ON THE OCCURRENCE OF COMMENSURABLE MEAN MOTIONS IN THE SOLAR SYSTEM

II. The Mirror Theorem

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(Communicated by the Director, University Observatory, Glasgow)

(Received 1954 December 6)

Summary

In Part I it is shown that the preference for near-commensurability of mean motions demonstrated in a previous paper extends to the small satellites of Jupiter, including the retrograde ones, and to the retrograde satellite of Saturn, implying that stability of near-commensurable configurations is the reason for such a preference. In Part II it is proved that if, at a certain epoch, a system of n gravitating point-masses has each radius vector from the (assumed stationary) centre of mass of the system perpendicular to every velocity vector (hereinafter called a "mirror configuration"), then the behaviour of each of the point-masses under the internal gravitational forces of the system after the epoch will be a mirror image of its behaviour prior to the epoch. It is further shown that, if a mirror configuration of the system exists at two separate epochs, then the orbit of each point-mass is periodic. The authors argue that such periodic orbits are the more stable (under the action of external forces) the shorter the interval of time between mirror configurations, and they show that the frequent occurrence of mirror configurations between any two point-masses requires that the mean motions of the two point-masses be nearly commensurable. In Part III, various near-commensurable pairs of orbits in the solar system are shown to behave according to the above arguments. The relationship of the present work to the more restricted " symmetry theorem " of Griffin is discussed.

Part I

1. *Introduction.*—In a previous paper (1), hereinafter referred to as Paper I, the present authors have shown that, in the solar system, among the planetary and satellite systems, the occurrence of commensurability between pairs of mean motions is more frequent than in a chance distribution. It was suggested that the observed distribution was due either to a property of the mechanism of formation of the solar system, or to an inherent stability of commensurable configurations.

More recent work has led to the adoption of the latter hypothesis. The authors were led to investigate this latter alternative by a consideration of bodies in the solar system not included in Paper I, namely Jupiter's satellites VI to XII and Saturn IX.

2. *The outer satellites of Jupiter and Saturn.*—Using the nomenclature of Paper I, we have

$$\epsilon = \left|\frac{n_2}{n_1} - \frac{A_2}{A_1}\right|, \tag{2.01}$$

where n_1, n_2 are the mean motions of the two bodies, and A_1 and A_2 are integers (which, in Paper I, were less than or equal to 7).

The Sun disturbs Jupiter's satellites considerably, and the orbits of the retrograde satellites J VIII, J IX and J XI are not even approximately elliptic. Yet they have approximately equal mean motions, and if we consider the Sun as a satellite of Jupiter, the ratio of its mean motion to those of the retrograde ones is within the range of 1 to 7 previously considered. As is seen from Table I, these satellites are nearly commensurable in mean motion with the Sun, the ratio being 1 to 6. Indeed, the Sun's mean motion about Jupiter is almost exactly one-sixth of the mean of the mean motions of J VIII, J IX and J XI.

TABLE I

Satellites	$\frac{n_2}{n_1}$	A_2	A_1	$\frac{n_2}{n_1}-\frac{A_2}{A_1}$
Sun J VIII	0·170 55	1	6	+0·003 88
Sun J IX	0·171 96	1	6	+0·005 29
Sun J XI	0·159 82	1	6	−0·006 84
Sun and mean of J VIII, J IX, J XI	0·167 25	1	6	+0·000 59
Sun J VI	0·057 84	1	17	−0·000 98
Sun J VII	0·060 02	1	17	+0·001 20
Sun J X	0·058 67	1	17	−0·000 15
Sun and mean of J VI, J VII, J X	0·058 83	1	17	−0·000 01
Sun J XII	0·145 64	1	7	+0·002 78
Iapetus Phoebe	0·144 11	1	7	+0·001 25

Note : $\frac{1}{16}-\frac{1}{17}=0{\cdot}003\,68$

$\frac{1}{6}-\frac{1}{7}=0{\cdot}023\,81$

It is convenient to consider the other satellites of Jupiter here. The Sun's mean motion about Jupiter is nearly 1/7 of the mean motion of J XII, while the satellites J VI, J VII and J X form a group whose mean motions are nearly commensurable with the Sun's at 1/17, and once again their average mean motion is almost exactly commensurable with that of the Sun. Finally, Phoebe (Saturn IX), disturbed by Iapetus, pursues a retrograde orbit whose mean motion is commensurable with that satellite at 1/7. In Table I the mean motions are derived from the *Handbook of the British Astronomical Association for* 1953, except for those of J XII, Iapetus, Phoebe and the Sun. The last three were obtained from the *Connaissance des Temps* for 1954, while the mean motion of J XII was from Hilton and Brady's orbit No. IV (**2**).

