Portfolio Diversification, Futures Markets, and Uncertain Consumption Prices

Peter Berck and Stephen G. Cecchetti

This paper examines the robustness of the Keynes-Hicks backwardation hypothesis for futures markets in a model that admits diversification and inflation protection as speculative motives. It presents a criterion in terms of the correlation of futures price with anticipated consumption net of other asset holdings for the Keynes-Hicks proposition to be true. The paper finds the effect of changes in net wealth and commodity demand on the risk premium, spread, open interest, and storage.

Key word: futures.

The conditions for a futures market to have contracts outstanding, for hedgers to pay speculators to accept risk, and for the futures markets to affect commodity storage are related to the characteristics of both the underlying commodity market and the assets market. The important features of the commodity market include the demand in each period and the costs of storage, while the characteristics of the assets market include the size, composition, and return on the economy’s wealth. In the context of a two-period model with inflation, uncertain prices, and uncertain asset returns, this paper studies the way in which these components of economic structure affect observed outcomes in a futures market.

One important futures market outcome is the risk premium. In the Keynes-Hicks (KH) view of futures markets, hedgers pay speculators a risk premium for the insurance services the speculators offer. As a theoretical proposition, the KH view depends on the premise that speculators increase their risk when they buy futures. However, empirical evidence, such as that presented in the Cootner (1960a,b)-Telser debate, lends only weak support to the KH proposition and its premise. The robustness of the KH proposition is examined in an equilibrium model of cash and futures markets that admits diversification and inflation protection as speculative motives, and, therefore, need not yield the KH proposition. In this general model, a simple criterion for the validity of the KH proposition is presented. It is found that the size of the risk premium, and whether or not the KH hypothesis holds, depend on the size and sign of the covariance of the futures price with (risk-adjusted) anticipated second-period consumption net of other asset holdings.

The success or failure of a futures market is measured by the size of the open interest—the number of contracts outstanding. Economic attributes that lead to a large open interest include a high level of demand in future periods, low storage costs, and the ability of individuals in the economy to trade successfully. Risk trades are most successful when the commodity represents a small percentage of aggregate real wealth and consumption and the individuals are sufficiently risk averse.

The paper is organized in five parts, beginning with this introduction. The second part presents the pricing of assets in a two-period world where both second-period asset payoffs and second-period consumption prices are uncertain. It is an extension of both Stein’s model, where storage is endogenous, and Stoll’s model of futures as part of a complete
capital market. It makes use of Berck and Cecchetti’s earlier results on consumers which are very similar to those of Stiglitz. In the third section, the model is solved for the equilibrium values of storage, open interest, the risk premium, and other variables. The fourth section presents the conditions under which the KH proposition is true and examines how the open interest, risk premium, etc., change with demand, storage costs, and wealth. The next section discusses the results and presents some further conclusions. Finally, there are two appendices that provide the formal mathematical arguments for the fourth section.

Asset-Pricing Equations

Consider the portfolio problem of dividing wealth, \( W \), among assets, \( z \), at purchase prices, \( s \), to maximize expected utility when both second-period asset prices and consumption prices are uncertain. Expanding the first-order conditions for this problem in a Taylor series (as Arrow and Pratt did for the direct utility function) gives both a risk premium and an asset-pricing equation in terms of expected real return (for a demonstration that third-order terms could matter - we assume away for the sake of the algebra in the last section—see Stiglitz). The expected real return is approximately the expected nominal return less the covariance of the asset’s return with the cost of a unit step along the income expansion path, which is the inflation premium. The risk premium can be characterized by the covariance of the asset’s return with the entire asset portfolio net of anticipated consumption. When applied to pricing futures, these equations are found to differ from constant consumption price asset-pricing equations in their inclusion of terms related to second-period consumption. They differ from the equations of Grauer and Litzenberger in their use of a Taylor expansion and their admission of nonhomothetic utility functions as well as their emphasis on second-period consumption rather than on covariance of marginal utility and real price.\(^2\)

The wealth holder’s problem is to maximize utility subject to an endowment constraint in the face of uncertainty posed by unknown asset and consumption goods prices. Define the \( N \)-dimensional vector \( p \) with the first \( M \) elements containing the stochastic asset prices and the remaining elements containing the stochastic consumption goods prices. Similarly, the vectors \( z \) and \( s \) are \( N \)-dimensional with their last \( N-M \) elements equal to zero.

The utility maximization problem can be stated formally as

\[
\text{max } E[v(p'z, p)] \quad \text{subject to } s'z = W
\]

where \( E \) is the expectation operator; \( v \) is the indirect utility function defined as the maximum utility attainable at a given level of income, \( y = p'z \); and \( p \) is prices.

