Lagrangian Coherent Structures in the Planar Parabolic/Hyperbolic Restricted Three-Body Problem

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Abstract: It is clarified that the parabolic/hyperbolic restricted three-body problem (PRTBP/HRTBP) can be adopted to provide a simple description of the dynamics of flyby asteroids or the close encounters between different galaxies. For these reasons, PRTBP and HRTBP have been investigated for long intervals of time. However, they are quite different from CRTBP due to the time-dependent and non-periodic dynamics. The Lagrangian coherent structures (LCS), as a useful tool to analyze the time-dependent dynamical system, could be applied to explain some mechanics of the PRTBP and HRTBP. In this paper, we verify the invariant manifolds on the boundary manifolds of PRTBP by analyzing the LCS in proper Poincaré sections, which shows that it works in such a non-periodic problem. One of contributions is to investigate the LCS in the complete phase space of PRTBP, and then some natural escape and capture trajectories from or to two main bodies can be obtained in this way. Another contribution is to establish and study the dynamics of HRTBP and its boundary dynamics. The LCS can be introduced into this system reasonably to work as the analogues of the invariant manifolds. And the similar natural escape and capture trajectories corresponding to the two main bodies can also be obtained in the complete phase space of HRTBP. As a typical technique applied to fluid flows to identify transport barriers in the time-dependent system, the LCS provides an effective way to determine the time-dependent analogues of invariant manifolds for the PRTBP/HRTBP.

Keywords: Methods: numerical – Celestial mechanics – Galaxy: kinematics and dynamics

I. Introduction

It is commonly known that asteroids may contain some original materials from the solar nebula where a solar system formed. Thus, they can reveal the formation processes of the planets in the solar system. However, most of the asteroids are located on the asteroid belt between the orbits of
Mars and Jupiter, and it will be a great cost to explore such an asteroid or even capture it. There are still some asteroids whose orbit can bring it into proximity with Earth, which can be defined as Near-Earth Asteroid (NEA). Usually, the closest approach from NEA to Earth is less than 0.3 astronomical unit. Therefore, the cost to explore or capture it will be decreased a lot due to its so close approach. In this case, the circular/elliptic restricted three-body problem (CRTBP/ERTBP) cannot work well, while the parabolic and hyperbolic models are supposed to be an approximate engineering one for this kind of problem. Besides, the parabolic and hyperbolic models have also been used in the study of close encounters between different galaxies. In this kind of problems, only two galaxies are involved in an encounter, and each galaxy is idealized as a disk of non-interacting test particles, which moves on the parabolic or hyperbolic orbit. By exploring the dynamics of this kind of problems, the mechanics of the formation of bridges and tails can be explained in some degree.

The parabolic restricted three-body problem (PRTBP) and hyperbolic restricted three-body problem (HRTBP) have been investigated for long intervals of time. In comparison to the circular and elliptical problems, the parabolic and hyperbolic ones are quite different due to the greater motion range and non-periodic behaviors. Indeed, the two different kinds of problems (the periodic problem and non-periodic problem) possess completely different characteristics. For example, there are some periodic orbits and quasi periodic orbits around the collinear equilibrium points in CRTBP and ERTBP, which play an important role in the recent studies. However, there exists no periodic orbits or quasi periodic orbits in the vicinity of equilibrium points in PRTBP and HRTBP, thus some valuable conclusions obtained in CRTBP and ERTBP cannot be applied immediately into the parabolic or hyperbolic problems. Alvarez et al (2006) proved the existence of some special types of motions in the PRTBP, i.e. the type-exchange, emission-capture, and emission-escape with close passages to collinear and equilateral triangle configuration. In the study, the Jacobi function of this dynamical system was proved to have a gradient-like property, thus the existence of the transfers between the points that possess different energy can be verified. Further, based on the gradient-like property, a geometric criterion for capture was introduced and compared with the similar ones given by Merman (1953) and Kocina (1954). Besides, they also obtained some connections of the invariant manifolds associated to the collinear configurations, and stable/unstable sets associated to binary
collision on the boundary manifolds. Based on the study of Alvarez, Barrabés et al (2015) obtained more connections between the invariant manifolds associated to the equilibrium points, restricted to the invariant boundaries. Also, the connections in the whole phase space were studied numerically, especially the capture and escape orbits. Toomre and Toomre (1972) argued that the bridges and tails seen in some multiple galaxies are just tidal relics of close encounters. In their paper, the encounter was considered to involve only two galaxies and to be roughly parabolic, thus the problem turned to be a PRTBP. It was proved that the two-sided distortions provoked by gravity alone in such circumstances can indeed evolve naturally into some remarkably narrow and elongated features.