These results provide additional evidence for the conclusions of Paper I, and suggest that the reason for the preference for near-commensurability is that such configurations are relatively stable. For, if the retrograde satellites of Jupiter and Saturn are captured bodies, it is difficult to understand why they should all have mean motions approximately commensurable with other bodies, unless they have reached positions of relative stability.

Part II

3. *The mirror theorem.*—In this section it is proved that, if n point-masses are acted upon by their mutual gravitational forces only, and at a certain epoch each radius vector from the (assumed stationary) centre of mass of the system is perpendicular to every velocity vector, then the orbit of each mass after that epoch is a mirror image of its orbit prior to that epoch.

Consider an isolated system of n point-masses, centre of mass G, and let their masses be m_i, $i=1, 2, 3, \ldots n$. Suppose that at a certain time the radius vector from G to m_i be $\boldsymbol{R}_i$, etc., and the velocity vector of m_i be $d\boldsymbol{R}_i/dt$, etc. Then the equation of motion of m_i relative to G is given, in gravitational units, by

$$\frac{d^2\boldsymbol{R}_i}{dt^2} = -\sum_{j=1}^{n} m_j \frac{(\boldsymbol{R}_i - \boldsymbol{R}_j)}{[(\boldsymbol{R}_i - \boldsymbol{R}_j) . (\boldsymbol{R}_i - \boldsymbol{R}_j)]^{3/2}}, \quad j \neq i. \tag{3.01}$$

The geometrical shapes and sizes of the orbits arising from this configuration, and their relative geometrical disposition, are completely specified by the masses m_i together with magnitudes and relative configuration of the radii vectores $\boldsymbol{R}_i$ and velocity vectors $d\boldsymbol{R}_i/dt$, i.e. by

$$\left.\begin{array}{rl} m_i & \text{for all values of } i, \\ \boldsymbol{R}_i . \boldsymbol{R}_j & \text{for all values of } i \text{ and } j, \\ (d\boldsymbol{R}_i/dt) . \boldsymbol{R}_j & \text{for all values of } i \text{ and } j, \\ \text{and} \quad (d\boldsymbol{R}_i/dt) . (d\boldsymbol{R}_j/dt) & \text{for all values of } i \text{ and } j. \end{array}\right\} \tag{3.02}$$

Consider now the geometrical shapes, sizes and relative geometrical disposition of the orbits arising from the same configuration of radii vectores and velocity vectors, but with the new independent variable $\tau = -t$. These orbits are completely specified by

$$\left.\begin{array}{rl} m_i & \text{for all values of } i, \\ \boldsymbol{R}_i . \boldsymbol{R}_j & \text{for all values of } i \text{ and } j, \\ (d\boldsymbol{R}_i/d\tau) . \boldsymbol{R}_j & \text{for all values of } i \text{ and } j, \\ \text{and} \quad (d\boldsymbol{R}_i/d\tau) . (d\boldsymbol{R}_j/d\tau) & \text{for all values of } i \text{ and } j; \end{array}\right\} \tag{3.03}$$

the equation of motion (3.01) being invariant under the transformation of t to τ. The geometrical shapes, sizes and relative disposition of the orbits arising from the configuration in τ are identical with those arising from the same configuration in t if conditions (3.03) are identical with conditions (3.02). Unless certain symmetry relationships hold between the masses m_i, conditions (3.03) are identical with conditions (3.02) if, and only if

$$(d\boldsymbol{R}_i/d\tau) . (d\boldsymbol{R}_j/d\tau) = (d\boldsymbol{R}_i/dt) . (d\boldsymbol{R}_j/dt) \text{ for all values of } i \text{ and } j, \tag{3.04}$$

and

$$(d\boldsymbol{R}_i/d\tau) . \boldsymbol{R}_j = (d\boldsymbol{R}_i/dt) . \boldsymbol{R}_j \quad \text{for all values of } i \text{ and } j. \tag{3.05}$$

But, from the definition of τ, (3.04) holds, and

$$(d\boldsymbol{R}_i/d\tau) . \boldsymbol{R}_j = -(d\boldsymbol{R}_i/dt) . \boldsymbol{R}_j \text{ for all values of } i \text{ and } j. \tag{3.06}$$

Condition (3.06) is equivalent to condition (3.05) if, and only if

$$(d\boldsymbol{R}_i/d\tau) . \boldsymbol{R}_j = 0 = (d\boldsymbol{R}_i/dt) . \boldsymbol{R}_j \text{ for all values of } i \text{ and } j. \tag{3.07}$$

Thus, if condition (3.07) holds at a certain epoch $t = \tau = 0$, the motion of the system of n bodies after this epoch is geometrically similar to the motion of the system prior to the epoch. At a mirror configuration the following conditions arise from the proof :—

(i) For all values of i, the geometrical shape and size of the orbit of the ith body after the mirror configuration is equivalent to the geometrical shape and size of the orbit of the ith body prior to the mirror configuration, but described in the opposite sense.

