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Solving Vening Meinesz-Moritz inverse problem in isostasy

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SUMMARY

Vening Meinesz' inverse problem in isostasy deals with solving for the Moho depth from known Bouguer gravity anomalies and "normal" Moho depth (T_0, known, e.g. from seismic reflection data) using a flat Earth approximation. Moritz generalized the problem to the global case by assuming a spherical approximation of the Earth's surface, and this problem is also treated here. We show that T_0 has an exact physical meaning. The problem can be formulated mathematically as that of solving a non-linear Fredholm integral equation of the first kind, and we present an iterative procedure for its solution. Moreover, we prove the uniqueness of the solution.

Second, the integral equation is modified to a more suitable form, and an iterative solution is presented also for this. Also, a second-order approximate formula is derived, which determines the Moho depth to first/linear order by an earth gravitational model (EGM), and the remaining short-wavelength/non-linear part, of the order of 2 km, can be determined by iteration. A direct, second-order formula, in principle accurate to the order of 25 m, combines the first-order solution from an EGM with a second-order term, which may include terrestrial Bouguer gravity anomalies around the computation point.

Key words: Inverse theory; Gravity anomalies and Earth structure; Crustal structure; Mechanics, theory, and modelling.

1 INTRODUCTION

The Mohorovicic discontinuity (or shortly Moho), discovered by the Croatian seismologist A Mohorovicic in 1931, is a well-known density contrast between the Earth's crust and mantle. The densities of the Earth's crust and mantle are traditionally assumed to be of the orders of 2.67 and 3.27 g cm^{-3}, respectively, and seismic reflection technique has revealed that the depth to the Moho is of the order of 30–100 km beneath continents, while it reduces to the order of 8 km under ocean basins. However, it should be emphasized that the seismic Moho does not necessarily agree with the gravimetric one, as they are based on different assumptions and hypotheses. Except for its interests among geologists, geophysicists and geodesists, the Moho has lately also gained an interest for such a disparate issue as the implementation of Article 76 of the United Nation's Convention for the Law of the Sea from 1982, which is concerned with the definition and extension of the continental shelf (e.g. Cook & Carleton 2000; ABLOS 2006).

Assuming that, at least in general, the Earth's gravity field is isostatically compensated by the variable depth to the Moho, Vening Meinesz (1931) formulated an inverse problem of isostasy to solve for the Moho depth from observed Bouguer gravity anomalies at the Earth's surface and the known density contrast at the Moho boundary. In contrast to previous studies with pure local topographic compensation, for example, J.H. Pratt's theory from 1854 and 1859 (Heiskanen & Moritz 1967, p. 134), Vening Meinesz approach considers a regional isostatic compensation. As numerous later studies on this problem, for example, by Parker (1972), Oldenburg (1974) and Granser (1986), Vening Meinesz provided a solution in planar approximation of sea level. However, in principle, the solution is related with the global gravity field, and recent (mostly satellite derived) earth gravitational models (EGMs) should be important data sources in such solutions. The Vening Meinesz' problem was generalized to a global, spherical earth model by Moritz (1990, section 8.3.2), and our investigation is closely related with this study, which we hereafter name the Vening Meinesz-Moritz (VMM) Problem of Isostasy.

2 DERIVATION OF THE INTEGRAL EQUATION

2.1 The basic integral equation

The VMM problem (Vening Meinesz 1931; Moritz 1990) is to determine the Moho depth T such that the compensating attraction $A_C(T)$ totally compensates the Bouguer gravity anomaly (Δg_B)on the Earth's surface (approximated by a sphere of radius R), implying that the isostatic anomaly ($\Delta g_{\rm I}$) vanishes for each point on the Earth's surface, that is,

$$\Delta g_{\rm I} = \Delta g_{\rm B} + A_{\rm C}(T) = 0. \tag{1}$$

It is assumed that the Moho density contrast ($\Delta\rho$) is a known constant, yielding that the compensation potential at an arbitrary point P can be written as the Newton integral

$$V_{\rm C}(P) = k \iint_{\sigma} \int_{R-T}^{R-T_0} \frac{r^2 {\rm d}r}{l_P} {\rm d}\sigma, \tag{2}$$

where $k = G\Delta\rho$, G is the gravitational constant, T_0 is the *'normal'* Moho depth (to be exactly defined in Section 2.3), σ is the unit sphere and $l_{\rm P} = \sqrt{r_{\rm P}^2 + r^2 - 2rr_{\rm P}t}$. Here, $r_{\rm P}$ and r are the geocentric distances to the computation point and integration point, respectively, and t is the cosine of the geocentric angle (to be denoted ψ) between these two points. Eq. (2) can also be written

$$V_{\rm C}(P) = V_{\rm C0}(P) + {\rm d}V_{\rm C}(P), \tag{3a}$$

where

$$V_{\rm C0}(P) = k \iint_{\sigma} \int_{R}^{R-T_0} \frac{r^2 {\rm d}r}{l_{\rm P}} {\rm d}\sigma \tag{3b}$$

is the potential contribution from the mean Moho potential (given by the normal Moho surface), and

$${\rm d}V_{\rm C}(P) = k \iint_{\sigma} \int_{R-T}^{R} \frac{r^2}{l_{\rm P}} {\rm d}r\, {\rm d}\sigma \tag{3c}$$