Based on the study of Toomre, Barrabés et al (2017) studied the case that two galaxies make a close encounter further. They presented a mechanism, applying techniques of dynamical systems theory to explain the formation of bridges and tails between galaxies in a simple model. In their paper, massive numerical simulations were carried out, and some previous results were also recovered. Liu et al (2018) studied the case that the mass parameter $\mu$ is tiny (i.e. the masses of the two primaries are quite different), especially the case that a binary asteroid pair make a close encounter with a planetary body. Then, it was assumed that the binary pair consists of a smaller asteroid in orbit about a larger one, and the mass of the smaller one was so tiny that could be ignored. Thus, the problem turned to be a PRTBP as well. In order to avoid the singularity, the related regularized dynamical equations were developed in the vicinity of two primaries. The approximate analytical solution was obtained by solving the linearized model. Further, the capture region for the problem was given by developing initial condition maps. Finally, two capture strategies were proposed to engineer and extend the possibility for capture of the minor asteroid in binary pair. Faintich (1972) introduced the dynamical equations of the HRTBP, and applied the equations to a hypothetical star-sun-comet system to determine the effect of the stellar encounter on the orbit of the comet. Cors et al (1995) proposed a novel hyperbolic model. In their paper, two mass points move under Newton’s law of attraction in a non-collision hyperbolic orbit, while a third mass point whose mass can be ignored moves on the straight line perpendicular to the plane of motion of the first two mass points and passing through their center of mass. Their main result is the characterization of the global flow of this problem. Lukyanov (2010) constructed the surfaces of minimum energy in the elliptic, parabolic and hyperbolic restricted three-body problem, which are a generalization of the surfaces of zero
velocity known in the circular problem. Naturally, the Hill stability, conditional stability and instability criteria have been established as well.

In the field of fluid mechanics, some scholars regarded the Lagrangian coherent structures (LCS) as the invariant manifolds of the time-dependent system to study the dynamics in the phase space. Haller and Shadden proposed and developed the concept of LCS almost simultaneously. Haller and Yuan (2000) introduced a Lagrangian definition for the boundaries of coherent structures in two-dimensional turbulence, and derived an analytic criterion that could be used to extract LCS from numerical data sets. Haller (2001) proved analytic criteria for the existence of finite-time attracting and repelling material surfaces and lines in three-dimensional unsteady flows, and proposed two approached to extract distinguished LCS from three-dimensional velocity data. Shadden et al (2005) defined the LCS as ridges of Finite-Time Lyapunov Exponent (FTLE) fields, which remained applicable to flows with arbitrary time dependence. Gawlik et al (2009) investigated LCS in the elliptic restricted three-body problem (ERTBP), and provided a convenient means of determining the time-dependent analogues of these invariant manifolds for the ERTBP. Short (2010) employed interactive visualization, numerical methods, and parallel computation to obtain FTLE data and the associated LCS, and showed that LCS is a useful tool in CRTBP. Haller (2011) developed a mathematical theory that clarifies the relationship between observable LCS and invariants of the Cauchy–Green strain tensor field, and introduced the notion of a Constrained LCS that eliminates normal repulsion or attraction under constraints. Oettinger et al (2016) introduced an approach to identify elliptic transport barriers in three-dimensional time-a periodic flows, and applied this approach to visualize elliptic LCSs in steady and time-a periodic ABC-type flows. Short et al (2015) exploited the anisotropy of the growth or decay of perturbations to the trajectories, building on recent ideas from the theory of hyperbolic LCS. Qi and Huang (2016) explored the transport mechanism in the bi-circular four-body problem numerically, and investigated the properties of LCS which are useful for revealing transport mechanism in the four-body problem. Lin et al (2017) proposed a block decomposition algorithm developed on Compute Unified Device Architecture (CUDA) platform for the computation of the LCS of multi-body gravitational regimes, and they showed that this GPU-based algorithm could satisfy double-precision accuracy requirements and greatly decrease the computation time. In this paper, we also use the parallel computing technique to obtain the LCS.
This paper consists of two parts. In the first part, we review the specific definitions of FTLE field and LCS based on the previous studies. Then, the existing heteroclinic connections on the boundary manifolds of PRTBP are verified by analyzing the LCS, which shows that the LCS is an effective tool to study the non-periodic problems. Further, we investigate the LCS in the complete phase space of PRTBP, and obtain some natural escape and capture trajectories from or to two primary bodies. It can help to design the transfer trajectories to a flyby asteroid on the parabolic orbit when the mass parameter is tiny, which can be regarded as one of the contributions of this paper.

Another contribution is acquired in the second part. We establish the dynamical equations of HRTBP and its boundary dynamics, and make a serious of coordinate transformations and normalized processing. Subsequently, we obtain the FTLE fields in the proper Poincaré sections and the corresponding LCS in the HRTBP model. And it is proved that the LCS can work as the analogues of the invariant manifolds for the HRTBP, which actually act as the separatrix of the collision manifolds of the two main bodies. Then, we investigate the LCS in the complete phase space of HRTBP, and obtain the natural escape and capture trajectories from or to two main bodies as well, which are quite similar with those in PRTBP.

