Geophys. J. R. astr. Soc. (1980) **60**, 67–75

Rapid computation of magnetic anomalies with demagnetization included, for arbitrarily shaped magnetic bodies

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Received 1979 May 16; in original form 1978 September 22

Summary. The potential function ϕ for a magnetic body of susceptibility μ in a medium of susceptibility μ^* satisfies the integral equation

$$\phi = \phi^* + \frac{k}{2\pi} \int_s \frac{\partial \phi}{\partial n} \frac{1}{R} \, ds.$$

Here ϕ^* is the potential function for the region without the heterogeneity and R is the distance from the point of observation to the point on the surface, s, of the body. $\partial\phi/\partial n$ is the normal derivative, in the direction of the outward normal. The equation allows for the effects of demagnetization. For numerical purposes the surfaces can be divided into N facets over which $\partial\phi/\partial n$ is a constant. The unknown quantities $\partial\phi/\partial n_j$ can be found from the system of equations defined by:

$$\frac{\partial \phi}{\partial n_i} + \frac{k}{2\pi} \sum_{j=1}^{N}{}' \frac{\partial \phi}{\partial n_j} \frac{\partial}{\partial n_i} \int_j \frac{1}{R_{ij}} \, ds = \frac{\partial \phi^*}{\partial n_i}$$

The prime on the summation sign denotes that the summation does not include the ith element. The magnetic field in the direction of the unit vector $\mathbf{P}(P_1, P_2, P_3)$ is given by:

$$\mathbf{H}_P = -\nabla\phi^* \cdot \mathbf{P} - \frac{k}{2\pi} \sum_{j=1}^{N} \frac{\partial \phi}{\partial n_j} \int \nabla\left(\frac{1}{R}\right) \cdot \mathbf{P} \, ds.$$

1 Introduction

As early as the mid-1950's the late L. A. Richardson was actively engaged with C. B. Kirkpatrick in modelling the magnetic gold bearing ore bodies of the Tennant Creek field of Australia. For these model studies an ellipsoidal model was chosen so as to cater for the largest class of shapes possible by the one modelling program. A further element in the modelling program was to allow for the effect of demagnetization. Subsequent experience by Farrar (1978) has shown that this refinement is a necessity in the area because of the very large percentage of magnetic minerals present in the ore bodies. What was being sought was a method that would allow for the direction of the magnetic field within the body to be different from the inducing field. Allowing for demagnetization then would allow better estimates of the dip of the magnetic bodies to be made. The method chosen for these studies was limited to ellipsoidal shapes only because of the limitations of boundary value theory (Kirkpatrick & Richardson 1974).

Later Sharma (1966, 1977) also appreciated the need to calculate magnetic profiles across magnetic bodies and at the same time allow for the effects of demagnetization. The method chosen by Sharma was to write down the equations for the profiles in terms of an integral equation that required the field to be found throughout the magnetic structure. The integral equation was then solved by summing over a series of small rectangular prisms that collectively defined the total structure.

The disadvantage with Sharma's method was that only relatively small structures could be solved if the total number of unknowns was not to grow too large. The method developed below is also based on an integral equation. However, this time the integral involved is a surface integral taken over the surface of the magnetic structure. The number of unknown quantities, then, that are required to solve the equation by the numerical method grows as L^2 and not L^3. Here L is a length parameter of the body.

A further advantage of the method is that the proposed algorithm has very strong similarities with a previously described algorithm for modelling the induced polarization and resistivity methods of prospecting. It is therefore very easy to implement as the code for this case is available (Barnett 1972).

A disadvantage with all the methods is that they do not allow for the effects of remanent magnetism which can also affect dip estimates (Green 1960). Despite this criticism the following analysis does provide a useful tool for magnetic interpretation when remanent magnetism can be ignored but demagnetization cannot be.