is the potential contribution from the variable Moho depth T. (At this point it deserves to be mentioned, that the potential $V_{\rm C0}$, and therefore also the attraction $A_{\rm C0}$, that follows below, does not contribute to the isostatic compensation.) The compensation attraction at point P can thus be decomposed into the terms

$$A_{\rm C}(P) = A_{\rm C0}(P) + {\rm d}A_{\rm C}(P) \tag{4}$$

with the components

$$A_{\rm C0}(P) = -\frac{\partial V_{\rm C0}}{\partial r_{\rm P}} = k \iint_{\sigma} \int_{R}^{R-T_0} \frac{r^2(r_{\rm P} - rt)}{l_{\rm P}^3} {\rm d}r\, {\rm d}\sigma, \tag{5}$$

and

$$\begin{aligned} {\rm d}A_{\rm C}(P) &= -\frac{\partial {\rm d}V(P)}{\partial r_{\rm P}} = k \iint_{\sigma} \int_{R-T}^{R} \frac{r^2\,(r_{\rm P} - rt)}{l_{\rm P}^3} {\rm d}r\, {\rm d}\sigma \\ &= k \iint_{\sigma} [J_2(r_{\rm P}, t, T) - J_3(r_{\rm P}, t, T)] {\rm d}\sigma, \end{aligned} \tag{6a}$$

where (see, e.g. Bois 1961)

$$J_2(r_{\rm P}, t, T) = r_{\rm P} \int_{R-T}^{R} \frac{r^2}{l_{\rm P}^3} {\rm d}r = -r_{\rm P} \left[\frac{r_{\rm P}t + (1 - 2t^2)x}{1 - t^2} \frac{1}{l_{{\rm P}x}} \right]_{x=x_1=R-T}^{x=R} + r_{\rm P} \ln \frac{R - r_{\rm P}t + l_{\rm P0}}{R - T - r_{\rm P}t + l_{{\rm P}x_1}} \tag{6b}$$

and

$$\begin{aligned} J_3(r_{\rm P}, t, T) = t \int_{R-T}^{R} \frac{r^3}{l_{\rm P}^3} {\rm d}r &= \left[\left\{ x^2 + \frac{6t^2 - 5}{1 - t^2} r_{\rm P} x t + \frac{(2 - 3t^2)}{1 - t^2} r_{\rm P}^2 \right\} \frac{t}{l_{{\rm P}x}} \right]_{x=x_1=R-T}^{x=R} \\ &\quad + 3r_{\rm P}t^2 \ln \frac{R - r_{\rm P}t + l_{\rm P0}}{R - T - r_{\rm P}t + l_{{\rm P}x_1}}. \end{aligned} \tag{6c}$$

Here, we have introduced the abbreviations $l_{\rm P0} = \sqrt{r_{\rm P}^2 + R^2 - 2Rr_{\rm P}t}$ and $l_{{\rm P}x} = \sqrt{r_{\rm P}^2 + x^2 - 2r_{\rm P}xt}$. In particular, for $r_{\rm P} = R$ and $\tau = T/R$ one obtains

$$J_2(R, t, \tau) = R^2 \left[\frac{\left(1 - 2t^2\right)(1 - \tau) + t}{\left(1 - t^2\right) l_{{\rm P}x_1}} - \frac{\left(1 - 2t^2\right) + t}{\left(1 - t^2\right) l_{\rm P0}} \right] + R \ln \Psi \tag{7a}$$

and

$$J_3\left(R, t, \tau\right) = \frac{R^2 t}{1 - t^2} \left[\left\{ (x/R)^2 \left(1 - t^2\right) - \left(5 - 6t^2\right) \frac{x}{R} t + 2 - 3t^2 \right\} \frac{1}{l_{{\rm P}x}} \right]_{x=x_1}^{x=R} + 3Rt^2 \ln \Psi, \tag{7b}$$

where

$$\Psi = \frac{1 - t + l_{\rm P0}/R}{1 - t - \tau + l_{{\rm P}x_1}/R}. \tag{7c}$$

For $r_P = R$ and introducing $s = 1 - \tau$, we thus obtain the following global surface integral from eq. (6a):

$$\mathrm{d}\tilde{A}_C(P) = \mathrm{d}A_C(P)_{r_P=R} = kR\iint_\sigma K(\psi, s)\mathrm{d}\sigma, \tag{8}$$

where the integration function can thus be written

$$K(\psi, s) = \frac{1}{R}\left[J_2(R, t, \tau) - J_3(R, t, \tau)\right]. \tag{9}$$

In eq. (16) of Section 2.2, we present $K(\psi, s)$as a series in Legendre's polynomials.

Based on the above derivations the integral equation for $\tau = T/R = 1 - s$ and $\tau_0 = T_0/R$ becomes

$$\Delta g_B + \tilde{A}_C(P) = 0, \tag{10}$$

where $\tilde{A}_C(P) = A_C(P)_{r_P=R}$, and $A_C(P)$ was given by eqs (4)–(6a), which yields also

$$\mathrm{d}\tilde{A}_C(P) = -\Delta g_B - A_{C0}(P). \tag{11}$$

In eq. (11), the unknown function τ is implicitly hidden in the integral of $\mathrm{d}\tilde{A}_C(P)$of eq. (8), and the right-hand side is regarded as a given function (which requires that τ_0 be given). By inserting eq. (8) into eq. (11), we get the non-linear Fredholm integral equation of the first kind

$$R\iint_\sigma K(\psi, s)\mathrm{d}\sigma = f(P), \tag{12a}$$

where

$$f(P) = -\left[\Delta g_B(P) + A_{C0}(P)\right]/k, \tag{12b}$$

whose solution for s may not necessarily be unique or even exist. In Section 5, we will dwell upon the uniqueness of the solution.