Equation (1) differs from the usual expected utility problem because it includes prices twice. Prices change both the value of income-producing assets and the cost of purchased goods. For instance, the value of wheat producers’ assets depends greatly on the \( p_w \) of wheat, but the gain from a price increase is partially offset by an increase in food prices.

Assuming that the first and \( i \)th assets are both included in an optimal portfolio and that \( s_i 
eq 0 \), the first-order condition for this problem is:

\[
\frac{Ev_{p_i} }{Ev_{p_1} } = s_i.
\]

A futures contract is an asset that (in its perfect form) costs nothing (\( s_F = 0 \)) and pays off its closing or settlement price, \( p_F \), less its opening price, \( p_F^o \). Applying this definition to equation (2) gives a more general form (not restricted by homotheticity) of the Grauer-Litzenberger asset-pricing equation for futures,

\[
p_F^o = \frac{Ev_{p_F v_y} }{Ev_y}.
\]

This first-order condition can be extended to the case of an imperfect future (one for which \( s_F 
eq 0 \)) by pricing with respect to a nominal bond. A nominal bond is an asset that costs a dollar and pays off \( n \) dollars in all states of nature. Carrying out the algebra, one obtains

\[
p_F^o = \frac{Ev_{p_F v_y} }{Ev_y} - ns_F.
\]

Equation (4) is a general pricing equation.
that allows both for costs that occur in the first period (brokerage fees for futures and purchase price for stocks) and for a return based on an opening (or striking) price.

These pricing equations can be interpreted if \( v_u \) and \( p_F \) are expanded in a Taylor expansion about their means. The interpretation will be in terms of the bundle of goods purchased at expected prices, which will be called the anticipated bundle, \( x(E_y, E_p) \). It is this anticipated bundle that makes the choice of assets dependent on “real” magnitudes. The algebra proceeds by expanding \( v_u \) as follows:

\[
(5) \quad v(y, p) = v_y + (v_{yy} z' + v_{yp})\hat{p},
\]

where \( \hat{p} = p - \bar{p} \) and the right-hand side is evaluated at the expected prices, \( \bar{p} = E_p \). Differentiating Roy’s identity (\( -x = \frac{v_y}{v_u} \)) with respect to \( y \) and substituting in equation (5),

\[
(6) \quad v(y, p) = v_u + [v_{yy}(z - x) - v_{xy}]\hat{p},
\]

where the right-hand side is, again, evaluated at \( \bar{p} \). Rearranging equation (4),

\[
(7) \quad E[v_e(p_F - p_t - n_S)] = 0,
\]

expanding \( p_t \) about its mean, defining absolute risk aversion (\( r = -\frac{v_{uu}}{v_u} \)), and using equation (5) gives the pricing equation in terms of the first and second moments of \( p \):

\[
(8) \quad p_F^0 = \bar{p}_F - x'E\hat{p}_F - r \bar{x}'E\hat{p}_F + r x'E\hat{p}_F - n_S.
\]

The price of a future is its expected (nominal) price less the covariance of the cost of income expansion with the futures price, less the coefficient of risk aversion times the covariance of the portfolio and the futures price, plus the coefficient of risk aversion times the covariance of anticipated consumption and the futures price, less the nominal rate of interest times the “up front” costs.

One can say a little more about the first two terms of (8) for the homothetic utility function, \( v = h[f(p) \phi] \). By the usual Taylor series expansion, expected real return (\( E[p_F] \) is approximately \( f(\bar{p}) \) \( [p_F - x'\hat{p} \hat{p}_F] \)). Thus, the futures price is approximately the expected closing price divided by the deflator less the risk term.

Expression (8) differs in two ways from the result without uncertain consumption prices. First, one subtracts \( x'\hat{p} \hat{p}_F \) from the expected return. This term adjusts the mean return downward so that high nominal payoffs when prices are high are reflected accurately as low real payoffs. Second, from the covariance of the market portfolio with expected return, one subtracts the covariance of consumption, a term that adjusts the covariance of return so that it becomes the covariance with the portfolio net of anticipated consumption.

Futures Market Equilibrium

A reasonable representation of a futures market requires equilibrium in the futures (or asset) market itself, in the first-period cash market, and in the second-period cash market. This section makes use of the asset-pricing equation (8) to give a symmetrical view of hedgers and speculators. What distinguishes these agents is that the speculators (“wealth holders”) own the wealth of the economy but do not own physical stocks, while the hedgers (“storers”) own the stocks but do not own the economy’s wealth. The institutional factors that keep these agents from diversifying their portfolios through the obvious means of selling shares in stocks will be discussed below.

The remaining two markets are quite simple. There is a linear demand for the stored good in both the first and second periods. In the first period, the good is split between storage and satisfying the demand; in the second period, this storage is added to whatever may be stochastically produced and is then consumed. The solution of the model proceeds by substituting the other equations into the equation for the futures market until it becomes an equation in storage alone. Given equilibrium storage, it is easy to find the open interest, spread, and risk premium.

