 \times 10^{-22} (n_g/10^{-12} n_H) = 4 \times 10^{-22} f_d \quad \mathrm{s}^{-1}, \tag{3·12}$$

where n_g is the number density of dust grains and f_d is a parameter representing the degree of their depletion from the interstellar value, $10^{-12} n_H$ (most of grains have sedimented at stages considered).

The balance between the ionization and the recombination is written as

$$n_H(\zeta_{\mathrm{CR}} + \zeta_{\mathrm{RA}}) = \langle \sigma v \rangle_{ig} n_i n_g, \tag{3·13}$$

where $\langle \sigma v \rangle_{ig}$ is the collision rate of an ion with a grain, for which we take, for simplicity, a value of 4×10^{-5} cm^3 s^{-1} corresponding to the grain radius 2×10^{-5} cm and the mean ion velocity 3×10^4 cm s^{-1} (which corresponds to the temperature 100°K and the ion mass number 25). From Eqs. (3·11)~(3·13) the ionization degree is expressed in the simple form

$$x = (1/4n_H)(e^{-l/l_0}/f_d + 4 \times 10^{-5}), \tag{3·14}$$

where n_H is in units of cm^{-3}.

In the above, we have estimated the magnitudes of the electric conductivity σ_e and the mean collision times τ_e and τ_i, which are contained in Eq. (3·1). It is to be noticed that τ_e and τ_i is given by $1/n_{H_2}\langle \sigma v \rangle_{eH_2}$ and $1/n_{H_2}\langle \sigma v \rangle_{iH_2}$, respectively. Now, we compare the magnitudes of the two terms contained in the square bracket in Eq. (3·1). In order of magnitude, the ratio of the two terms is equal to $(\omega\tau)_e(\omega\tau)_i$ and using Eqs. (3·7) and (3·8) we have numerically

$$\{(\omega\tau)_e(\omega\tau)_i\}^{1/2} = 5.5 \times 10^{13} H(\mathrm{Gauss})/n_{H_2}(\mathrm{cm}^{-3}). \tag{3·15}$$

For the gas density given by Eq. (2·7) and the magnetic field H_1 given in Table I, the right-hand side of Eq. (3·15) have a value of the order of unity for the regions of the terrestrial planets. Then, the magnetic diffusion term in Eq. (3·1), which is proportional to $(\omega\tau)_e(\omega\tau)_i$, may be neglected as compared with the Joule loss term, $\boldsymbol{\nabla H}$, in so far as the magnetic field is smaller than H_1. It is to be noticed that, in the case of giant molecular clouds, the gas density is so low that $(\omega\tau)_e(\omega\tau)_i$ has a very large value.

Now, we estimate the decay time of magnetic fields due to the Joule loss in each region of the solar nebula. As a typical length which characterizes magnetic configurations in the nebula, we adopt the half-thickness z_0 given by Eq. (2·9). In this case, the decay time is given by

$$t_d = 4\pi\sigma_e z_0^2/c^2 . \qquad (3\cdot16)$$

Then, putting $l=\rho_s$ in Eq. (3·14), we calculate the ionization degree and the decay time at the equator for the two cases where the dust depletion factor f_d is 1 and 1×10^{-4}. The results are shown in Table II for regions of the four planets. The last two columns indicate the decay time in units of the Keplerian period, $t_K=2\pi/\Omega_K$, for each region. As will be shown in § 4, the ratio t_d/t_K indicates the degree of amplification of seed magnetic fields due to differential rotation.

In regions of the terrestrial planets, cosmic rays do not reach the equatorial plane and the ionization is due mostly to radioactivities. Correspondingly, as is shown in Table II, the ionization degree is extremely low and magnetic fields decay very rapidly if they exist at some time. This is not the case for the regions of the giant planets, especially Uranus and Neptune, where ionization is due mainly to cosmic rays. Further, it is probable that the depletion factor is of the order of 1×10^{-4} or even smaller and, in this case, seed magnetic fields are able to be amplified by a large factor as shown in the last column in Table' II. In the above, we have considered the regions, $z\lesssim z_0(r)$, where most of the mass of the nebula is contained. In the outer regions, $z>z_0(r)$, where the gas density is small as shown in Fig. 2 and also in regions inside Mercury's orbit, the ionization degree is much higher than the values given in Table II and, correspondingly, the decay time of magnetic fields is much greater.