II. Lagrangian coherent structures

In the field of fluid mechanics, LCS can be regarded as the generalizations of invariant manifolds of the time-dependent system to study the dynamics in the phase space. In this part, the concept of LCS is introduced. Firstly, the concept of the FTLE is proposed as follows. For a given dynamical system:

\[
\begin{align*}
\dot{x}(t) &= v(t) \\
x(0) &= x_0
\end{align*}
\]

(1)

The point \(x\) at time \(t_0\) will reach \(\phi(t_0 + T; x)\) at time \(t_0 + T\), where \(\phi\) denotes the flow of Eq.(1). Supposing that an infinitesimal perturbation \(\delta x(t_0)\) is applied to \(x\) at time \(t_0\), the corresponding variation at time \(t_0 + T\) is given by
\[
\delta x(T) = \phi(t_0 + T; t_0; x + \delta x(t_0)) - \phi(t_0 + T; t_0; x)
= \frac{\partial \phi}{\partial x} \cdot \delta x(t_0) + O(\delta x^2(t_0))
\]

(2)

After ignoring the infinitesimal of higher order, the following equation can be obtained:

\[
\|\delta x(T)\| = \sqrt{(\delta x(t_0), \Delta \cdot \delta x(t_0))}
\]

(3)

where \(\Delta = \left(\frac{\partial \phi}{\partial x}\right)^T \cdot \frac{\partial \phi}{\partial x}\). Further, it can be obtained that

\[
\max_{\delta x(0)} \|\delta x(T)\| = \lambda_{\text{max}} \cdot \xi
\]

(4)

where \(\lambda_{\text{max}}\) is the maximum eigenvalue of \(\Delta\), and \(\xi\) is the corresponding characteristic vector. Then, the FTLE can be defined as

\[
\sigma(x) = \frac{1}{|T|} \ln \lambda_{\text{max}}
\]

(5)

Subsequently, the FTLE field can be obtained by assigning to each point in the whole domain a measure \(\sigma\). And it shows the rate of divergence of trajectories with neighboring initial conditions and measures the initial value sensitivity of the system.

Then the LCS are defined as ridges (more generally, codimension-1 surfaces in system with arbitrary dimension in the domain whose images in the graph of the FTLE field satisfy certain conditions that formalize intuitive notions of a ridge) in the FTLE field of the system. For \(T > 0\), the LCS can be regarded as the non-autonomous analogues of stable manifolds, while for \(T < 0\), the LCS can be regarded as the non-autonomous analogues of unstable manifolds.

III. The parabolic restricted three-body problem and its LCS

III.I. Dynamical equations of PRTBP

In this section, the planar parabolic restricted three-body dynamics is proposed to serve as one of basic models in this paper, to verify the effectiveness of the LCS method in the non-periodic problem. In Barrabés et al (2015), the non-periodic problem refers to the problem where the gravitational attraction of the primaries is non-periodic. It means that the energy of the primaries is
non-negative, and they move in parabolic or hyperbolic orbits. And the non-periodic dynamics is used to describe the dynamics of the non-periodic problem. The coordinate systems used in this model are defined as follows. For the inertial frame, the barycenter of the system is defined as the origin; the direction from \( m_2 \) to \( m_1 \) at time \( t = 0 \) is defined as the \( x \)-axis; the \( y \)-axis is determined by the right-hand rule. For the rotating and pulsating frame, the barycenter of the system is defined as the origin; the direction from \( m_2 \) to \( m_1 \) is defined as the \( x \)-axis; the \( y \)-axis is determined by the right-hand rule. Clearly, the two primaries are fixed on the \( x \)-axis in the rotating and pulsating frame.

\[
\begin{align*}
[M] &= m_1 + m_2 \\
[L] &= \|R\| \\
[T] &= \left(\|R\|^2 / G (m_1 + m_2)\right)^{1/2}
\end{align*}
\]  

(6)

where \( G \) is the gravitational constant. \( \|R\| \) is the Euclidean norm of \( R \). According to the new unit defined above, the gravitational constant \( G \) turns to be 1. Without loss of generality, it can be supposed that \( m_1 \geq m_2 \), and then the mass parameter \( \mu \) can be defined as

\[
\mu = \frac{m_2}{m_1 + m_2} \in (0, 0.5]
\]  

(7)
Instead of choosing the time $t$ or the true anomaly $f$ as the independent variable, we perform the following reparametrization of time

$$\frac{dt}{ds} = \frac{1}{n\left(\frac{R}{q}\right)^{\frac{3}{2}}}$$

(8)

After the introduction of the new variable $s$, $\sigma$ could be expressed by

$$\begin{align*}
\sigma &= \sinh(s) \\
\frac{d\sigma}{ds} &= \sqrt{\sigma^2 + 1}
\end{align*}$$

(9)

Obviously, when the time $t$ tends to $\pm\infty$, $s$ tends to $\pm\infty$ as well. Then dynamical equation in the rotating and pulsating frame can be obtained as

$$\begin{align*}
\left\{ 
&x' + x' \tanh(s) + 4y' \text{sech}(s) = \Omega, \\
y' + y' \tanh(s) - 4x' \text{sech}(s) = \Omega,
\right. 
\end{align*}$$

(10)

where $(\cdot)'$ represents the derivative with respect to the variable $s$. $\Omega$ is the potential function, given by

$$\Omega = x^2 + y^2 + \frac{2(1 - \mu)}{\sqrt{(x - \mu)^2 + y^2}} + \frac{2\mu}{\sqrt{(x + 1 - \mu)^2 + y^2}}$$

(11)

In order to transform the non-autonomous system into an autonomous one, the independent variable $s$ must be involved into the system variable. Thus the following transformation is introduced naturally:

$$\sin(\theta) = \tanh(s)$$

(12)