2.2 The kernel $K(\psi, s)$expressed as a series

The inverse distance of eq. (2) can be expressed by the following series in Legendre's polynomials, $P_n(t)$:

$$l_P^{-1} = \sum_{n=0}^{\infty}\frac{r^n}{r_P^{n+1}}P_n(t); t = \cos\psi, \text{ for } r_P \geq R \geq r, \tag{13}$$

which leads to the following expressions of the residual compensation potential and attraction of eqs (3c) and (6a), respectively:

$$\mathrm{d}V_C(P) = k\sum_{n=0}^{\infty}\iint_\sigma\int_{R-T}^{R}\frac{r^{n+2}}{r_P^{n+1}}\mathrm{d}r\, P_n(t)\,\mathrm{d}\sigma$$

$$= k\sum_{n=0}^{\infty}\frac{R^2}{n+3}\left(\frac{R}{r_P}\right)^{n+1}\iint_\sigma\left[1 - s^{n+3}\right]P_n(t)\,\mathrm{d}\sigma, \tag{14}$$

and

$$\mathrm{d}\tilde{A}_C = (\mathrm{d}A_C)_{r_P=R} = kR\sum_{n=0}^{\infty}\frac{n+1}{n+3}\iint_\sigma\left[1 - s^{n+3}\right]P_n(t)\,\mathrm{d}\sigma. \tag{15}$$

Comparing eqs (8) and (15), we thus obtain

$$K(\psi, s) = \sum_{n=0}^{\infty}\frac{n+1}{n+3}\left(1 - s^{n+3}\right)P_n(t). \tag{16}$$

This formula can be rewritten in a closed form. To derive that we first rewrite the equation as

$$K(\psi, s) = \sum_{n=0}^{\infty}\left(1 - s^{n+3}\right)P_n(t) - 2\left[S(1) - S(s)\right], \tag{17a}$$

where

$$S(s) = \sum_{n=0}^{\infty}\frac{1}{n+3}s^{n+3}P_n(t) = s\int_0^s\sum_{n=0}^{\infty}x^{n+2}P_n(t)\mathrm{d}x = s\int_0^s\frac{x^2}{L_x}\mathrm{d}x$$

$$= \frac{s}{2}\left[(x+3t)L_x + \left(3t^2 - 1\right)\ln(x - t + L_x)\right]_{x=0}^{x=s} \tag{17b}$$

with $L_x = \sqrt{1 + x^2 - 2xt}$. Inserting eq. (17b) into eq. (17a), one finally arrives at

$$K(\psi, s) = \frac{1}{L_0} - \frac{s^3}{L} + 3t(1-s) + \left(s^2 + 3st\right)L - (1+3t)L_0$$

$$+ \left(3t^2 - 1\right)\left(s\ln\frac{s - t + L}{1 - t} - \ln\frac{1 - t + L_0}{1 - t}\right), \tag{18}$$

where we have also introduced

$$L_0^{-1} = \sum_{n=0}^{\infty} P_n(t) = \frac{1}{\sqrt{2(1-t)}} = \frac{1}{2\sin\frac{\psi}{2}}, \tag{19a}$$

and

$$L^{-1} = \sum_{n=0}^{\infty} s^n P_n(t) = \frac{1}{\sqrt{1-2st+s^2}}. \tag{19b}$$

As $s < 1$, eq. (18) shows that $K(\psi, s)$ is a regular function all over the sphere. Also, as s is always close to 1, the function is always close to zero, except for $t = 1$. This property will be useful in solving eq. (12a).

2.3 The definition of T_0

The *normal* Moho depth T_0 will now be precisely defined from a mathematical point of view and given a physical meaning. If we assume that the Bouguer gravity anomaly has the global average zero, that is, no zero-degree spherical harmonic, it follows from eqs (1) and (4) that

$$\tilde{A}_{C0} + \frac{1}{4\pi}\iint_\sigma \mathrm{d}\tilde{A}_C \mathrm{d}\sigma = 0, \tag{20}$$

where we use the notation $\tilde{X} = (X)_{r_P=R}$. As the first term of this equation can be simplified to

$$\tilde{A}_{C0} = \frac{4\pi kR}{3}\left[(1-\tau_0)^3 - 1\right], \tag{21}$$

it follows from eqs (5) and (15) that

$$\frac{4\pi kR}{3}\left(1-\frac{T_0}{R}\right)^3 - \frac{kR}{3}\iint_\sigma \left(1-\frac{T}{R}\right)^3 \mathrm{d}\sigma = 0, \tag{22}$$

with the following solution for T_0:

$$\frac{T_0}{R} = 1 - \sqrt[3]{\frac{1}{4\pi}\iint_\sigma (1-T/R)^3\,\mathrm{d}\sigma}, \tag{23}$$

with a first-order approximation equal to the global mean Moho depth

$$T_0 \approx \bar{T} = \frac{1}{4\pi}\iint_\sigma T\mathrm{d}\sigma. \tag{24}$$

For the practical realization of T_0, we may use Moho depths T determined from seismic data.