The storers of commodities—especially agricultural commodities—are (or, traditionally, were) closely held firms dependent on loans for operating capital and on futures markets for insurance against price changes. They hold title to storage facilities and they pay off mortgages. In addition to these fixed investments, they choose three other assets for their portfolios: (a) a commodity, \( S \), purchased at price, \( p_c \), stored at (possibly uncertain) cost, \( c(S, t) \), where \( t \) is factor prices and hold for price, \( p_S \); (b) loans at the gross nominal rate of interest, \( n \); and (c) futures \( z_F \), without margin or brokerage, at opening price, \( p_F^0 \), and clos-

---

1 Technically, this model has only a single agent in each group; the generalization to \( n \) identical agents per group is tedious and trivial.
The storer’s choice problem leads to the economy’s demand for storage, to its supply of futures, and to a relationship between the cash and the futures price. An interior solution to the optimum problem (9) is assured by the assumption that $S > 0$. This means that the first-order conditions to (9) hold with strict equality. To find the relationship between prices, take the derivative of $E v$ with respect to $S$ and $z_F$ and set them equal to zero and each other:

$$
E v_o \cdot (P_F - n p_c - c\prime) = E v_u \cdot (P_F - P_F^0) = 0
$$

where $c\prime$ is marginal cost.

On rearranging (10), noting that $n p_c$ is nonstochastic, using the definition $\text{cov}(c\prime, v_d) = E(c\prime v_d) - E(c\prime)E(v_d)$, and dividing by $E v_u$, we get:

$$
P_F^0 = n p_c + E c\prime + \frac{\text{cov}(c\prime, v_d)}{E v_u}.
$$

When there is no risk of fire or shortage of transportation (or whatever would make $c\prime$ stochastic) and when, as assumed previously, there is no basis risk, then

$$
P_F^0 = n p_c + c\prime(S, t),
$$

which can be inverted to give the demand for storage,

$$
S = c^{-1}(P_F^0 - n p_c).
$$

Finding the supply of futures requires the use of the approximate pricing equation (8). The storers, denoted by $A$, have asset bundle $z^A$, which is the vector with the $F$th element $c^{-1}(P_F^0 - n p_c) + z_F^A$ and all other elements zero. Solve for $z_F^A$ to yield the supply of futures:

$$
z_F^A = \frac{E p_F - P_F^0 + r^A x^A E \tilde{p}_F - x_u^A E \tilde{p}_F - \frac{1}{\sigma^2} c^{-1}(P_F^0 - n p_c)}{\sqrt{\sigma^2}}
$$

where $\sigma^2 = E \tilde{p}_F \tilde{p}_F$.

The demand side of the futures market consists of a financier (wealth holder), who is denoted as $B$. He is not in the physical storage business; therefore, $S$ is zero in this portfolio. He may choose any futures holding, $z_F^B$. The remainder of his portfolio is the economy’s endowment of claims on real capital, $z^*$. The wealth holder does not hold shares in the physical storage firms because syndicating so many firms would involve a very large transactions cost, much larger than the transactions costs of creating a single futures market. Although futures are imperfect claims on the firm (they fail to diversify the risks of physical storage), they are good enough claims to dominate the use of the more costly shares. The argument for the financier to hold the economy’s real capital is the familiar portfolio argument of the capital asset-pricing model: The supply of real capital is completely inelastic within a given period, and someone must own it. The wealth holder’s demand for futures can be found by letting $z$ in the pricing equation (8) be $z_F^B + z^*$ and solving for $z_F^B$:

$$
z_F^B = \frac{E p_F - P_F^0 + r^B (x^B - z^*)' E \tilde{p}_F - x_u^B E \tilde{p}_F}{\sqrt{\sigma^2}}.
$$

The risk premium, open interest, and quantity stored are equilibrium notions. They are determined by finding the prices and quantities that simultaneously clear the asset market, the first-period cash market, and the second-period cash market. To find these quantities and examine how they change with changes in exogenous parameters, the Taylor-series approximations will be used, and homotheticity on the assumed identical consumption preferences of the agents will be imposed. The asset

---

* Brokerage and margin costs are small, and they are neglected in the analysis for the algebraic simplicity. Basis risk is the risk that $p_t \neq p_s$, usually because of delivery point or grade. There is no convenience yield—gain to having inventory to sell from—in a two-period model.