2 Derivation of the integral equation

If **H** is the magnetic field intensity, **B** the magnetic induction, **J** the current density and μ the permeability, then:

$$\nabla \times \mathbf{H} = \mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0 \tag{1}$$

$$\mathbf{B} = \mu \mathbf{H}. \tag{2}$$

Consider the geometry shown in Fig. 1 where there is a body of permeability μ in a medium of permeability μ^*. No currents flow so

$$\nabla \times \mathbf{H} = 0 \tag{3}$$

and there exists a potential function ϕ such that

$$H = -\nabla \phi. \tag{4}$$

Hence from equations (1) and (2)

$$\nabla \cdot \mu \nabla \phi = 0$$

$$= \nabla \mu \cdot \nabla \phi + \mu \nabla^2 \phi \tag{5}$$

or

$$\nabla^2 \phi = -(\nabla \mu \cdot \nabla \phi)/\mu. \tag{6}$$

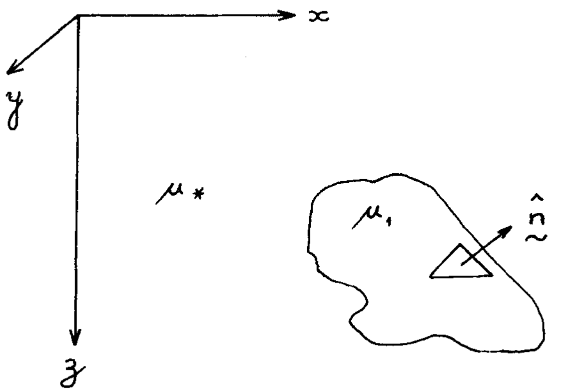

Figure 1. The geometry for the integral equation.

If ϕ^* is the potential function for the medium without the magnetic intrusion, as shown in Fig. 1 and μ^* the corresponding value of the magnetic permeability then

$$\nabla^2(\phi - \phi^*) + \nabla\mu^* \cdot \frac{\nabla(\phi - \phi^*)}{\mu^*} = 0 \tag{7}$$

outside the inhomogeneity, and

$$\nabla^2(\phi - \phi^*) + \frac{[\nabla(\mu) \cdot \nabla(\phi - \phi^*)]}{\mu^*} = -\frac{\nabla\phi \cdot \nabla\mu}{\mu} \tag{8}$$

in the inhomogeneity.

Let G be a solution of the following equation and subject to the same boundary conditions as ϕ^*.

$$\nabla^2 G + \frac{\nabla G \cdot \nabla\mu^*}{\mu^*} = -\delta(x - x')\delta(y - y')\delta(z - z'). \tag{9}$$

The point (x', y', z') lies within the inhomogeneity.

An integral equation can be found by multiplying equations (7) and (8) by G and equation (9) by $\phi - \phi^*$. Next subtract equation (9) from equations (7) and (8) and integrate over all space. Whence

$$\phi = \phi^* + \int_v G\nabla\phi \cdot \frac{\nabla\mu}{\mu}\, dv \tag{10}$$

Here v denotes the volume of the inhomogeneity. Following an analogous procedure to that used by Lee (1972) equation (11) is simplified to

$$\phi = \phi^* + 2k \int_s \frac{\partial\phi}{\partial n} G\, ds. \tag{11}$$

In equation (11), s is the surface of the inhomogeneity, $k = (\mu^* - \mu)/(\mu^* + \mu)$, μ is the permeability of the inhomogeneity and the partial derivative is with respect to the outward normal, n, of the surface s.

Equation (11), then, is the integral equation describing the magnetic potential for the geometry in Fig. 1. As a check on this equation the known potential function for a spherical inhomogeneity will be found from equation (11) from a method advocated by Lee (1975) for solving analogous resistivity problems.

3 Potential function for a magnetic sphere

Let the sphere of radius b be subjected to a magnetic field of strength H_0 which is in the z direction.

Therefore

$$\phi^* = H_0 z = H_0 r \cos\theta$$
$$= H_0 r P_1(\cos\theta) \quad (12)$$

and

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi^*}{\partial n} + \frac{2k}{4\pi}\int_s \frac{\partial\phi}{\partial n'}\frac{\partial}{\partial n}\left(\frac{1}{R}\right) ds'. \quad (13)$$

In equation (13) the primes denote the variable of integration.