The physical meaning of the solution for T_0 becomes obvious if we rearrange eq. (22) as follows:

$$\frac{4\pi\rho_0}{3}\left[R^3 - (R-T_0)^3\right] = \rho_0 \iint_\sigma \int_{R-T}^{R} r^2 \mathrm{d}r, \tag{25a}$$

which implies that T_0 should be chosen under the law of mass conservation of the compensation masses, that is, the total mass between the undisturbed Moho with constant radius T_0 and the Earth's surface S (approximated by the sphere of radius R) is equal to the mass between the real Moho and S.

Alternatively, eq. (25a) can be written in the form

$$\Delta\rho_0 \iint_\sigma \int_{R-T_0}^{R-T} r^2 \mathrm{d}r\mathrm{d}\sigma = 0, \tag{25b}$$

which means that normal Moho at depth T_0 is defined such that the total mass anomalies caused by the true Moho above and below this surface are the same.

3 A SOLUTION TO THE ORIGINAL INTEGRAL EQUATION FOR τ

In this section, we look for a solution procedure to the non-linear integral equation of the first kind, eq. (12a). According to Chambers (1976, p. 148) '*finding a solution*' (of such an equation) '*is entirely a matter of luck, trial and error*'. First, we will derive a (approximate) solution for τ, which can serve as an initial value for the iterative solution that follows.

3.1 An approximate solution for τ

Considering that τ is only of the order of 0.01, eq. (15) can be approximated to first order by

$$\mathrm{d}\tilde{A}_{\mathrm{C}}(P) \approx kR\sum_{n=0}^{\infty}\frac{n+1}{n+3}\iint_{\sigma}\{1-[1-(n+3)\tau]\}\,P_n(t)\mathrm{d}\sigma \approx 2\pi kR(\tau_{\mathrm{P}}+\tau_0), \tag{26}$$

where we have used Dirac's delta function on the sphere

$$\delta(t) = \sum_{n=0}^{\infty}(2n+1)P_n(t)/(4\pi),$$

with the property

$$\iint_{\sigma}\delta\left(\cos\psi_{\mathrm{PQ}}\right)h_{\mathrm{Q}}\mathrm{d}\sigma_Q = h_{\mathrm{P}} \tag{27}$$

for any continuous function h_{Q} on the sphere, and $\tau_0 = T_0/R$. Hence, eqs (26), (1), (4) and (21) yield

$$\tau_{\mathrm{P}} \approx -\frac{\Delta g_{\mathrm{B}}+\tilde{A}_{\mathrm{C0}}}{2\pi kR} \approx -\frac{\Delta g_{\mathrm{B}}}{2\pi kR}+\tau_0. \tag{28}$$

Eq. (28) thus suggests that τ can be uniquely estimated from Δg_{B} provided that τ_0 is known.

3.2 Successive approximations

A tentative iteration procedure for eq. (12a) can be written

$$\tau^{i+1}(P) = \tau^i(P) + c\left[f(P)/R - \iint_{\sigma}K\left(\psi, s^i\right)\mathrm{d}\sigma\right]; i = 0, 1, 2, \ldots, \tag{29a}$$

or, for $T^i = R\tau^i$,

$$T^{i+1} = T^i + c^*\left[f(P) - R\iint_{\sigma}K(\psi, s^i)\mathrm{d}\sigma\right]; i = 0, 1, 2, \ldots \tag{29b}$$

where c is a suitably selected constant and $c^* = cR$. Once $s^i = s$, it implies that eqs (29a) and (29b) have converged to the unique solution of eq. (12a). Unfortunately, there is no safe way to choose the constant c (not even its sign) to warrant convergence of the equations, but only trials remain. (In some numerical tests we found $c = 0.35$ appropriate.)

4 A MODIFIED INTEGRAL EQUATION

As discussed above, the iterative solutions to the original integral equation, eq. (12a), do not necessarily lead to convergence. Here, we will reformulate the basic equation and present new iterative solutions from it. The results will be based on the following proposition.

$$\text{Proposition 1}: 2f(P) - M_{\mathrm{Q}}\left[H\left(\psi_{\mathrm{PQ}}\right)f(Q)\right] = 4\pi RM_{\mathrm{Q}}\left[B\left(\psi_{\mathrm{PQ}}, s_{\mathrm{Q}}\right)\right], \tag{30a}$$

where

$$M_{\mathrm{Q}}[\,] = \frac{1}{4\pi}\iint_{\sigma}[\,]\mathrm{d}\sigma_{\mathrm{Q}}, \tag{30b}$$

$$H(\psi) = \sum_{n=0}^{\infty}\frac{2n+1}{n+1}P_n(\cos\psi) \tag{30c}$$

and

$$B(\psi, s) = \sum_{n=0}^{\infty}\frac{2n+1}{n+3}\left(1-s^{n+3}\right)P_n(\cos\psi). \tag{30e}$$