* $z_F^A$ and $z_F^B$ are used both as the vector quantity with a nonzero $F$th place and as a scalar quantity, the $F$th element.
market clears when the net futures position of the economy is zero:

$$z_{F^A} + z_{F^B} = 0.$$  

The assumption of identical homothetic preferences implies that $x_{y^A} = x_{y^B}$. Adding equation (14) to equation (15) and solving for $p_{F^B} - Ep_F$ expresses equilibrium in the futures market:

$$p_{F^0} - Ep_F = \frac{\left(\frac{r^A + r^B}{r^A}\right) (x^A - S e_{S} + x^B - z^*) - x_{y}}{E \hat{p}_{F}}$$  

where $e_i$ is the unit vector in the $i$th direction.

The first-period cash market clears when the store’s bid price and the consumer’s demand price are the same. Let

$$p_c = a_1 - b_1(S_0 - S)$$  

be the consumer’s bid price in the first period. By equation (17)—the asset-pricing equation for hedgers [equation (8)]—and equation (12), the first-period cash market clears when

$$n[a_1 - b_1(S_0 - S)] + c' = Ep_F - x_{y} E \hat{p}_{F} + r^A(x^A - S e_{S} - z_{F} e_{F})' E \hat{p}_{F}.$$  

The second-period cash market is cleared when storage equals demand,

$$p_S = a_2 - b_2S + \epsilon$$  

where $\epsilon$ is a random variable with expectation zero and variance $\sigma^2$.

The equilibrium system [equations (16) through (19)] can be solved by substituting the other equations into equation (16). Combining the cash futures spread, equation (12), and the first-period demand equation (17) yields

$$p_{F^0} = \frac{n[a_1 - b_1(S_0 - S)] + c'}{\sigma^2 (r^A + r^B)} [r^B(x^B - z^*)] - r^A(x^A - S^* e_{S})' E \hat{p}_{F}.$$  

It is these futures market-clearing equations, not the first-order conditions (2) and (8), that determine the quantities and prices in the futures market.

**Comparative Statics**

Under reasonable assumptions, the equilibrium futures model of the last section yields the intuitive results usually attributed to Keynes and Hicks and further explored by Stein. This section sets out sufficient conditions for the risk premium $(Ep_F - p_{F^0})$ to be positive, for an increase in the cost of storage to decrease storage, and for increases in speculation to decrease the risk premium. It also examines the spread $(p_{F^0} - p_c)$ and the effects on other variables of changes in either storage or demand. Finally, the consequences of increasing the size of total wealth are explored.

Equation (16) can be used to derive the conditions under which the KH hypothesis holds. Since homothetic preferences imply $x_y = (x/y)$, the risk premium will be positive whenever

$$c' + na_1 - nb_1S_0 - a_2 + (nb_1 + b_2)S^* = \left[\left(\frac{r^A + r^B}{r^A}\right) (x^A - S^* e_{S} + x^B - z^*) - x_{y}\right] E \hat{p}_{F}.$$  

Note that $x^A$, $x^B$, and $x_y$ are functions of $Ep_F$ and, by equation (21), of $S$. With these substitutions, equation (22) contains only the single endogenous variable, $S$, with $S^*$ as its solution. Open interest,

$$z_{F} = z_{F^B},$$  

is computed by solving equations (14) and (15) for

$$(Ep_F - p_{F^0} - x_{y} E \hat{p}_{F}),$$  

eliminating the common term, and solving for $z_{F}$ as follows:

$$z_{F} = \frac{1}{\sigma^2 (r^A + r^B)} [r^B(x^B - z^*)] - r^A(x^A - S^* e_{S})' E \hat{p}_{F}.$$  

Richard and Sundaresan derive a condition similar to (24) in a continuous time framework. Their model, however, does not allow simple computation of comparative static results.
The sign of the risk premium depends on the covariances involving anticipated consumption and asset holdings. Therefore, the expression does not necessarily imply that the KH hypothesis is true. First, a special case in which the KH hypothesis is true will be provided, and then a more general case in which it may not be true will be discussed.

The simplest case is that of Stoll. Anticipated consumption does not matter. Formally, it is the assumption that the left-hand side of equation (24) is zero,

\[ E(\bar{P}F) < (S^*e_S + z^*)' E\bar{p}F. \]  

(24) \[ x^A(1 - \frac{1}{r^AyA}) + x^B(1 - \frac{1}{r^ByB}) \]

Equation (25) states that the covariance of the whole market portfolio and the futures price need to be positive for the KH hypothesis to hold. The whole market portfolio is the value of physical storage \((S^*e_S)\), which certainly correlates positively with the futures price plus the value of the economy's capital stock \((z^*/\bar{p})\). The value of the capital stock is usually represented by an index such as the S&P 500. As an empirical matter, one expects the covariance of the S&P 500 and the futures price to be positive—so the lack of a motive to hedge future consumption seems to guarantee a positive risk premium.8

The general case at hand includes the possibility that wealth holders and storers will have an interest in stabilizing their real second-period income. This consumption effect is the weighted covariance of the anticipated bundle and the futures price. The weights depend on \(r_y\), the measure of relative risk aversion. In the demand theory literature, this measure is the Frisch parameter, and its size depends upon wealth. Dervis, de Melo, and Robinson summarize other studies that show that the Frisch parameter is between 1.5 and 2.0 for developed countries and much higher for developing countries. So long as \(r^Ay^A\) and \(r^By^B\) are both greater than one, which is in accord with the empirical evidence, the terms on the left-hand side of (24) in parentheses are simply fractions less than one. These fractions multiply the covariance of anticipated consumption and the futures price.