Table II. Ionization degree and the ratio of the magnetic decay time t_d to the Kepler time t_K on the equatorial plane of the nebula. The factor of dust depletion is denoted by f_d.

Region	z_0 (a.u.)	n_H (cm^{-3})	ρ_S/ρ_0	x (ioniz. deg.)		t_d/t_K	
				$f_d=1$	$f_d=10^{-4}$	$f_d=1$	$f_d=10^{-4}$
Mercury	0.014	1×10^{16}	69	9×10^{-22}	9×10^{-22}	9×10^{-10}	9×10^{-10}
Earth	0.047	8×10^{14}	17	1×10^{-20}	2×10^{-19}	3×10^{-8}	5×10^{-7}
Jupiter	0.37	9×10^{12}	1.4	7×10^{-15}	7×10^{-11}	9×10^{-2}	9×10^{2}
Neptune	3.3	7×10^{10}	0.10	3×10^{-12}	3×10^{-8}	2×10^{2}	2×10^{6}

§ 4. Growth and decay of seed magnetic fields

Gases of the solar nebula as studied in § 2 are, more or less, in turbulent motions which are caused, for example, by convection due to temperature differences and also by friction due to velocity differences, which are existent between gases and solid bodies. Turbulences are especially strong in the surface layers of the nebula which are irradiated directly by the strong solar wind in the T Tauri stage of the sun, as was pointed out by Elmegreen[13] (see Fig. 2). Gases of the solar nebula are nearly in circular Kepler motion and there exist shear motions due to differential rotation. Then, if there exist some seed magnetic fields due to the turbulences, they are amplified by the winding and stretching of magnetic lines of force while they decay by the Joule dissipation as described in § 3.

In order to know the degree of amplification of magnetic fields, we try to solve Eq. (3·1) where the term proportional to $(\omega\tau)_e(\omega\tau)_i$ is omitted, in the approximation that the conductivity σ_e is constant for a region of the solar nebula considered. Here we consider a region in the neighborhood of an equator with the coordinates, $r=r_0$ and $z=0$, and for the gas velocity we adopt the circular Kepler velocity, i.e., we neglect the back reaction of magnetic fields to gas motion as represented by the last term in Eq. (2·10). Then, the Kepler angular velocity $\Omega(r)$ is written in the form

$$\Omega(r)=\Omega_0+\left(\frac{d\Omega}{dr}\right)_0(r-r_0)=\Omega_0-\frac{3\Omega_0}{2r_0}(r-r_0). \tag{4·1}$$

In terms of the abbreviated notations

$$r-r_0=x,\quad r_0\varphi=y,\quad z=z\,, \tag{4·2}$$

Eq. (3·1), with the omission of the last term, is written in the form

$$\left\{\frac{\partial}{\partial t}+\Omega_0\left(r_0-\frac{3}{2}x\right)\frac{\partial}{\partial y}-\frac{c^2}{4\pi\sigma_e}\nabla\right\}\begin{pmatrix}H_x\\H_y\\H_z\end{pmatrix}=\begin{pmatrix}0\\-\frac{3}{2}\Omega_0H_x\\0\end{pmatrix}, \tag{4·3}$$

where H_x, H_y and H_z denote the components of a magnetic field in the directions of r, φ and z, respectively.