Then Eq.(10) is transformed to

$$\begin{align*}
\theta' &= \cos \theta \\
x' &= v_x \\
y' &= v_y \\
v_x' &= -v_x \sin \theta - 4v_y \cos \theta + \Omega_s \\
v_y' &= 4v_x \cos \theta - v_y \sin \theta + \Omega_s
\end{align*}$$

(13)
III.II. LCS on the boundary dynamics of PRTBP

Barrabés et al (2017) introduced the definition of the upper/lower boundary problems, and the corresponding equations are given as follows respectively:

\[
\begin{align*}
{x'} &= v_x \\
{y'} &= v_y \\
{v_x}' &= \mp v_y + \Omega_x \\
{v_y}' &= \mp v_x + \Omega_y
\end{align*}
\]  
(14)

where the negative sign is corresponded with the upper boundary problem, and it is corresponded with the case that \( t \to +\infty \). While the positive sign is corresponded with the lower boundary problem, and it is corresponded with the case that \( t \to -\infty \). The boundary dynamics is used to describe the dynamics of the boundary problem. In this paper, the upper boundary problem will be picked as an instance to verify the heteroclinic connections mentioned in Barrabés (2017).

Firstly, we can analyze the invariant manifolds of the collinear equilibrium points. It can be easily verified that the dimension of the unstable manifold is one. After substituting \( y = v_y = 0 \) into Eq.(14), it can be derived that \( v_y' = 0 \). Therefore, the unstable manifolds of the collinear equilibrium points are along with the \( x \)-axis. And the dynamical system denoted by Eq.(14) can be reduced to a two-dimensional one, which is

\[
\begin{align*}
{x'} &= v_x \\
{v_x}' &= -v_x + \Omega_x
\end{align*}
\]  
(15)

And the Hamiltonian system corresponding to this reduced one is

\[
\begin{align*}
{x'} &= v_x \\
{v_x}' &= \Omega_x
\end{align*}
\]  
(16)

Then the FTLE fields of the dynamical systems denoted by Eqs.(15)-(16) are obtained respectively, which are shown in Fig.2.
The FTLE field of the dynamical system denoted by Eq. (15) (shown in Fig. 2a) is only slightly distorted from that of its corresponding Hamiltonian system (shown in Fig. 2b). Therefore, the introduction of the friction term $-v_x$ almost has no effect on the dynamics of the collinear equilibrium points.

In Fig. 2a, the FTLE field is divided into four different regions based on the final evolutions. In the Region I, the particle will tend to $-\infty$ when $t \to +\infty$; in the Region II, the particle will tend to $m_1$ when $t \to +\infty$; in the Region III, the particle will tend to $m_2$ when $t \to +\infty$; in the Region IV, the particle will tend to $+\infty$ when $t \to +\infty$. Specifically, the ridges of the above FTLE field represent LCS, and they act as the boundaries between different regions in the whole field, which are quantitative. Taking these points on the line segment A in Fig. 2a as an example, we show the final evolution orbits integrated from the above points. In the following Fig. 3, the red curves represent the orbits whose integral starting points belong to the intersection of the line segment A and the Region II (i.e. A∩II in Fig. 2a), while the blue curves represent the orbits whose integral starting points belong to the intersection of the line segment A and the Region I (i.e. A∩I in Fig. 2a). It can be observed that there is a precise initial value making its evolution orbit tend asymptotically to $L_1^+$, which is represented by the black line in Fig. 3. In other words, the points on both sides of LCS possess completely different dynamic properties. Inspired by this characteristic of the LCS, a novel method to extract the LCS from the FTLE field is proposed based on the particle swarm optimization (PSO), which is considered as one of the contributions in this paper. Traditionally, the dichotomy method is...
adopted to extract the LCS from the FTLE field. However, it is inefficient when the shape of LCS is spiral, while the proposed method in this manuscript can overcome this problem.

In order to extract the boundary between Region I and Region II, the detailed algorithm is outlined as follows.

**Table.1 Pseudocode to extract the LCS from the FTLE field**

1. Select a section $\Gamma = \{(x, y, v_x, v_y) | y = 0, v_y = 0\}$;

2. Select N discrete points that satisfy $x \in [lb, ub]$ on the section $\Gamma$ (where $lb$ and $ub$ represent the lower and upper bound respectively.);

3. for each discrete point,

4. Take it as the starting point to make the integration;

5. When $x$ is equal to $x_{L1}$, the integral is terminated;

6. Define the objective function as the distance between $L_1^+$ and the integral terminal point;

7. Determine the optimization variable as the $x$ component of the initial value;

8. Particle swarm optimization method is employed to get the exact $x$ component which makes the objective function reaches the minimum value;

9. end for
The set of the orbits that start from the points on the line segment A

The boundary (i.e. LCS) between different regions can be constructed by all these initial values obtained from the above algorithm, which are shown in Fig.4.

\[ U = \{(x, v_x)\mid y = 0, v_y = 0\} \]
In Fig. 4, the curve denoted by $a$ is the set of the initial values corresponding to the orbits that tend to $L_1^+$ when $t \to +\infty$; the curve denoted by $b$ is the set of the initial values corresponding to the orbits that tend to $L_2^+$ when $t \to +\infty$; the curve denoted by $c$ is the set of the initial values corresponding to the orbits that tend to $L_3^+$ when $t \to +\infty$. In addition, the curves denoted by $d$ and $e$ imply the singularities caused by $m_1$ and $m_2$ respectively. In other words, the curves denoted by $a$, $b$, and $c$ are the stable manifolds in the section $U = \{(x, v_y) | y = 0, v_y = 0\}$ of the three collinear equilibrium points.