When \(r_y > 1\), the KH hypothesis that the risk premium is positive is true whenever the portfolio effect—covariance of whole market and futures—is larger than the consumption effect—weighted covariance of anticipated consumption and futures price.

Again, looking at (24)—given the conditions above—it is immediate that an increase in risk aversion of either group increases the left-hand side of the expression and makes the risk premium smaller.

Turning this around, the risk premium will be negative when \(r^Ay^A\) and \(r^By^B\) are large, and anticipated consumption correlates better with the futures than the market correlates with the futures. When futures are useful to wealth holders because they “hedge” inflation and relative price changes, there is no need, even in theory, for the KH hypothesis of normal backwardation.

The solution to the model in the third section permits one to study the effects of three actions: (a) increasing first-period demand, (b) increasing second-period demand, and (c) increasing storage costs. To do this requires three additional assumptions. The first assumption is that changing the price of the stored good affects the covariance of the cost of the anticipated consumption bundle and the stored good only through its effects on the stored good,

\[ E\left(\frac{d(x^A)}{d\bar{p}_F} \bar{p}_F\right) = \sigma^2 \frac{d(x)_F}{d\bar{p}_F} \]

where \((x)_F\) is the consumption of the stored good. The second assumption is that the consumers have identical homothetical preferences. Thus, the slope of the second period market-demand curve is proportional to the slope of the demand curve of any agent. Letting \(y^T\) be the sum of the agent’s incomes and \(f\) be the deflator described in section 3,

\[ \frac{1}{b^2} = y^T \frac{d(\bar{p}_F/f)}{d\bar{p}_F}. \]

Finally, it is assumed that all agents have constant absolute risk aversion.

With these additional assumptions: If the first period market demand increases or the second period market demand decreases (or storage costs increase), then (a) the optimal level of storage falls, (b) the risk premium decreases, and (c) open interest decreases; but (d) the decrease in open interests is less than the decrease in storage. In addition, (e) a decrease in second period demand decreases the
spread. The effects of changes in first period demand and costs on the spread are ambiguous.

These comparative statics results are derived in appendix 1 by the usual methods. Since all the results depend upon the same arguments, the next few paragraphs will provide a short, less mathematical derivation of the results pertaining to an increase in storage costs.

First, whenever the equilibrium storage increases, so does the open interest. Equation (23) gives the open interest as a function of equilibrium storage consumption, both directly and through the endogenous variable. The direct effect of increasing storage is that it increases the amount of risk that can be traded and, therefore, directly leads to increased open interest [e.g., the term with \( e_F \) in equation (a) of appendix 1]. The indirect effects come through the consumption bundles and partially offset each other (the terms with \( x^a \) and \( x^B \)). More storage means lower second-period price and higher second-period consumption of the storable and, therefore, under the additional assumptions, a larger covariance of the anticipated bundle with the futures price. To hedge this additional consumption risk, both storers and wealth holders would like to have a larger claim on the storable. The wealth holder does this by buying more futures, while the storer accomplishes it by selling fewer; so the consumption effects are partially offsetting. Given the homothetic preferences assumption, the strength of the effects is proportional to the group's share of total income. It turns out that the direct effect of having more storage to hedge outweighs the storers need to hedge more of his consumption, so the result follows.

Increasing storage increases the risk premium which is shown by differentiating equation (16) [appendix 1, equation (f)]. Again, there is a direct effect of increasing the variance in the nominal income of the economy as a whole which adds to the risk that must be borne by the agents and drives up the price charged for bearing this risk—the risk premium (this is the term with \( e_F \)). There is the offsetting indirect effect that both agents wish to hold more of the claims against the storable asset to hedge their anticipated consumption (these are the terms with \( x^a + x^B \)). The direct effect is on the order of total income, while the indirect effects are on the order of the share of the groups; so the net effect is to drive up the risk premium. There is an additional term in \( x_u \) that further drives up the premium.