Following Lee (1975) we suppose that $\partial\phi/\partial n$ is represented by $AP_1(\cos\theta)$. Since $z = b \cos\theta$ on the sphere it follows that:

$$AP_1(\cos\theta) = H_0 b P_1(\cos\theta) + \frac{k}{2\pi}\int_{s'} \frac{\partial}{\partial n}\frac{1}{R}\cdot AP_1(\cos\theta)\, ds'. \quad (14)$$

The integral in equation (14) is easily evaluated by expanding $1/R$ in spherical harmonics.

Multiplying though by $P_1(\cos\theta)\sin\theta$ and integrating over the sphere yields:

$$A = \frac{3H_0 b(\mu^* - \mu)}{\mu + 2\mu^*}. \quad (15)$$

The potential function outside the sphere, then, is readily found from equations (15) and (11) to be given by

$$\phi = \phi^* + \frac{b^3}{r^2}\frac{[H_0(\mu^* - \mu)]}{\mu + 2\mu^*} P_1(\cos\theta). \quad (16)$$

Equation (16) is the familiar solution from boundary value theory (Ward 1966, p. 66, equations A2–17).

4 Numerical solution of the integral equation

Equation (11) is analogous to the integral equation that Barnett (1972) has provided an elegant numerical solution for. In view of this only the briefest details are given below.

For the magnetic case under consideration

$$\phi^* = H_x x + H_y y + H_z z$$

and

$$G = \frac{1}{4\pi R}$$

where

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$

Hence

$$\phi = \phi^* + \frac{k}{2\pi} \int_s \frac{\partial}{\partial n} \phi \frac{1}{R} \, ds'. \tag{17}$$

Barnett solves this equation by dividing up the surface of the heterogeneity by a number of triangular facets over each one of which the quantity $\partial\phi/\partial n$ is a constant.

To construct an equation suitable for numerical purposes we require that $(\partial\phi/\partial n)\mu$ is continuous over the surface. If the body is divided up into N facets and $\mu(\partial\phi/\partial n)$ is constant over each of them then the above integral equation yields for the ith point (at either side of the boundary) the equation

$$\phi_i = \phi^* + \frac{k}{2\pi} \sum_{j=1}' \int \frac{1}{R} \frac{\partial\phi_i}{\partial n_j} \, ds_j + \frac{k}{2\pi} \int_i \frac{\partial\phi_i}{\partial n_i} \frac{1}{R} \, ds_i. \tag{18}$$

The prime on the summation sign indicates that the sum is over all the facets but does not include the ith facet.

If subscripts 1 and * denote terms within and outside the boundary respectively, then the boundary conditions require that

$$\frac{\partial\phi_i}{\partial n_*} \mu_* - \frac{\partial\phi_i}{\partial n_1} \mu_1 = 0 \tag{19}$$

at the boundary.

An equation suitable for numerical purposes, then, can be found by calculating $(\partial\phi/\partial n)\mu$ at both sides, but not at the ith facet, and letting the points at which the quantity is calculated approach the same point on the ith facet.

Therefore

$$\frac{\partial\phi_i}{\partial n_i} + \frac{k}{2\pi} \sum_j' \frac{\partial\phi}{\partial n_j} \frac{\partial}{\partial n_i} \int \frac{1}{R} \, ds = \frac{\partial\phi^*}{\partial n_i}. \tag{20}$$

In proceeding to the limit we have used the result of Sternberg & Smith (1964, p. 140, equation 20).

Equation (20) defines a system of equations for which the unknown quantities $\partial\phi/\partial n_j$ can be determined.

Once these quantities have been determined the magnetic field $\mathbf{H}_P$ in the direction of a unit vector $\mathbf{P}(P_1, P_2, P_3)$ is found from equations (4) and (8) to be given by:

$$\mathbf{H}_P = -\nabla\phi^*\mathbf{P} - \frac{k}{2\pi} \sum_{j=1}^{N} \frac{\partial\phi}{\partial n_j} \int \nabla\left(\frac{1}{R}\right) \cdot \mathbf{P} \, ds. \tag{21}$$

5 An example to check the method

Equation (16) forms the basis for calculating the magnetic field about a sphere which has a high susceptibility. Equations (20) and (21) are the two equations that provide the basis for a numerical method of calculating the same quantity. Fig. 2 shows the result of using equations (20) and (21) to approximate the integral equation for the case of a spherical shaped structure. This choice was dictated because simple analytical results were available to check the numerical procedures.