Similarly, we can analyze the invariant manifolds of the triangular equilibrium points. After substituting $x = v_x = 0$ into Eq. (14), it can be derived that $v_x' = 0$. And the dynamical system denoted by Eq. (14) can be reduced to a two-dimensional one, which is

$$\begin{cases} y' = v_y' \\
v_y' = -v_y + \Omega_x \end{cases} \quad (17)$$

The Hamiltonian system corresponding to this reduced one is

$$\begin{cases} y' = v_y \\
v_y' = \Omega_x \end{cases} \quad (18)$$

Then the FTLE fields of the dynamical systems denoted by Eqs. (17)-(18) are obtained respectively, shown in Fig. 5.

![Fig. 5 The FTLE fields of the two 2-dimensional systems](https://academic.oup.com/mnras/advance-article-abstract/doi/10.1093/mnras/staa199/5715472)
It can be observed that the two FTLE fields are quite different from each other. Thus, the introduction of the friction term \( -v_y \) has vital effects on the dynamics of the triangular equilibrium points. In Fig.5a, the FTLE field corresponding to the original system is still divided into three regions based on the final evolutions. In the Region I, the particle will tend to \(-\infty\) when \( t\to\infty \); in the Region II, the particle will tend to \( L_2^+ \) when \( t\to\infty \) due to the introduction of the friction term \(-v_y\); in the Region III, the particle will tend to \(+\infty\) when \( t\to\infty \). It means that the heteroclinic connection between \( L_4^+ \) and \( L_5^+ \) in its Hamiltonian system is destroyed by the friction term \(-v_y\), and two new heteroclinic connections (i.e. from \( L_4^+ \) to \( L_2^+ \), from \( L_5^+ \) to \( L_2^+ \)) are obtained, which is shown in Fig.6. Similar to the method to extract the LCS corresponding to the invariant manifolds of \( L_i^+ \), we can extract the LCS from the FTLE filed shown in Fig.5a based on the different qualitative behaviors as well, which is shown in Fig.7. And the curves denoted by \( a \) and \( b \) in Fig.7 are the stable manifolds in the section \( U = \{(y,v_x)|x=0,v_y=0\} \) of the two triangular equilibrium points \( L_4^+ \) and \( L_5^+ \). In Fig.5b, the FTLE field corresponding to the Hamiltonian system is still divided into three regions based on the final evolutions. In the Region I, the particle will tend to \(-\infty\) when \( t\to\infty \); in the Region II, the particle will travel along with the period orbits in the vicinity of \( L_2^+ \); in the Region III, the particle will tend to \(+\infty\) when \( t\to\infty \). Specifically, the ridges of the above FTLE field represent LCS, and they act as the boundaries between different regions in the whole field. Similar to the method to extract the LCS corresponding to the invariant manifolds of \( L_i^+ \), we can extract the LCS from the FTLE filed shown in Fig.5b based on the different qualitative behaviors as well, which is shown in Fig.8.
Fig. 6 The heteroclinic connections from $L_4^+ / L_5^+$ to $L_2^+$.

Fig. 7 The LCS of the original system in section $U = \{ (y, v_y) | x = 0, v_x = 0 \}$.
The LCS of the Hamiltonian system in section \( U = \{(y, v_y) | x = 0, v_x = 0\} \)

Since the invariant manifolds of the PRTBP have been examined in considerable detailed by Barrabés et al (2015) with the parameterization method, it helps to compare the results of the two methods with published data. Clearly, the results obtained in our paper are exactly the same as those of Barrabés. Consequently, the LCS method works effectively in such a non-periodic problem.

III.III. LCS on the complete dynamics of PRTBP

In Barrabés et al (2015), they focused on escaping and captured orbits in the inertial coordinate system and introduced a criterium (the so-called \( C \)-criterium) to classify an orbit in the synodic coordinate system. While in Barrabés et al (2017), they applied techniques of dynamical systems theory to explain the formation of tails and bridges between galaxies in a simple model. In this section, we try to obtain the FTLE fields in the complete phase space of PRTBP and investigate the LCS, which can also be regarded as a criterium that distinguishes different trajectories. Inspired by the special quality of the LCS, a low-energy transfer method can be proposed to design the transfer trajectories between the two main bodies. This method used here is similar with the one proposed by Koon et al (2001), but the LCS are regarded as the analogues of invariant manifolds in this paper.

Without loss of generality, the corresponding transfer trajectories with close passage to centroid of the system are shown as numerical instances. After making forward and backward integration
respectively, the FTLE fields at the centroid are obtained as follows, where Fig.9a presents the FTLE field corresponding to forward integration, and Fig.9b presents the FTLE field corresponding to backward integration. Based on the different final evolutions, the FTLE fields in Fig.9 are divided into three different regions, which are shown in Fig.10.