Finally, there is the matter of the effect of increased costs (or demand) on storage. One finds these by taking the total derivative of equation (22) which gives equation (h) of the appendix. Here the algebra is more complicated, but the ideas are similar. The left-hand side of (22) uses the equal profitability of holding futures or the commodity itself to express the negative of the risk premium in terms of equilibrium storage. Clearly, increasing storage costs drive the profits from storing down and makes the left-hand side of (22) greater than the right-hand side. Storing less drives the profits back up and tends to equilibrate the system in its effects on the real side of the market. On the financial, or right-hand side of (22), decreasing storage has exactly the effects described above. Directly, it drives down the risk premium while, through consumption, it tends to drive it up. The direct effect dominates so the net effect on the right-hand side is also equilibrating. Thus, an increase in costs decreases storage. Decreased storage decreases both open interest and risk premium. Similar arguments can be made for all the other comparative statics results.

The second major result of this section is derived by considering the consequences of an increase in the size of the nonagricultural sector. In the KH view of futures markets, an increase in the number (or wealth) of speculators should decrease the risk premium. There are several ways in which such an increase in speculation could occur. Agents in group B might become richer in an exogenous fashion such as a spurt in GNP growth, a change in trade policy, or a shift in labor laws that is unfavorable to workers. Another possibility is that agents formerly in the excluded group would be able to exercise their demands in the futures market. This could occur in a less developed economy with the development of other organized capital markets and, particularly, with syndicating of firms that were formerly held in private. For a country such as the United States, this would require that claims be created against the wealth of the excluded agents—presumably claims against their human capital—so that those agents could participate in the capital markets. 9

9 The forward sale of human capital in the United States is not unheard of, but it is rare. While professional athletes can sell their labor forward, as can people qualifying for student loans, steelworkers cannot.
nally, an increase in wealth might simply correspond to the development process itself. Any of these cases could correspond to an increase in speculation, and any of these changes are modeled by considering the case in which the portfolio owned by the wealth holder expands proportionately so that \( z^* \) becomes \( k z^* \).

The result is as follows. If the covariance of the wealth holder's anticipated second-period consumption net of their asset holdings is positive, \( E(x^b - z^*)'pF > 0 \), then an increase in wealth will increase both optimal storage and open interest and decrease the risk premium. The comparative statics leading to this proposition are in appendix 2. The intuition is that of the KH hypothesis. When the wealth holders use the futures as a second-period consumption hedge, increasing the size of wealth will increase the demand for the future (increasing the supply of what are normally called "speculators") and drive down the risk premium. From the perspective of the storers, this lowers the cost of storage, encouraging them to hold the commodity into second period. The increase in open interest naturally follows.

Conclusion

A model of futures markets in a two-period world requires equality of supply and demand in three markets: the first-period cash market, the second-period cash market, and the market for futures contracts. This formulation considers a futures contract as a standard financial asset, and it naturally gives rise to a supply of speculation. Speculation depends on expected return and covariance of the future with the market portfolio net of risk adjusted anticipated consumption. The size of the risk premium embodied in these contracts depends \( \text{inter alia} \) on how large the anticipated consumption covariance or consumption hedging effect is. Because the correlation of commodity prices and the market portfolio can be as large as the correlation of any asset price and the market portfolio and because consumption has the same value as income, there is every reason to believe that the anticipated consumption effects will be large, at least in a two-period model.\(^\text{10}\) Thus, the KH hypothesis that storers will pay speculators a risk premium need not hold true even in theory.

A large open interest, synonymous with a successful futures markets, depends on both the economic circumstances that lead to large storage and the character of the remaining assets market. Of course, increased second-period demand, decreased storage costs, and decreased first-period demand lead to higher storage and open interest. Increasing the size of other wealth, which increases the supply of speculation, also increases open interest. Finally, it is conjectured, but not proven, that decreasing the correlation between the market return and the return on a futures also increases open interest.

The last section presented a set of comparative static results regarding the effect of changes in structural economic conditions on futures market outcomes. One of the uses of these results is in a comparison of advanced industrial and developing countries. In futures markets wholly internal to a country, one would expect the different characteristics of developed and developing countries to lead to a different size risk premium and different levels of storage and open interest.\(^\text{11}\) Developing countries have much higher Frisch parameters (higher degree of relative risk aversion) than developed countries do. Therefore, the effects of anticipated consumption are more strongly felt in the developing countries. All else the same, a higher Frisch parameter—and a strong positive correlation of consumption and the futures market price—leads to a smaller risk premium. The large share of gross national product (GNP) in developing countries originating in the agricultural sector and the large portion of consumption derived from agriculture creates differences as well. In a developing country the covariances of both anticipated consumption and other assets with the futures price are likely to be large, positive, and, perhaps, not too different from each other. The open interest and risk premia will be small. When much of wealth derives from the same sources, namely, agriculture, the portfolios of all agents will change together—it is of little use for individuals with the same risks to try to trade risk. In the developed economy case, the value of the assets of the

\(^\text{10}\) In a multiperiod model, one might expect these effects to be attenuated by the small percentage of wealth devoted to consump-

\(^\text{11}\) Of course, some developing counties simply fix internal terms of trade making futures markets for internal purposes useless.
storers and the value of the market portfolio, as well as anticipated consumption, are likely to correlate differently with the futures price. Hence, there is a greater possibility for a KH risk trade. A developed country derives a larger share of its wealth from the industrial sector and this, by itself, leads to a larger open interest and lower risk premium for obvious reasons that, when nonagricultural wealth is larger, there are "more" claims on other assets which can be used to diversify the risk associated with commodity storage. Taken together, these arguments show why futures markets are more likely to be developed world phenomena.