(ii) For all values of i and j, the relative geometrical disposition of the ith and jth orbits after the mirror configuration is equivalent to the relative geometrical disposition of these orbits prior to the mirror configuration.

(iii) For all values of i, the ith orbit is continuous at the mirror configuration. These three conditions serve to define uniquely the nature of the "reflection" of the orbits.

4. *The two possible cases of a mirror configuration.*—Consider two of the n point-masses. Let NM_1A_1 be the plane containing G and the instantaneous velocity vector of m_1, and NM_2A_2 be the plane containing G and the instantaneous velocity vector of m_2, where M_1, M_2 are the projections, on the celestial sphere centred on G, of the radii vectores Gm_1 and Gm_2 respectively, and A_1, A_2 are the projections, on the same celestial sphere, of the instantaneous directions of motion of m_1 and m_2 respectively. Then, if $d\boldsymbol{R}_i/dt \,.\, \boldsymbol{R}_j = 0$,

$$M_2A_2 = M_2A_1 = M_1A_2 = M_1A_1 = 90^\circ.$$

Thus, either M_1 and M_2 are poles of the great circle A_1A_2, or A_1 and A_2 are poles of the great circle M_1M_2. Thus, if longitudes are measured along the plane of the appropriate orbit, from the common node N, for a mirror configuration either

(i) longitude of $M_1 = 0^\circ$ or 180°, longitude of $A_1 = \pm 90^\circ$ and longitude of $M_2 = 0^\circ$ or 180°, longitude of $A_2 = \pm 90^\circ$, or

(ii) longitude of $M_1 = \pm 90^\circ$, longitude of $A_1 = 0^\circ$ or 180° and longitude of $M_2 = \pm 90^\circ$, longitude of $A_2 = 0^\circ$ or 180°.

Since for any pair of point-masses, either (i) or (ii) must hold, for a mirror configuration of the n point-masses, it is necessary either (i) that all the radii vectores $\boldsymbol{R}_i$ are parallel or anti-parallel, and all the velocity vectors $d\boldsymbol{R}_i/dt$ are perpendicular to all the $\boldsymbol{R}_i$, or (ii) that all the velocity vectors $d\boldsymbol{R}_i/dt$ are parallel or anti-parallel, and all the radii vectores $\boldsymbol{R}_i$ are perpendicular to all the $d\boldsymbol{R}_i/dt$.

For any two of the n point-masses, the above cases correspond to a conjunction or opposition (with respect to G) with each body at either pericentron or apocentron. In case (i), the conjunction or opposition must occur at the common nodes of the two orbits, while in case (ii) the conjunction or opposition must occur 90° from the common node.

To distinguish more clearly between these two cases, we discuss the nature of the "reflection" of the orbits for each case separately.

The equation of motion (3.01) may be written

$$\left.\begin{aligned}
\ddot{x}_i &= -\sum_{j=1}^{n} m_j \frac{(x_i - x_j)}{[(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2]^{3/2}}, \qquad j \neq i \\
\ddot{y}_i &= -\sum_{j=1}^{n} m_j \frac{(y_i - y_j)}{[(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2]^{3/2}}, \qquad j \neq i \\
\ddot{z}_i &= -\sum_{j=1}^{n} m_j \frac{(z_i - z_j)}{[(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2]^{3/2}}, \qquad j \neq i
\end{aligned}\right\} \tag{4.01}$$

where x, y, z are a rectangular set of axes with origin G, the centre of mass of the system of n point-masses.