Proof: First, we consider the orthogonality of Legendre's polynomials when integrated over the sphere:

$$M_{\mathrm{Q'}}\left[P_n(\cos\psi_{\mathrm{PQ'}})P_k\left(\cos\psi_{\mathrm{Q'Q}}\right)\right] = \begin{cases}\dfrac{P_n\left(\cos\psi_{\mathrm{PQ}}\right)}{2n+1}, & \text{if } n = k\\ 0 & \text{otherwise.}\end{cases} \tag{31}$$

Inserting the right-hand side of eq. (12a) into the left member (LM) of eq.(30a), and considering eq. (31), we obtain

$$\mathrm{LM} = 8\pi R\sum_{n=0}^{\infty}M_{\mathrm{Q}}\left[\frac{n+1}{n+3}\left(1-s^{n+3}\right)P_n\left(\cos\psi_{\mathrm{PQ}}\right)\right] - 4\pi R\sum_{n=0}^{\infty}M_{\mathrm{Q}}\left[\frac{1-s^{n+3}}{n+3}P_n\left(\cos\psi_{\mathrm{PQ}}\right)\right], \tag{32}$$

which equals the right member of the proposition.

The kernel function $H(\psi)$, presented in eq. (30c) and frequently denoted Hotine's function in the honour of late M. Hotine, can be written in the closed form

$$H(\psi) = \frac{2}{L_0} - \ln\left(\frac{1 - t - L_0}{1 - t}\right). \tag{33a}$$

or

$$H(\psi) = \cos ec\,(\psi/2) + \ln\left[1 + \cos ec\,(\psi/2)\right]. \tag{33b}$$

Similarly, the function B becomes in a closed form:

$$B(\psi, s) = \frac{2}{L_0} - 2\frac{s^2}{L} + 5S(s) - 5S(1), \tag{34}$$

where L_0 and L were defined below eq. (8b), and $S(s)$ was defined in eq. (17b).

Below we consider two solutions to eq. (30a).

4.1 Successive approximations

With reference to Proposition 1, it is obvious that the global integral of the kernel $B(\psi, s)$ is very much dominated by the value of the kernel around point P, while the contribution from other points on the sphere is small. Considering also that

$$M_Q\{B\,(\psi, s_P)\} = \frac{1}{3}\left[1 - (1 - \tau_P)^3\right], \tag{35}$$

we easily obtain from the proposition above:

$$\tau_P = F(P)/R + \tau_P^2 - \tau_P^3/3 + M_Q\left\{B\,(\psi, s_P) - B(\psi, s)\right\}, \tag{36a}$$

or

$$T_P = F(P) + \frac{T_P^2}{R} - \frac{T_P^3}{3R^2} + RM_Q\left\{B\,(\psi, s_P) - B\,(\psi, s)\right\}, \tag{36b}$$

where

$$F(P) = [2f(P) - M_Q\left\{H(\psi_{PQ})f(Q)\right\}]/(4\pi). \tag{36c}$$

The modified integral eq. (36b) leads to the iterative process

$$\tau_P^{k+1} = F(P)/R + (\tau_P^k)^2 - \frac{\left(\tau_P^k\right)^3}{3} + M_Q\left\{B\left(\psi, s_P^k\right) - B\left(\psi, s^k\right)\right\}, k = 1, 2, \ldots \tag{37}$$

with $\tau^0(Q) = RF(Q)$.

Using eq. (30e) with the binomial series

$$s^{n+3} = (1 - \tau)^{n+3} = 1 - (n+3)\,\tau + (n+3)\,(n+2)\,\tau^2/2 - \ldots \tag{38}$$

and

$$s_P^{n+3} = \left(1 - \tau_P\right)^{n+3} = 1 - (n+3)\,\tau_P + (n+3)\,(n+2)\,\tau_P^2/2 - \ldots \tag{39}$$

we can see that the integral of eq. (36a) is of order $\tau^2 \approx 10^{-4}$, while $F(P)/R$ is of order 10^{-2}. We also expect the Moho to be a rather smooth surface. Hence, the process should converge well for all practical applications (in the sense that the resolution is small but finite).

4.2 A solution to order τ^2

We now expand $B(\psi, s)$ of eq. (30e) to order τ^2 (and we neglect higher order terms). The result for its global average is

$$\begin{aligned} M_Q[B\,(\psi, s)] &= \sum_{n=0}^{\infty}(2n+1)M_Q\left[\left(\tau - \frac{n+2}{2}\tau^2\right)P_n\,(\cos\psi)\right] \\ &= \tau - \tau^2 - M_Q\left[W(\psi)\tau^2\right], \end{aligned} \tag{40}$$

where (*cf.* Heiskanen & Moritz 1967, p. 39)

$$M_Q\left[W\,(\psi)\,\tau^2\right] = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)nM_Q\left[\tau^2 P_n(\cos\psi)\right] = -\frac{1}{8}M_Q\left[\frac{\tau^2 - \tau_P^2}{\sin^3(\psi/2)}\right]. \tag{41}$$