By analyzing the LCS, it is found some natural transfer trajectories between the primary bodies, or from one of the primary bodies to infinity, or from infinity to one of the primary bodies based on the theory of LCS. In Fig.10a, the particle in the Region I will tend to \( m_1 \) when \( t \to +\infty \); the particle in the Region II will tend to \( m_2 \) when \( t \to +\infty \); the particle in the Region III will tend to infinity when \( t \to +\infty \). In Fig.10b, the particle in the Region IV will tend to \( m_1 \) when \( t \to -\infty \); the particle in the Region V will tend to \( m_2 \) when \( t \to -\infty \); the particle in the Region VI will tend to infinity when \( t \to -\infty \).

In this problem, all the trajectories are divided into nine types. When the particle is in the intersection of Region I and Region IV (I∩IV), it will escape from \( m_1 \) and be re-captured by \( m_1 \), which is shown.
in Fig.11a. When the particle is in the intersection of Region I and Region V (I∩V), it will escape from \( m_2 \) and be captured by \( m_1 \), which is shown in Fig.11b. When the particle is in the intersection of Region I and Region VI (I∩VI), it will be captured by \( m_1 \) from the infinity, which is shown in Fig.11c. When the particle is in the intersection of Region II and Region IV (II∩IV), it will escape from \( m_1 \) and be captured by \( m_2 \), which is shown in Fig.11d. When the particle is in the intersection of Region II and Region V (II∩V), it will escape from \( m_2 \) and be re-captured by \( m_2 \), which is shown in Fig.11e. When the particle is in the intersection of Region II and Region VI (II∩VI), it will be captured by \( m_2 \) from the infinity, which is shown in Fig.11f. When the particle is in the intersection of Region III and Region IV (III∩IV), it will escape to the infinity from \( m_1 \), which is shown in Fig.11g. When the particle is in the intersection of Region III and Region V (III∩V), it will escape to the infinity from \( m_2 \), which is shown in Fig.11h. When the particle is in the intersection of Region III and Region VI (III∩VI), it will come from infinity and go to the infinity again, which is shown in Fig.11i.

![Fig.11 Nine types of natural transfer trajectories](https://academic.oup.com/mnras/advance-article-abstract/doi/10.1093/mnras/staa199/5715472)
IV. The hyperbolic restricted three-body problem and its LCS

IV.I. Dynamical equations of HRTBP

References show that few studies about HRTBP have been achieved in the past years for some dynamical characteristics including the aperiodicity, nonlinearity, the boundedness of the true anomaly and so on. However, the HRTBP is more significant compared with the PRTBP, because it is so rigorous that the eccentricity is exactly equal to one. In this section, the planar hyperbolic restricted three-body dynamics is proposed to serve as another basic model in this paper, to search for some new results and guide the transfer trajectories design in the HRTBP. The coordinate systems used in this model are defined as follows. For the inertial frame, the barycenter of the system is defined as the origin; the direction from $m_2$ to $m_1$ at time $t = 0$ is defined as the $x$-axis; the $y$-axis is determined by the right-hand rule. For the rotating and pulsating frame, the barycenter of the system is defined as the origin; the direction from $m_2$ to $m_1$ is defined as the $x$-axis; the $y$-axis is determined by the right-hand rule. Similarly, the two primaries are fixed on the $x$-axis in the rotating and pulsating frame as well.

In the inertial frame, the dynamical equation of the third mass $m_0$ is given by

$$\ddot{\mathbf{r}} = -Gm_1 \frac{\mathbf{r}_1}{R_1^3} - Gm_2 \frac{\mathbf{r}_2}{R_2^3}$$ (19)
\( \mathbf{R}_1 = \mathbf{R} - \mathbf{R}'_1, \ \mathbf{R}_2 = \mathbf{R} - \mathbf{R}'_2, \ \mathbf{R} \) represents the position of \( m_0 \) in the inertial frame, \( \mathbf{R}'_1 \) represents the position of \( m_1 \), and \( \mathbf{R}'_2 \) represents the position of \( m_2 \).

After a serious of coordinate transformations and normalized processing, the dynamical equation of \( m_0 \) is given by

\[
\begin{align*}
\ddot{x} - 2\dot{y}\dot{\dot{x}} - \dot{y}^{2}x - \dot{y}^{2} = U_x, \\
\ddot{y} + 2\dot{x}\dot{\dot{y}} - \dot{x}^{2}y + \dot{x} = U_y,
\end{align*}
\]

where \( \dot{x} = h, \ \dot{y} = 2h^{2}\left( -\frac{e\sin f}{1 + e\cos f} \right), \ h = \sqrt{1 + e\cos f}, \ U = \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}. \) And \( \dot{x} = \frac{dx}{dt} \) represents the derivate with respect to time \( t \).

When the true anomaly \( f \) is chosen to be the independent variable, the dynamical equation can be turned to be

\[
\begin{align*}
x'' - 2y' = \frac{1}{1 + e\cos f}(U_x + x), \\
y'' + 2x' = \frac{1}{1 + e\cos f}(U_y + y),
\end{align*}
\]

Clearly, the dynamical equation of HRTBP is similar with that of ERTBP, but the eccentricity is larger than one and the periodicity of the system is destroyed.