The above equation can be brought into a readily-integrable form by means of a Fourier expansion of the form

$$\boldsymbol{H}(\boldsymbol{r},t)=\sum_{\boldsymbol{k}}\boldsymbol{H}_{\boldsymbol{k}}(t)\exp i\left[k_xx+k_y\left\{y+\left(\frac{3}{2}x-r_0\right)\Omega_0t\right\}+k_zz\right], \tag{4·4}$$

where $\boldsymbol{k}=(k_x,k_y,k_z)$ denotes the wave vector. Originally, the above form of the exponent has been found by applying a transformation to a co-moving coordinate system where the effect of shear motions disappears so that the exponent takes a usual form of a plane wave not depending on t (see Goldreich and Lynden-Bell[14] for the co-moving system). Putting Eq. (4·4) into Eq. (4·3), we have for each Fourier amplitude (the subscript $\boldsymbol{k}$ being omitted in the following)

$$\frac{dH_x}{dt} = -\omega H_x, \quad \frac{dH_z}{dt} = -\omega H_z, \tag{4·5}$$

$$\frac{dH_y}{dt} = -\omega H_y - \frac{3}{2}\Omega_0 H_x, \tag{4·6}$$

where ω is given by

$$\omega = \frac{c^2}{4\pi\sigma_e}\left\{\left(k_x + \frac{3}{2}k_y\Omega_0 t\right)^2 + k_y^2 + k_z^2\right\}. \tag{4·7}$$

It will be seen that the equation, div $\boldsymbol{H}=0$, is expressed in the form

$$\left(k_x + \frac{3}{2}k_y\Omega_0 t\right)H_x + k_y H_y + k_z H_z = 0, \tag{4·8}$$

and this equation is always satisfied as a result of Eq. (4·6) if satisfied at $t=0$. A solution to Eq. (4·6) for the initial amplitude, $\boldsymbol{H}(0)$, of seed magnetic fields is written as

$$H_x(t) = H_x(0)\,e^{-\lambda(t)}, \quad H_z(t) = H_z(0)\,e^{-\lambda(t)}, \tag{4·9}$$

$$H_y(t) = \left\{H_y(0) - \frac{3}{2}H_x(0)\,\Omega_0 t\right\}e^{-\lambda(t)}, \tag{4·10}$$

where

$$\lambda(t) = \int_0^t \omega(t')\,dt'. \tag{4·11}$$

The above solution indicates that the only component which grows by differential rotation is H_y, i.e., the φ-component and that the decay of H_y is slowest for a mode with $k_y=0$, i.e., an axisymmetric mode which has a field pattern (projected to the x-z plane) as illustrated by Fig. 3. In the following, we consider only this mode which has the smallest decay constant as given by

$$\omega = \frac{\lambda(t)}{t} = \frac{c^2}{4\pi\sigma_e}(k_x^2 + k_y^2). \tag{4·12}$$

Further, for the solar nebula, we consider the wave numbers, k_x and k_z, for which the decay is slowest, i.e., a seed magnetic field is amplified to the greatest extent. This mode is given by choosing for k_z its minimum value $\pi/2z_0$ (see Fig. 3 for the relation of k_z with the half-thickness z_0 of the nebula) and for k_x a value such that k_x is considerably smaller than k_z. In this case, for the decay constant we have from Eq. (4·12)

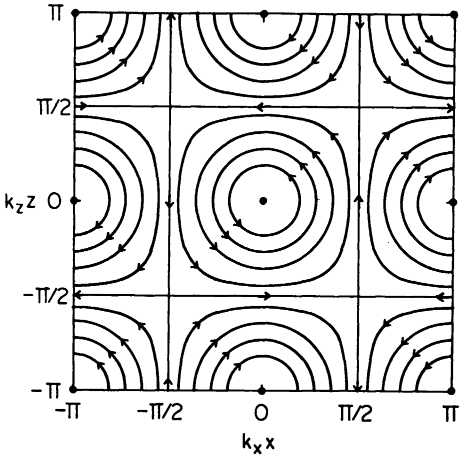

Fig. 3. Magnetic lines of force for a field with the wave numbers k_x and k_z, which is given by $H_x=\partial\varphi/\partial z$ and $H_z=-\partial\varphi/\partial x$ with $\varphi=\cos(k_x x)\cos(k_z z)$.

$$\omega=\frac{\pi}{16}\left(\frac{c}{\sigma_e z_0}\right)^2, \tag{4·13}$$

and the decay time $1/\omega$ is seen to be nearly equal to t_d given by Eq. (3·16). Further, it is seen from Eq. (4·10) that the part of $H_y(t)$ which is proportional to $H_x(0)$ takes, at a time $t=1/\omega$, the maximum value

$$\left\{\frac{H_y(t)}{-H_x(0)}\right\}_{\max}=\frac{3}{2e}\frac{\Omega_0}{\omega}\simeq\frac{t_d}{t_K}, \tag{4·14}$$

and, afterwards, it decays nearly exponentially with time. Thus, the amplification factor of seed magnetic fields due to differential rotation is given by t_d/t_K, as mentioned at the end of § 3.