For these calculations the sphere was approximated by 48 facets and the susceptibility was taken to be 0.1. Curves 1 and 3 are based upon the numerical procedures described above while curves 2 and 4 were calculated from the known formulae. The first two curves (1, 2) are for the case where effects of a demagnetization are ignored and the last two (3, 4) are for the case where demagnetization is included. For the cases where demagnetiza-

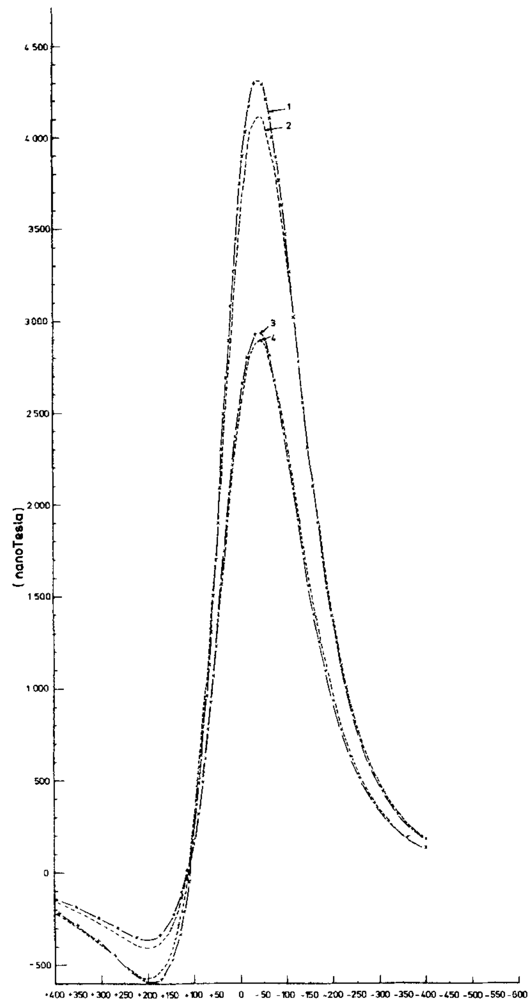

Figure 2. A comparison of two sets of calculations for a north–south profile of the total magnetic field intensity across a buried sphere. Curves 1 and 3 are based on numerical methods while curves 2 and 4 are based on an analytical method. The inducing field has a strength of 58429 nanoTesla and a dip of $-64.2°$.

tion was ignored the curves were calculated from equation (22) with $\partial\phi/\partial n_j = \partial\phi^*/\partial n_j$. In all cases the radius of the sphere is 100 m and its depth to centre is 200 m.

The results show that it is possible to use the formulation given above to obtain excellent numerical results. As Fig. 2 shows, the greatest difference in the curves is over the top of the sphere. However, even here the ratio of the field strengths is still 0.67 against an expected 0.70. The difference is due to the point on the top of the faceted sphere.

For more irregularly shaped structures, however, care should be taken to ensure that there is a sufficient number of facets in areas of high curvature. If this is not done the unknown function will be poorly approximated and in such cases the calculated results will have large errors.