Inserting eqs (40) and (41) into eq. (30a), one arrives at the (non-linear) Fredholm equation:

$$\tau_P = \tau_P^2 + F(P)/R - \frac{1}{8}M_Q\left[\frac{\tau^2 - \tau_P^2}{\sin^3\frac{\psi}{2}}\right], \tag{42a}$$

or

$$T_P = F(P) + \frac{T_P^2}{R} - \frac{1}{8R}M_Q\left[\frac{T^2 - T_P^2}{\sin^3(\psi/2)}\right], \tag{42b}$$

which lends itself to iteration. The third term needs some special care to compute, as it has a strong singularity at the computation point. Hence, for the near-zone (within the cap size ψ_0) we may use a plane approximation to the integral (I), that is,

$$I = \frac{1}{32\pi R} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\psi_0} \frac{T^2 - T_P^2}{\sin^3 \frac{\psi}{2}} \sin\psi \, d\psi \, d\alpha \approx \frac{1}{4\pi} \int_{\alpha=0}^{2\pi} \int_{D=0}^{D_0} \frac{T^2 - T_P^2}{D^2} dD d\alpha, \tag{43}$$

where we have assumed that $R \sin\psi \approx 2R \sin\frac{\psi}{2} \approx R\psi \approx D$ and $R^2 d\sigma \approx D dD d\alpha$, and D is the surface distance between computation and integration point. Furthermore, by assuming that T varies by the polynomial

$$T = T_P + D\left(T_x \cos\alpha + T_y \sin\alpha\right) + \frac{D^2}{2}\left(T_{xx}\cos^2\alpha + T_{xy}\sin 2\alpha + T_{yy}\sin^2\alpha\right), \tag{44}$$

where all T_i and T_{ij} are horizontal derivatives in the x- and y-directions of T, it follows from eq. (43) that

$$I \approx \frac{D_0}{8}\left[T_P\left(T_{xx} + T_{yy}\right) + 2\left(T_x^2 + T_y^2\right)\right]. \tag{45}$$

The horizontal derivatives of T can be regarded as unknown coefficients in the surface polynomial representation of eq. (44), and they can thus be determined by least squares from the (approximately) known values of T of the grid around the computation point.

Let us take a closer look at the first-order approximation of eq. (42b):

$$T(P) \approx F(P) = [2f(P) - M_Q\{H(\psi_{PQ}) f(Q)\}]/(4\pi). \tag{46}$$

Inserting eq. (12b) with $k = G\Delta\rho$, we arrive at

$$T(P) \approx \frac{R}{4\pi\Upsilon} \iint_\sigma H(\psi)\Delta g_B d\sigma - R\frac{(2\Delta g_B + A_{c0})_P}{\Upsilon} \approx T_0 - \frac{2R\Delta g_B}{\Upsilon} + \frac{R}{4\pi\Upsilon} \iint_\sigma H(\psi)\Delta g_B d\sigma, \tag{47}$$

where

$\Upsilon = 4\pi G R \Delta\rho \approx 2011$ Gal for $R = 6371$ km and $\Delta\rho = 0.6$ g cm $^{-3}$.

For comparison, the geoid height N can be determined from gravity disturbances δg by Hotine's formula

$$N = \frac{R}{4\pi\gamma} \iint_\sigma H(\psi)\delta g d\sigma, \tag{48}$$

where γ is normal gravity at sea level of the order of 981 Gal. The similarities of eqs (46) and (45) are obvious. (Note that eq. 48 could also include a zero-degree term, corresponding to the first two terms of eq. 47).

4.3 A solution including spherical harmonics

The function $f(P)$ of eq. (12b) is, except for the constant, proportional to the Bouguer anomaly. Considering the expansion of $f(P)$ in a series of fully normalized spherical harmonics, $Y_{nm}(\theta, \lambda)$, that is,

$$f(P) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm} Y_{nm}(\theta, \lambda), \tag{49a}$$

where

$$f_{nm} = \frac{1}{4\pi} \iint_\sigma f Y_{nm} d\sigma, \tag{49b}$$

as well as the series expansion of $H(\psi)$ of eq. (30c), it follows that $F(P)$ defined in eq. (36c) can be expanded as an approximation to the Moho depth

$$\begin{aligned} T \approx T_1 = F(P) &= \frac{1}{4\pi}\left[2f(P) - \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{m=-n}^{n} f_{nm} Y_{nm}(\theta, \lambda)\right] \\ &= \frac{1}{4\pi}\left[\sum_{n=0}^{\infty}\left(2 - \frac{1}{n+1}\right) \sum_{m=-n}^{n} f_{nm} Y_{nm}(\theta, \lambda)\right]. \end{aligned} \tag{50}$$

Hence, a first-order approximation of T can be directly computed from a global set of spherical harmonics of the Bouguer gravity anomaly, f_{nm}, where the upper limit of eq. (50) thus is finite. In practice, the free-air gravity anomaly (Δg) is directly available from an EGM, for example, EGM2008 complete to degree and order 2160. With access to a corresponding spherical harmonic representation H_{nm} of the solid Earth rock equivalent topographic height H (with negative heights in oceanic areas) and assuming constant topographic density, it is straightforward to generate the set of spherical harmonics of the Bouguer anomaly, simply by subtraction the coefficients $2\pi\mu H_{nm}$ from the free-air anomaly coefficients (G_{nm}). However, a better solution would require that the lateral density variations of topographic density be considered. In the latter case the result is

$$f_{nm} = \begin{cases} [2\pi(\mu H)_{00} - A_{C0}]/(4\pi k), & \text{if } n = 0 \\ \left[2\pi(\mu H)_{nm} - G_{nm}\right]/(4\pi k) & \text{otherwise} \end{cases}, \tag{51}$$

where we have assumed that the zero-degree harmonic of the free-air gravity anomaly, G_{00}, is zero. The solid Earth rock equivalent topographic series is given by the formula

$$(\mu H)_{nm} = \frac{1}{4\pi} \iint_{\sigma} (\mu H)^{\mathrm{req}} Y_{nm} \mathrm{d}\sigma, \tag{52a}$$

where the rock equivalent column of mass/unit area is

$$(\mu H)^{\mathrm{req}} = \begin{cases} \mu H, & \text{if} \quad H \geq 0 \\ (\mu_w - \mu) H & \text{if} \quad H < 0. \end{cases} \tag{52b}$$