§ 5. Angular momentum transport and density redistribution

We estimate in this section the rate of density redistribution in the solar nebula which is caused by the transport of angular momentum due to the presence of magnetic viscosity and also mechanical turbulent viscosity. The rate will be important for the study of formation as well as dissipation of the nebula. Now, the effect of the mechanical viscosity will be included in Eq. (2·10) if the gas pressure p_g is replaced by the pressure tensor

$$p_{ik}=\left(p_g+\frac{2}{3}\mu\,\mathrm{div}\,\boldsymbol{v}\right)\delta_{ik}-\mu\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right), \tag{5·1}$$

where μ is the coefficient of turbulent viscocity. Then, in the axisymmetric case under consideration, Eq. (2·6) for v_φ or the equation for the specific angular momentum

$$j=rv_\varphi=r^2\Omega\,, \tag{5·2}$$

is written in the form

$$\rho\left(\frac{\partial j}{\partial t}+v_r\frac{\partial j}{\partial r}+v_z\frac{\partial j}{\partial z}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r^2H_rH_\varphi}{4\pi}+\mu r^3\frac{\partial\Omega}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{rH_zH_\varphi}{4\pi}+\mu r^2\frac{\partial\Omega}{\partial z}\right). \tag{5·3}$$

The right-hand side of Eq. (5·3) represents the transfer of angular momentum. In order to compare the effects of the two viscosities, magnetic and mechanical, we consider a case where H_r, H_φ and H_z are given by H_x, H_y and H_z, respectively, as described in § 4 and a case where H_y takes the maximum value given by Eq. (4·14), i.e.,

$$H_y(t)=-\frac{3}{2}\frac{\Omega_0}{\omega}H_x(t)=\frac{r}{\omega}\frac{\partial\Omega}{\partial r}H_x(t), \tag{5·4}$$

where we have put $H_y(0)=0$ for simplicity and noticed that, in general cases of differential rotation including the Kepler case, Ω_0 is to be replaced by $-(2/3)r\partial\Omega/\partial r$ according to Eq. (4·1). Then, the two viscosity terms in Eq. (5·3) are combined into a simple form, i.e.,

$$\frac{r^2H_rH_\varphi}{4\pi}+\mu r^3\frac{\partial\Omega}{\partial r}=\left(\frac{H_r^2}{4\pi\omega}+\mu\right)r^3\frac{\partial\Omega}{\partial r}, \tag{5·5}$$

which indicates that $H_r^2/4\pi\omega$, where ω is the decay rate as given by Eq. (4·12), may be called the coefficient of magnetic viscosity.

In order to compare the orders of magnitude of the two viscosity coefficients, we put

$$\mu=\frac{1}{3}\rho l_t v_t,\quad l_t=\alpha z_0,\quad v_t=\beta c_s, \tag{5·6}$$

where l_t and v_t are the mean size and velocity of turbulences, respectively, α and β are both non-dimensional constants (of the order of, say, 1/10 or 1/100), z_0 is the half-thickness of the nebula as given by Eq. (2·1) and c_s is the sound velocity given by Eq. (2·2). Further, we put for the magnetic field

$$H_r^2/4\pi=\gamma p_g=\gamma\rho c_s^2, \tag{5·7}$$

where γ is a non-dimensional constant which is unity if the magnetic pressure is equal to the gas pressure. Then, for the ratio of the two viscosities we have from Eqs. (5·5) $\sim$ (5·7)

$$\frac{H_r^2/4\pi\omega}{\mu}=3\sqrt{2}\frac{\gamma}{\alpha\beta}\frac{\Omega_K}{\omega}, \tag{5·8}$$

where we have used Eq. (2·1) which represents the hydrostatic equilibrium condition in the z-direction. It is seen from Eq. (5·8) that the relative importance of the magnetic viscosity is measured essentially by the factor $\Omega_K/\omega\,(\simeq t_d/t_K)$ which takes greatly different values depending upon the regions of the nebula considered, as was shown in Table II.