6 Discussion

The theory given above establishes an integral equation and outlines an algorithm that will give its approximate solution. Those results have been used to model a vertical magnetic

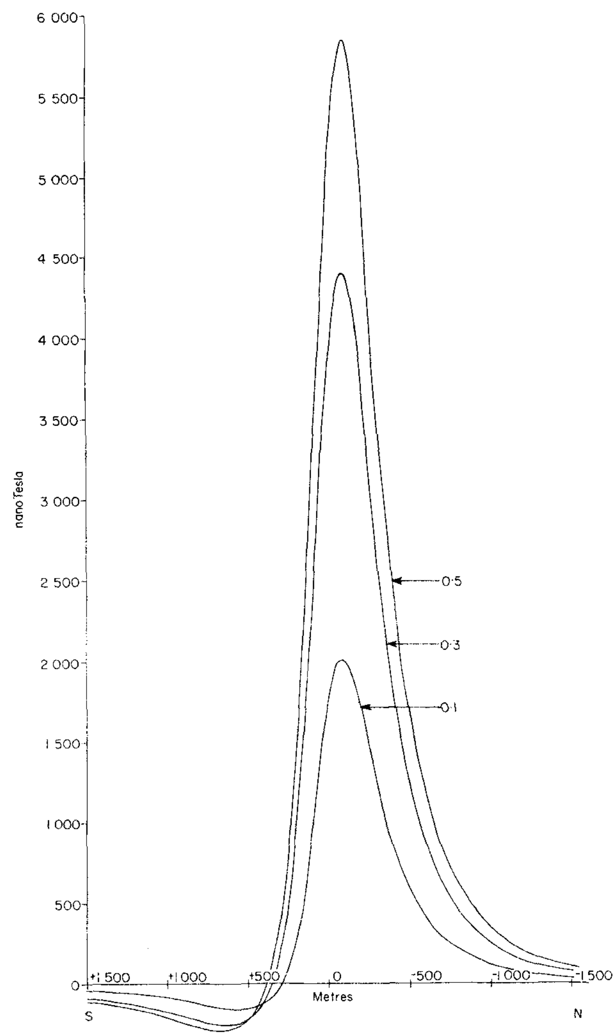

Figure 3. A north–south profile across a vertical dyke. The inducing field has a strength of 58429 nano-Tesla and a dip of – 64.2°.

dyke of depth extent of 1000 m, width 500 m, and thickness 100 m. The depth to the centre of the dyke is 750 m. The results, which are shown in Fig. 3, are for a north–south profile and give the value of the magnetic field in the direction of the inducing field. In the calculations the susceptibility was 0.1, 0.3 and 0.5 respectively.

The calculations show just how strongly the inducing field can be bent along the direction of the dyke. Routine interpretation procedures, assuming induced magnetization in the field direction, would indicate a southerly dip for the dyke and make it appear to be much thinner than it is.

The theory given above provides the basis of a useful modelling program.

Acknowledgments

The author would like to acknowledge Mrs J. Farrar for help with the computing, Mr R. L. Richardson for suggesting the problem and to L. A. Richardson and associates for funding the project.

Typographical errors in Lee (1975)

Equation reads		Correction
T1	eq. 2 2π	$2\pi/d$
10	$1/P_1 + 1/P_2$	$P_1 + P_2$
15	1	k_1^m
20	$2\frac{\partial}{\partial b}K_n(\alpha b)$	$2\frac{\partial}{\partial b}I_n(\alpha b)\cdot K_n(\alpha b)$
21	$1 - 2k_1\frac{\partial}{\partial b}K_n(\alpha b)$	$1/b - 2k_1\frac{\partial}{\partial b}I_n(\alpha b)\cdot K_n(\alpha b)$
22	$\bar{A}_n[$	$\bar{A}_n I_n(\alpha b)[$
22	$-\lambda(h-z)$	$+\lambda(h-z)$
22	$d\beta$	$\frac{d\beta}{\lambda}$
23	$\frac{\partial}{\partial b}K_n(\alpha b)$	$b\frac{\partial}{\partial b}I_n(\alpha b)K_n(\alpha b)$
23	$\int_{-\infty}^{\infty}$	$b\int_{-\infty}^{\infty}$
23	$(-1)^{n+m}$	$(-1)b^{n+m}$
27	$1/(2\pi\gamma)$	$1/\lambda$
28	$-\int_{-\infty}^{\infty}$	$\int_{-\infty}^{\infty}$
29	$1/(2\pi)$	1
30	$b\lambda^n$	$(b\lambda)^n$
31	$d\lambda$	$d\alpha d\beta$

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