It can be concluded that the true anomaly \( f \) can only vary in a limited range due to the hyperbolic orbit, while the real time will tend to infinity when the main bodies are at infinity. And it will bring some problems in the process of integration. Therefore, a reparametrization of the true anomaly is defined as follows.

Let \( \alpha = 1 + e\cos f \), and substitute it into Eq.(20), as

\[
\begin{align*}
\ddot{x} - 2\sqrt{\alpha}\dot{y} - \alpha x - 2\frac{\dot{\alpha}}{\sqrt{\alpha}}y = U_x, \\
\ddot{y} + 2\sqrt{\alpha}\dot{x} - \alpha y + 2\frac{\dot{\alpha}}{\sqrt{\alpha}}x = U_y,
\end{align*}
\]
where $\alpha \in (0,1+e]$. And rewrite its expression in the complete phase space, as

\[
\begin{align*}
\dot{\alpha} &= -e \sin f \sqrt{1+e \cos f} \\
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= 2\sqrt{\alpha} v_y + \alpha x + 2 \dot{\alpha} y / \sqrt{\alpha} + U_x \\
\dot{v}_y &= -2\sqrt{\alpha} v_x + \alpha y - 2 \dot{\alpha} x / \sqrt{\alpha} + U_y
\end{align*}
\] (23)

Clearly, when the particle is located on the boundary, the true anomaly satisfied

\[
f = \pm \arccos \left( -\frac{1}{e} \right) \] (24)

Where the negative sign is corresponded with the upper boundary problem, while the positive sign is corresponded with the lower boundary problem. Substituting it into Eq.(23) to obtain the boundary dynamics as

\[
\begin{align*}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= -2 e \sin f \cdot y + U_x \\
\dot{v}_y &= 2 e \sin f \cdot x + U_y
\end{align*}
\] (25)

Clearly, the form of Eq.(25) also possesses the autonomous character similar to the Eq.(14) in the parabolic problem.

**IV.II. LCS on the boundary dynamics of HRTBP**

In this section, we will pay attention to the boundary dynamics of HRTBP. And in this paper, we only consider the case $\mu = 0.5$ in order to simplify the HRTBP further. In this section, the upper boundary problem will be picked as a numerical instance. And the lower boundary problem can be analyzed with the same method.

Based on Eq.(25), the upper boundary dynamics is a time-independent system. It can be easily verified that there is only one equilibrium point in this system, which is exactly located at the
original point. When the Poincaré section is chosen as \( U = \{(x, y)|v_x = 0, v_y = 0\} \), the FTLE field of the system is obtained in Fig.13. In this simulation, the value of the eccentricity is determined to be 1.2.

![FTLE field](image)

**Fig.13** The FTLE field in section \( U = \{(x, y)|v_x = 0, v_y = 0\} \) corresponding to forward integration.

In Fig.13, the FTLE field is divided into three regions clearly based on the final evolutions. In the Region I, the particle will tend to \( m_1 \) when \( t \to \infty \); in the Region II, the particle will tend to \( m_2 \) when \( t \to \infty \); in the Region III, the particle will tend to \( \infty \) when \( t \to \infty \). Specifically, the ridges of the above FTLE field represent LCS, and they act as the boundaries between different regions in the whole field. When we choose the points in the vicinity of LCS as the integral starting points, it can be clearly observed that the orbits are divided based on the final evolutions. And there is a precise initial value, such that the orbit tends asymptotically to the only equilibrium point, which is shown in Fig.14.
Fig. 14 The set of the orbits that start from the points in the vicinity of LCS

Based on this characteristic, the LCS in the FTLE field shown in Fig. 13 could be obtained in Fig. 15. And the LCS represent the stable manifold of the equilibrium point.

Fig. 15 The LCS of the upper boundary manifold in the section $U = \{(x, y) \mid v_x = 0, v_y = 0\}$
Similarly, when the backward integration is made, the FTLE field corresponding to the same section is obtained in Fig.16. And the corresponding LCS can be extracted in Fig.17, which represent the unstable manifold of the equilibrium point.

Fig.16 The FTLE field in section $U = \{(x, y) \mid v_x = 0, v_y = 0\}$ corresponding to backward integration

Fig.17 The LCS of the upper boundary manifold in the section $U = \{(x, y) \mid v_x = 0, v_y = 0\}$
IV.III. LCS on the complete dynamics of HRTBP

Similar with the process in Section III.III, we investigate the LCS in the complete phase space of HRTBP, and obtain the natural escaping and captured trajectories from or to two main bodies as well, which is quite similar with those in PRTBP. In this simulation, the value of the eccentricity is set to be 1.2 for demonstration. However, the value of the energy is not a constant, and it varies in a range due to the request of the calculating of LCS. Based on the results of Lukyanov, although the Jacobi integral does not exist in the restricted noncircular three-body problem, the Jacobi quasi-integral, an integral invariant relation, is known. The dynamical equations that describe the motions of a particle $m_0$ in a rotating and pulsating coordinate system are (Lukyanov, 2010)

$$
\begin{align*}
    x'' - 2y' &= \frac{\partial \Omega}{\partial x} \\
    y'' + 2x' &= \frac{\partial \Omega}{\partial y}
\end{align*}
$$

(26)

where

$$
\Omega = \rho \left[ \frac{1}{2} (x^2 + y^2) + p^3 \left( \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right) \right]
$$