Other uses of parts of this model include the home purchase decision and the "food security" of the developing nations. The home purchase decision is fundamentally a question of hedging the future consumption of housing. Since a home purchase is risk reducing in the sense of correlation with the portfolio net of anticipated consumption, the prospective homeowner would be willing to pay a "risk premium" over the expected cost of housing services for its purchase. The developing nations' problem of food security is whether to invest in agriculture to grow food or whether to invest in other sectors and trade for food. The advantage of growing food is that it is risk reducing when anticipated consumption is included in the notion of risk; the disadvantage is that it may not be the investment with the highest expected value. The problem is formally analogous to the decision problem at the beginning of the second section and illustrates that the portfolio problem, with uncertain consumption, has applications to many fields of economics besides finance.

[Received January 1984; final revision received August 1984.]

References


Appendix 1

Comparative Statics on Costs and Other Variables

An outline is provided here of the details of the comparative statics. The consequences of changes are examined in first-period demand, \( a_1 \), second-period demand, \( a_2 \), and storage costs, \( t \), on open interest, storage, and the \( p \) premium. Four intermediate results are derived. First, any action that increases equilibrium storage increases open interest but on a less-than-one-for-one basis. Equation (23) gives equilibrium open interest as a function of the equilibrium storage \( S^* \) both directly and through \( x \). Taking the total derivative of (23) gives

\[
\frac{dz}{ds^*} = \frac{1}{\sigma^2 (\sigma^2 + \rho^2)} \left[ \frac{dr}{ds^*} \frac{dp}{dr} \frac{dx}{ds^*} + \rho \frac{dr}{ds^*} \frac{dp}{dr} \right] + \frac{1}{\sigma^2 (\sigma^2 + \rho^2)} \frac{dF}{dx} \frac{dF}{ds^*} \frac{dx}{ds^*} \frac{dp}{dr} \frac{dr}{ds^*}
\]

where \( e_F \) is the unit vector in the Fth direction.

Use the additional assumptions mentioned in the text to simplify

\[
\frac{dp}{dr} \quad \frac{dF}{dx} \quad \frac{dF}{ds^*}
\]

and

\[
\frac{dF}{dx} \quad \frac{dF}{ds^*}
\]

where
Do the same for agent A to get

\[
\frac{dz_F}{dS^*} = \frac{r^a y^a}{(r^a + r^b) y^r} + \frac{r^b (y^r - y^a)}{(r^a + r^b) y^r}.
\]

Both terms are positive, so \(dz_F/dS^*\) is greater than zero.

The second intermediate result gives the change in the spread with respect to \(a_1\), \(a_2\), and \(t\). Combining (12) and (15) gives

\[ p_0^a - p_r = (n - 1)(a_1 - b_1 S_0 + b_2 S) + c'. \]

Take the derivatives of (e):

\[ \frac{d(p_0^a - p_r)}{dS} = \frac{\partial p_0^a - p_r}{\partial S} \frac{dS}{da_1} + n - 1 \]
\[ \frac{d(p_0^a - p_r)}{da_1} = \frac{\partial p_0^a - p_r}{\partial S} \frac{dS}{da_1} \]
\[ \frac{d(p_0^a - p_r)}{dt} = \frac{\partial p_0^a - p_r}{\partial S} \frac{dS}{dt} + \frac{\partial^2 c}{\partial S^2} \]

where

\[ \frac{\partial^2 c}{\partial S^2} = (n - 1)b_1 + \frac{\partial^2 c}{\partial S^2} > 0. \]

Because of the properties of cost functions, \(\partial^2 c/\partial S^2 \geq 0\) and \(\partial^2 c/\partial S^2 t > 0\).