Here, μ_w is the density of water. In principle, to reach a higher resolution in the estimation of T_1, the term $f(P)$of eq. (50) can be determined from the local gravity anomaly, while the remaining sum, that tapers off with the degree, is advantageously estimated by an EGM.

Note. In the global approach the simple Bouguer anomaly is related with a Bouguer shell rather than a Bouguer plate. The shell yields the factor 4π rather than 2π in eq. (51). However, the Bouguer shell is related with a much bigger terrain effect (not considered in eq. 51) than the Bouguer plate, and this fact made us choose the factor 2π for the topographic correction to achieve a simple Bouguer anomaly.

By combining eqs (50) and (42b), we thus obtain a second-order iterative solution to T:

$$T_{\mathrm{P}} = T_1 + \frac{T_{\mathrm{P}}^2}{R} - \frac{1}{8R} M_{\mathrm{Q}} \left(\frac{T^2 - T_{\mathrm{P}}^2}{\sin^3(\psi/2)} \right), \tag{53a}$$

or, to the same order of precision, a direct formula:

$$T_{\mathrm{P}} = (T_1)_{\mathrm{P}} + \frac{(T_1)_{\mathrm{P}}^2}{R} - \frac{1}{8R} M_{\mathrm{Q}} \left(\frac{\left(T_1^2\right)_{\mathrm{Q}} - \left(T_1^2\right)_{\mathrm{P}}}{\sin^3(\psi/2)} \right). \tag{53b}$$

Here, the integral should be significant only in a small region around the computation point. As eqs (53a) and (53b) include all terms of second order and $T \leq 100$ km, the approximation error is of the order of $100^3/6371^2$ km $\approx$25 m, which should be sufficiently small for most practical applications.

5 GENERALIZATION OF THE EQUATION AND UNIQUENESS OF THE SOLUTION

5.1 Generalization

Our basic integral equation of eq. (12a) was limited to the case with $r_{\mathrm{P}} = R$, that is, all the observations $f(P)$ were given on the sphere (S) of radius R. However, we can easily generalize this equation to $f(P)$ located at an arbitrary point P with $r_{\mathrm{P}} \geq R$ (on or outside the sphere). From eqs (1) and (4), we thus obtain

$$\mathrm{d}A_{\mathrm{C}}(P) = -[\Delta g_{\mathrm{B}}(P) + A_{\mathrm{C}0}(P)], \tag{54}$$

or, after inserting eq. (6a) and using the surface element $\mathrm{d}S = R^2 \mathrm{d}\sigma$ for the sphere:

$$\frac{1}{R} \iint_{\sigma} K(r_{\mathrm{P}}, \psi, \tau) \mathrm{d}S = f(P), \tag{55a}$$

where

$$K(r_{\mathrm{P}}, \psi, \tau) = \sum_{n=0}^{\infty} \frac{n+1}{n+3} \left(\frac{R}{r_{\mathrm{P}}} \right)^{n+2} (1 - s^{n+3}) P_n(t) \tag{55b}$$

and $f(P)$, given in eq. (12b), is generalized to any point P in the exterior of S. From eq. (9), we also obtain the closed form of eq. (55b):

$$K(r_{\mathrm{P}}, \psi, \tau) = [J_2(r_{\mathrm{P}}, \cos\psi, R\tau) - J_3(r_{\mathrm{P}}, \cos\psi, R\tau)], \tag{56}$$

where J_2 and J_3 were given in eqs (6b) and (6c).

5.2 Uniqueness of the solution

We will now prove that eqs (55a) and (55b) have a unique solution for τ. First, we notice that for any degree n it holds that $r_{\mathrm{P}}^{-n-1} P_n(t)$ is a harmonic function in the exterior of the sphere S. Hence, this applies also to the function $U = r_{\mathrm{P}} K(r_{\mathrm{P}}, \psi, \tau)$. Let us assume that eq. (55a) is satisfied by two solutions τ_1 and τ_2. Let us also introduce the notations $U_1 = r_{\mathrm{P}} K(r_{\mathrm{P}}, \psi, \tau_1)$, $U_2 = r_{\mathrm{P}} K(r_{\mathrm{P}}, \psi, \tau_2)$ and their difference $W = U_1 - U_2$. By inserting W into Green's first identity (e.g. Heiskanen & Moritz 1967, p. 11), we obtain

$$\iiint_{\upsilon} [W \Delta W + \mathrm{grad} W \bullet \mathrm{grad} W] \mathrm{d}\upsilon = \iint_{S} W \frac{\partial W}{\partial R} \mathrm{d}S = \frac{1}{2} \frac{\partial}{\partial R} \iint_{S} W^2 \mathrm{d}S. \tag{57}$$