(27)

$$
\rho = \frac{1}{1 + e \cos \varphi}
$$

(28)

$$
\begin{align*}
    r_1 &= \sqrt{(x-x_1)^2 + y^2}, \quad r_2 = \sqrt{(x-x_2)^2 + y^2}
\end{align*}
$$

(29)

$x_1 = -p\mu$ and $x_2 = p(1-\mu)$ are their abscissas, $e$ and $p$ are the eccentricity and focal parameter of their relative orbit, $\Omega$ is the effective potential function. The true anomaly is assumed to vary in the range $-f_a \leq f \leq f_a$, where $f_a = \arccos(-1/e)$. According to the results of Lukyanov (2010), the energy of the particle can be defined as

$$
E = \frac{1}{2} (x'^2 + y'^2) - \Omega
$$

(30)

Based on this definition, the range of the value of the energy is $[-0.909, 35.091]$ in this numerical simulation.
Without loss of generality, the corresponding transfer trajectories with close passage to centroid of the system are still shown as the numerical instances. After making forward and backward integration respectively, the FTLE fields at the centroid are obtained as follows, where Fig.18a represents the FTLE field corresponding to forward integration, and Fig.18b represents the FTLE field corresponding to backward integration. Based on the different final evolutions, each FTLE field in Fig.18 is divided into three different regions clearly.

![FTLE fields at the centroid of the system](image)

**Fig.18** The FTLE fields at the centroid of the system

By analyzing the LCS, it is found some natural transfer trajectories between the primary bodies, or from one of the primary bodies to infinity, or from infinity to one of the primary bodies based on the theory of LCS. In Fig.18a, the particle in the Region I will tend to \( m_1 \) when \( t \to +\infty \); the particle in the Region II will tend to \( m_2 \) when \( t \to +\infty \); the particle in the Region III will tend to infinity when \( t \to +\infty \). In Fig.18b, the particle in the Region IV will tend to \( m_2 \) when \( t \to -\infty \); the particle in the Region V will tend to \( m_1 \) when \( t \to -\infty \); the particle in the Region VI will tend to infinity when \( t \to -\infty \).

In addition, the Regions A, B, C and D imply the singularities caused by the two primary bodies. Similar with PRTBP, all the trajectories can be divided into nine types as well. When the particle is in the intersection of Region I and Region IV (I∩IV), it will escape from \( m_2 \) and be captured by \( m_1 \), which is shown in Fig.19a. When the particle is in the intersection of Region I and Region V (I∩V), it will escape from \( m_1 \) and be re-captured by \( m_1 \), which is shown in Fig.19b. When the particle is in the intersection of Region I and Region VI (I∩VI), it will be captured by \( m_1 \) from the infinity, which is shown in Fig.19c. When the particle is in the intersection of Region II and Region IV (II∩IV), it
will escape from $m_2$ and be re-captured by $m_2$, which is shown in Fig.19d. When the particle is in the intersection of Region II and Region V ($\text{II} \cap \text{V}$), it will escape from $m_1$ and be captured by $m_2$, which is shown in Fig.19e. When the particle is in the intersection of Region II and Region VI ($\text{II} \cap \text{VI}$), it will be captured by $m_2$ from the infinity, which is shown in Fig.19f. When the particle is in the intersection of Region III and Region IV ($\text{III} \cap \text{IV}$), it will escape to the infinity from $m_2$, which is shown in Fig.19g. When the particle is in the intersection of Region III and Region V ($\text{III} \cap \text{V}$), it will escape to the infinity from $m_1$, which is shown in Fig.19h. When the particle is in the intersection of Region III and Region VI ($\text{III} \cap \text{VI}$), it will come from infinity and go to the infinity again, which is shown in Fig.19i.

Fig.19 Nine types pf natural transfer trajectories
V. Conclusion

In general, PRTBP and HRTBP are considered as the ideal models to study the close encounters between different galaxies, and they are also regarded as practical engineering models to study the flyby asteroids. In this paper, we study the dynamics of PRTBP with the method of LCS, and verify the existing heteroclinic connections on the boundary manifolds of PRTBP in this way. It shows that the LCS is an effective method to study this kind of non-periodic problem. We investigate the LCS in the complete phase space of PRTBP, and obtain some natural escape and capture trajectories from or to two primary bodies. The results can also be regarded as a criterion that distinguishes different trajectories. Then, we establish the dynamical equations of HRTBP and its boundary dynamics. By analyzing the LCS in the boundary manifolds, the collision manifolds corresponding to the two primary bodies can be obtained respectively. Similar with the process of PRTBP, we investigate the LCS in the complete phase space of HRTBP, and obtain the natural escape and capture trajectories from or to two primary bodies as well. In the future, we will consider to apply these results to help to design the transfer trajectories from earth to a flyby asteroid, or to explain the mechanics of the formation of tails and bridges between galaxies in the frame of PRTBP or HRTBP.

Acknowledgements

The authors acknowledge the support of the National Natural Science Foundation of China (11772024 and 11432001), the Fundamental Research Funds for the Central Universities, and Key Laboratory of Spacecraft Design Optimization and Dynamic Simulation Technologies, Ministry of Education of China.

Reference