The third result is that any action that increases equilibrium storage also increases the gain to the long position which is the risk premium. Equation (16) gives the risk premium as a function of \(S\) and \(x\), as argued earlier, \(x\) is a function of \(p_F\) which, by equation (21), is a function of \(S\) so the right-hand side of equation (16) can be viewed as a function of the single endogenous variable, \(S\). Because \(a_1\), \(a_2\), and \(c\) do not appear in equation (16), the effect of changing these variables on the risk premium comes solely through \(S\). Carrying out the algebra,

\[ \frac{d(E_p - p_0^a)}{dS} = \left[ R \left( dx \frac{dp_F}{dp_F} + \frac{dx}{dp_F} \right) \right] dS^* + \frac{dx}{dp_F} dS^* + n - 1 \]

Using the assumptions and results derived above for \(ds_F/dS^*\), this expression simplifies to

\[ \frac{ds_F}{dt} = \frac{\partial^2 c}{\partial S^2} \left[ -\left( b_1 + b_2 \frac{\partial^2 c}{\partial S^2} \right) \right] \]

which is negative because \(y^r > y^a + y^b\), marginal cost increases with increasing factor costs, and all other symbols are positive.

One can use the same method to show that \(ds_F/dx > 0\), i.e., increasing second-period demand increases storage. Similarly, \(ds_F/dx < 0\), i.e., increasing first-period demand decreases storage. Putting these four results together yields the assertion in the text.

### Appendix 2

#### Comparative Statics on Wealth

In this appendix comparative statics of increasing the size of the nonagricultural sector of the economy are provided. The rest of the economy is scaled up by a factor of \(k\) from \(S^*\) to \(kS^*\). Three comparative statics experiments are examined: (a) the change in storage, (b) the change in risk premium, and (c) the change in the open interest all with respect to a change in \(k\).

In order to find \(ds_F/dk\), solve (22) for \(S^*\) and take its derivative with respect to \(k\) evaluated at \(k = 1\). By homotheticity, an increase in \(S^*\) implies a proportional increase in the wealth holders consumption so

\[ \frac{ds_F}{dk} = \left[ R \left( dx \frac{dp_F}{dp_F} + \frac{dx}{dp_F} \right) \right] dS^* + \frac{dx}{dp_F} dS^* + n - 1 \]

Using the additional assumptions gives

\[ \frac{ds_F}{dk} = \frac{R x^a - z^a)}{y^r - y^a} \left( E_y \frac{dp_F}{dp_F} - \frac{dx}{dp_F} \right) \]

and by using the additional assumptions, this simplifies to

\[ \frac{ds_F}{dk} = R \sigma^2 \left( \frac{y^r - y^a}{y^r} \right) + \frac{\sigma^2}{y^r} > 0. \]

An (a) increase in costs, (b) increase in first-period demand, or (c) decrease in second-period demand decreases optimal storage.

Recall the cost function \(c(S, t)\) where \(t\) represents cost increasing factors such as factor prices. Taking the total derivative of equation (22) with respect to \(t\) and solving yields

\[ \frac{dx}{dp_F} \]

Using the additional assumptions gives

\[ \frac{dx}{dp_F} = R \sigma^2 \left( x^a - z^a \right) \frac{dp_F}{dp_F} \left( 1 - \frac{y^a}{y^r} \right) + \frac{\sigma^2}{y^r} + nb_1 + b_2 + \frac{dc}{ds^*} \]

Note that each term of the denominator is positive so that
(c) \[ \text{sign} \left( \frac{dS^*}{dk} \right) = \text{sign} \left[ (x^b - z^*)' \hat{E} \hat{p}_F \right]. \]

The risk premium is

(d) \[ E\hat{p}_F - p_F = \left( -1 \right) \left[ c'(S^*, r) + na_1 - nb_1 S^* - a_2 + (nb_1 + b_2) S^* \right], \]

so

(e) \[ \frac{d(E\hat{p}_F - p_F)}{dk} = - \left( \frac{\partial c}{\partial S^*} + nb_1 + b_2 \right) \frac{dS^*}{dk}. \]

Since \( \frac{\partial^2 c}{\partial S^2}, n, b_1, \) and \( b_2 \) are positive; an increase in \( k \) decreases the risk premium if and only if

\[ E[(x^b - z^*)' \hat{p}_F] > 0. \]

Finally, from equation (23) the derivative of the open interest with respect to \( k \) is

(f) \[ \frac{dz_F}{dk} = \frac{\partial z_F}{\partial S} \frac{dS^*}{dk} + \frac{r_F}{(r^2 + r^*)^{1/2}} E(x^b - z^*) \hat{p}_F. \]

Since \( \frac{\partial z_F}{\partial S^*} > 0 \) and \( dS^*/dk \) has the same sign as \( E(x^b - z^*) \hat{p}_F \), an increase in \( k \) increases open interest if and only if \( E[(x^b - z^*)' \hat{p}_F] > 0. \) Putting all this together gives: If \( (x^b - z^*)' \hat{E} \hat{p}_F > 0 \), then, with an increase in wealth, (a) \( S^* \) rises, (b) the risk premium decreases, and (c) open interest rises.