Here, the volume υ is the exterior of S, Δ is the Laplace operator and $\bullet$ denotes scalar product of vectors. As $W = \Delta W = 0$ and $\mathrm{grad}W = 0$ in υ, we arrive at the equation

$$\frac{\partial}{\partial R}\iint_S W^2 \mathrm{d}S = 0, \tag{58}$$

which can be integrated to

$$\iint_S W^2 \mathrm{d}S = C, \tag{59}$$

where C is the integration constant. As U_1, U_2 and W are all regular at infinity, C takes on the value 0. Hence, the only solution of eq. (59) is $W = 0$ for each point on S, that is, $U_1 = U_2$ implying $\tau_1 = \tau_2$. Thus, we have proved that the integral equation (55a) has a unique solution for τ. This implies, that with respect to a fixed normal Moho depth T_0 and a known, constant density contrast $\Delta\rho$at the Moho, the VMM hypothesis, expressed mathematically by eqs (55a) and (55b), has a unique solution for the Moho.

Note. A mathematical problem is called an improperly posed problem if either or both conditions on uniqueness and stability of the solution is/are violated. The solution of the VMM problem belongs to this class, as its solution is unique but ill-conditioned.

6 PLANAR APPROXIMATIONS

If we assume that the radius R is infinite, the surface element changes according to $R^2\mathrm{d}\sigma \to \mathrm{d}D = \delta\mathrm{d}\delta\mathrm{d}\alpha$, where $\mathrm{d}D$ is the surface element on the plane with radial distance δ and azimuth α. Also, from eq. (18)

$$\frac{K(\psi, s)}{R} \to \frac{1}{\delta} - \frac{1}{\sqrt{\delta^2 + T^2}}, \tag{60}$$

so that an iterative form of eq. (29b) can be written

$$T_{\mathrm{P}}^{i+1} = -\frac{\Delta g_{\mathrm{B}} + A_{\mathrm{C0}}(P)}{4\pi k \Delta\rho} + \frac{c}{4\pi}\iint_D \left(\frac{1}{\sqrt{d^2 + (T^i)^2}} - \frac{1}{\sqrt{d^2 + (T_{\mathrm{P}}^i)^2}}\right)\mathrm{d}D, \tag{61}$$

where D is the infinite integration area of the plane.

Moreover, as

$$\frac{H(\psi)}{R} \to \frac{2}{\delta} \text{and} \frac{B(\psi, s)}{R} \to \frac{2}{\delta} - \frac{2}{\sqrt{\delta^2 + T^2}}, \tag{62}$$

eq. (30a) becomes

$$f(P) = \iint_D \left(\frac{1}{\delta} - \frac{1}{\sqrt{\delta^2 + T^2}}\right)\mathrm{d}D, \tag{63}$$

which is the same as the generating equation for the iterative solution of eq. (61).

7 CONCLUDING REMARKS

We have formulated the Vening Meinesz- Moritz inverse problem for the Moho depth by a global, non-linear integral equation similar to Moritz (1990). However, our treatment of the problem differs from Moritz approach in a number of ways. First, Moritz approximated the function $K(\psi, s)$ of eq. (18) by $1/L_0 - 1/L$. Second, he solved the integral equation by first expanding the term $s^{n+3} = (1-\tau)^{n+3}$of eq. (16) as a binomial series and then developing each term of this series as a Laplace harmonic series. Although the binomial series converges, this is not likely for the Laplace harmonic series of higher order terms of the binomial series. *The convergence behaviour seems to be similar to that of the Molodensky series, well known in physical geodesy: although the series may not be convergent in a mathematical sense, it is probably "practically convergent" in the sense that the first few terms give a good approximation provided that the data are suitably smoothed* (Moritz 1990, p. 262). Admittedly, although we have shown that the original integral eq. (9a) has a unique solution (and therefore also that of Proposition 1), this convergence problem occurs also in the solution to the integral equations.

If we limit the solution to second order, yielding an approximation of the order of 25 m (Section 4.2), the solution will be stable, provided that the infinite series of eq. (41) is convergent. Considering that Moho is a smooth surface, this convergence is likely. A linear/first-order solution to the problem can conveniently be obtained from an EGM, like EGM2008 to a resolution of about 5′. The remaining non-linear/short-wavelength contribution, which we suggest to be determined by an iterative process with the linear solution as its initial value, should not exceed 2 km.

Finally, it should be emphasized that the Earth's gravity field is generated by various sources, and the variable Moho topography is only one of them. In particular, the long-wavelength contribution to the gravity field is mainly due to deep mantle density variations, for example, related with mantle convection and postglacial rebound, as well as core–mantle topography variations. This implies that the long wavelengths of gravity, say to degree and order 10, will lead to systematic errors of the computed Moho topography. This problem could be reduced by excluding the long-wavelength harmonics from the integral equations, both for the Bouguer anomaly and the kernel functions $K(\psi, s)$and $B(\psi, s)$, respectively.

The theory developed in this paper will be used in several applications in estimating the Moho depth and density contrast, one of which is Sjöberg & Baherbandi (submitted).

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