Now, we estimate the redistribution of the surface gas density due to angular momentum transfer. In the following, the effect of mechanical viscosity alone will be considered, since the magnetic effect may be inferred from Eq. (5·8). In the approximation of a thin disk such that all the physical quantities depend on r but not on z and v_z is negligible, the equations of continuity and motion for axisymmetric gases of the nebula are given by

$$\frac{\partial \rho_s}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(rv_r\rho_s)=0, \tag{5·9}$$

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}=\frac{j^2}{r^3}-\frac{GM_\odot}{r^2}-\frac{1}{\rho}\frac{\partial p_g}{\partial r}, \tag{5·10}$$

$$\rho_s\left(\frac{\partial j}{\partial t}+v_r\frac{\partial j}{\partial r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(\nu\rho_s r^3\frac{\partial \Omega}{\partial r}\right), \tag{5·11}$$

where $\nu=\mu/\rho$ is the coefficient of kinematic viscosity and in Eq. (5·10) a small viscosity term has been omitted for simplicity.

The solar nebula together with the protosun with a radius of, say, $50R_\odot$ is supposed to have been formed as a result of collapse of a rotating gas cloud and, now, we consider stages just after the end of the collapse, where the centrifugal equilibrium has been nearly attained in the solar nebula, i.e., v_r is very small compared to v_φ and the main terms in Eq. (5·10) are j^2/r^3 and $GM_\odot/r^2$ (the pressure-gradient term being smaller even if the gas temperature is 10 times higher than that given by Eq. (2·3) with $L/L_\odot=1$). In this case, first we have from Eq. (5·10) approximately the Kepler relation

$$j=(GM_\odot r)^{1/2}, \tag{5·12}$$

and putting this into Eq. (5·11) we obtain

$$v_r=-\frac{3}{\rho_s r^{1/2}}\frac{\partial}{\partial r}(\nu r^{1/2}\rho_s), \tag{5·13}$$

which indicates that gases flow inwards or outwards depending upon the gradient of $\nu r^{1/2}\rho_s$, as illustrated schematically in Fig. 4. Inserting Eq. (5·13) into Eq. (5·9), we have a diffusion-type equation for ρ_s, i.e.,

$$\frac{\partial \rho_s}{\partial t}=\frac{3}{r}\frac{\partial}{\partial r}\left\{r^{1/2}\frac{\partial}{\partial r}(\nu r^{1/2}\rho_s)\right\}. \tag{5·14}$$

The resemblance to the diffusion equation will become more explicit, if we consider a case where the coefficient of kinematic viscosity is proportional

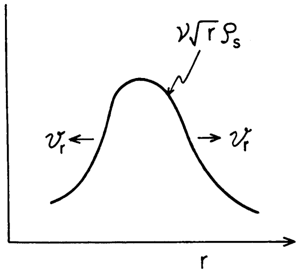

Fig. 4. Direction of the redistribution of surface density in the nebula.

to r (this being the case if Eq. (5·6) holds and the temperature is proportional to $r^{-1/2}$ as given by Eq. (2·3)). In this case, putting

$$\nu=\nu_0 r, \quad x=2(r/3)^{1/2}, \tag{5·15}$$

where ν_0 is a constant, we can rewrite Eq. (5·14) in the form

$$\frac{\partial}{\partial t}(r^{3/2}\rho_s)=\nu_0\frac{\partial^2}{\partial x^2}(r^{3/2}\rho_s), \tag{5·16}$$

which indicates that the surface density ρ_s changes with time in a direction to attain a distribution nearly proportional to $r^{-1.5}$. It is interesting to note that the distribution given by Eq. (2·6) has the same r-dependence and this suggests that the density distribution in the solar nebula as described in § 2 was a result of the above diffusion process. On the other hand, if we consider another case where Eq. (5·6) holds but the temperature is constant with respect to r, we have a distribution of the form r^{-2}, instead of $r^{-1.5}$. It is to be noticed that, from the distribution of the planetary mass, the surface density distribution proportional to r^{-2} cannot entirely be ruled out, in view of uncertainties involved in its derivation.

Now, in order to see whether the redistribution of gas density at the formation stage of the nebula was possible or not, we estimate a time required for the redistribution of ρ_s to proceed over a radial distance Δr. Either from Eq. (5·16) or from Eq. (5·14), this redistribution time is given by

$$t_r=(\Delta r)^2/3\nu, \tag{5·17}$$

and, if we adopt the viscosity law given by Eq. (5·6) and also T and z_0 given by Eqs. (2·3) and (2·9), respectively, we have

$$t_r=1.0\times10^2(\Delta r)^2/\alpha\beta\gamma \text{ yr}, \tag{5·18}$$

where both Δr and r are in a.u. This time-scale may be too large if α and β are of the order of 10^{-2} at the formation stage.

However, at the end of the collapse of the nebula, strong shock waves as well as turbulances are generated and the nebula is hot and oscillating, more or less, violently for a certain period of time. Accordingly, for this stage we put $\alpha=\beta=1$ and, further, we adopt a gas temperature which is ten times greater than that given by Eq. (2·3). In this case, we have

$$t_r=10t_K(\varDelta r)^2/r^{5/2}, \tag{5·19}$$

where t_K is the Kepler period for a region considered and both $\varDelta r$ and r are again in a.u. The above equation indicates that about ten times oscillations are required to attain the density redistribution. This number of oscillations may not be too large, but without detailed hydrodynamic computations of collapse and bounce it is not certain whether the effect of mechanical viscosity alone is sufficient or the aid of magnetic viscosity is necessary.

Anyhow, at the formation stage of the nebula, the ionization degree of gases is much greater than that given in Table II and the effect of magnetic viscosity on the density redistribution will be considerably large. Especially, hot gases in regions inside the orbit of Mercury are greatly affected by the magnetic viscosity due to differential rotation and this may give rise to a rapid decrease of gas density in these regions. On the other hand, at the stages considered, the sun is supposed to be an early T Tauri star with coronal magnetic activities which are, say, 10^5 times greater than the present sun. It is probable that the sun at these stages is losing its angular momentum by nearly isotropic ejection of strong winds into the outer space, a small part of it being transferred to the region of Mercury of the solar nebula.

Finally, we consider a long-term dissipation of the solar nebula which has a structure as described in § 2. The time of dissipation due to turbulent viscosity is given by Eq. (5·18) but, at present, it is difficult to estimate the precise values of α and β. On the other hand, the effect of magnetic and turbulent viscosities on the dissipation is expected to be important especially in outer low-density regions with $z>z_0$. Elmegreen[13] pointed out that gases in the outer regions, where dynamical pressure of the solar wind is comparable to gas pressure, are highly turbulent owing to the Helmholtz-Kelvin instability. He estimated the total time of dissipation due to viscosities in these layers while Sekiya, Nakazawa and Hayashi[15] estimated it from a simple energy consideration and obtained a result which agrees nearly with that of Elmegreen. Both of the results depend on the luminosity of the solar wind assumed and it is probable that the dissipation time lies in the range between 1×10^6 and 1×10^8 yr.

More precise estimation of the dissipation time is desirable for us since, according to our theory of planetary formation,[5),10)] all the planets except for Uranus and Neptune are considered to have been formed in stages before the dissipation of the solar nebula. Saturn is marginal is this respect; its core mass is nearly the same as Jupiter but the mass of its gaseous envelope is about three times smaller than that of Jupiter. This indicates that at a formation stage of Saturn the gas density in the nebula had already diminished by a certain factor and at the later stages where Uranus and Neptune were formed the gas of the nebula had been dissipated almost completely.

§ 6. Concluding remarks

We have estimated in §§ 3 and 4 the rates of decay and amplification of magnetic fields in the solar nebula which is in a state of thermal and mechanical equilibrium. The results indicate that magnetic effects on the structure of the nebula are negligible for regions of the terrestrial planets, except for the outermost layers of very low density, since the surface density in these regions is so high that ionization by cosmic rays is very small and magnetic fields decay very rapidly owing to the Joule dissipation. This is not the case for regions of the giant planets, especially for regions of Uranus and Neptune where magnetic fields can be amplified to considerable strengths owing to differential rotation of the nebula.

Further, we have studied in § 5 the redistribution of mass in the solar nebula due to the presence of magnetic and turbulent viscosities, with a result that the density distribution in the solar nebula was possibly determined by the effects of these viscosities at the formation stage of the nebula. In order to verify the above viscosity effects on the formation of the solar nebula, it is desirable to make in near future hydrodynamic computations of a collapsing gas cloud, where the viscosity effects will be duly taken into account. In this respect, it is to be pointed out that Tscharnuter[16] already computed a collapse of a rotating cloud taking into account the effect of turbulent viscosity.

Until now, a number of two- or three-dimensional computations have been made for the collapse of rotating gas clouds. Many of these computations have been performed for the initial condition of a rotating sphere with uniform density, which satisfies Jeans' condition for a sphere. In this respect, a comment will be given in the following. In giant molecular clouds where stars are born, there exist complex turbulent motions and even magnetic fields. If we consider a relatively small region in the clouds, there may exist a certain axis of symmetry as given by, say, the axis of rotation and it is plausible that condensation of matter first proceeds along this axis in a direction to form a rotating isothermal disk where, in the axial direction, pressure gradient balances with gravity. Surface density of this quasi-equilibrium disk increases with time and, finally, when the surface density reaches a critical value the disk will fragment into a number of smaller disks.

The half-thickness (in the direction of the rotating axis) of the parent and daughter disks considered above is given by

$$z = c_s^2/\pi G\rho_s\,, \tag{6·1}$$

and let the ratio of the radius r of a fragment to the above z be denoted by f, i.e.,

$$r = fz\,. \tag{6·2}$$

Then, according to Goldreich and Lynden-Bell,[17] we have about $f=6.7$ for a mode of fragmentation which grows most rapidly. For a fragment of this mode with mass $1M_\odot=\pi r^2\rho_s$, we have as the initial condition for collapsing

$$r=2.8\times10^{16}\left(\frac{6.7}{f}\right)\left(\frac{20}{T}\right)\mathrm{cm},\quad z=4.1\times10^{15}\left(\frac{6.7}{f}\right)^2\left(\frac{20}{T}\right)\mathrm{cm}\,,$$

$$\rho_s=0.83\left(\frac{f}{6.7}\right)^2\left(\frac{T}{20}\right)^2\mathrm{g\ cm^{-2}},\quad \rho=1.0\times10^{-16}\left(\frac{f}{6.7}\right)^4\left(\frac{T}{20}\right)^3\mathrm{g\ cm^{-3}}\,, \tag{6·3}$$

where we have put $\mu=2.34$ for the mean molecular weight and $\rho=\rho_s/2z$ is the gas density. As seen from the dependence of ρ on the factor f, the volume of this fragment is about 10^3 times smaller than that given by the spherical Jeans condition for the same temperature.

For the reasons mentioned above, it will be more plausible to adopt the above values for the initial condition rather than the spherical Jeans condition. The only remaining parameter is the angular velocity Ω which lies probably in the range between 1×10^{-14} and $1\times10^{-13}\ \mathrm{s^{-1}}$ as observed in giant molecular clouds. For such a value of Ω, the ratio of rotational energy to gravitational energy of the fragment is very small, i.e., about $4\times10^{-4}(\Omega/10^{-13}\ \mathrm{s^{-1}})^2$. It is expected that the collapse of the above fragment will soon become adiabatic, since ρ_s is relatively high from the first, and it ends at a stage where pressure-gradient or centrifugal force counteracts gravity. At this stage, generation of shock waves as well as large oscillations and, further, subsequent relaxations due to viscosities are expected to occur.

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