Case (i):—Suppose that at $t = \tau = 0$, each m_i is on the line $y = 0$, $z = 0$, and is moving in a plane parallel to yGz. Suppose also that

$$(x_i, y_i, z_i, t) \qquad \text{etc.} \tag{4.02}$$

is a solution of (4.01). Then

$$(\bar{x}_i = x_i;\ \bar{y}_i = -y_i;\ \bar{z}_i = -z_i;\ \tau = -t) \quad \text{etc.} \tag{4.03}$$

is also a solution of (4.01), which is unchanged by the substitution. Hence (4.02) and (4.03) represent possible orbits. The initial conditions of the set of orbits (4.02) are

$$(x_i, y_i, z_i;\ dx_i/dt, dy_i/dt, dz_i/dt;\ t) = (a_i, 0, 0;\ 0, g_i, h_i;\ 0) \quad \text{etc.} \tag{4.04}$$

By the transformation (4.03), these initial conditions for the set of orbits represented by (4.03) become

$$(\bar{x}_i, \bar{y}_i, \bar{z}_i;\ d\bar{x}_i/d\tau, d\bar{y}_i/d\tau, d\bar{z}_i/d\tau;\ \tau);\ = (a_i, 0, 0;\ 0, g_i, h_i;\ 0) \quad \text{etc.} \tag{4.05}$$

(4.04) and (4.05) are identical, and therefore (4.02) and (4.03) represent the same orbit. Hence the orbit of m_i for $t>0$ is the mirror image of the orbit for $t<0$ in the planes $y\mathrm{G}x$ and $z\mathrm{G}x$.

Case (ii):—Suppose that at $t=\tau=0$, each m_i is in the plane $x\mathrm{G}y$ and moving perpendicular to it. Suppose also that

$$(x_i, y_i, z_i, t) \quad \text{etc.} \tag{4.06}$$

is a solution of (4.01). Then

$$(\bar{x}_i = x_i;\ \bar{y}_i = y_i;\ \bar{z}_i = -z_i;\ \tau = -t) \quad \text{etc.} \tag{4.07}$$

is also a solution of (4.01), which is unchanged by the substitution. Hence both (4.06) and (4.07) represent possible orbits. The initial conditions of the set of orbits (4.06) are

$$(x_i, y_i, z_i;\ dx_i/dt, dy_i/dt, dz_i/dt;\ t) = (a_i, b_i, 0;\ 0, 0, f_i;\ 0) \quad \text{etc.} \tag{4.08}$$

By the transformation (4.07), these initial conditions for the set of orbits represented by (4.07) become

$$(\bar{x}_i, \bar{y}_i, \bar{z}_i;\ d\bar{x}_i/d\tau, d\bar{y}_i/d\tau, d\bar{z}_i/d\tau;\ \tau) = (a_i, b_i, 0;\ 0, 0, f_i;\ 0) \quad \text{etc.} \tag{4.09}$$

(4.08) and (4.09) are identical, and therefore represent the same orbit. Hence the orbit of m_i for $t>0$ is the mirror image of its orbit for $t<0$ in the plane $x\mathrm{G}y$.

5. *The periodicity theorem.*—We define periodicity as follows:—In a system of n point-masses moving under their mutual gravitational forces only, their orbits are said to be periodic if, at periodic intervals of time, the same relative configuration of radii vectores and velocity vectors occurs with no change in scale.

Hence the periodicity theorem:—If n point-masses are moving under their mutual gravitational forces only, their orbits are periodic if, at two separate epochs, a mirror configuration occurs.

For, if the mirror condition (3.07) holds at $t=-t_0$ and at $t=0$, then it holds at $t=+t_0$, the orbits after $t=0$ being the mirror images of the orbits prior to $t=0$. Hence the orbits are periodic, with period $2t_0$.

6. *Discussion of stability.*—The above section gives no information on the stability of such orbits against perturbations by external forces. The manner in which stability is treated here makes no pretence to being exhaustive or rigorous, but the present authors believe that it throws some light on the possible importance of the periodic orbits discussed above, and leads to conclusions which are subsequently confirmed by appeal to observation.

There are various accepted definitions of stability of an orbit. We will use Poincaré's conditions for stability (4), namely:

(i) the heliocentric distance of any planet cannot increase or decrease without limit,

(ii) the system repeatedly passes through the configuration it had at time t_0, say a times t_1, t_2, etc.,

(iii) close encounters of any pairs of planets are ruled out.

(The conditions for satellite orbits are analogous.)

Condition (iii) implies that a body occupying a stable orbit will not stray far radially from it.

The discussion in Section 5 does not of itself satisfy Poincaré's conditions. Although the orbits are periodic if the mirror condition is satisfied at two separate epochs, they might be unstable in the sense that a small change in their configuration would lead to non-periodic orbits and rapid departures from those orbits so far pursued.

Let us consider the classical case of a system consisting of two massive bodies of unequal masses M_1 and M_2 pursuing circular orbits about their common centre of gravity, with a small mass m moving about the larger massive body M_1, in the same plane as, and being perturbed by the other massive body, M_2. Initially, let M_1, M_2 and m satisfy the mirror condition, at time t_0 say. Then m will move in a perturbed orbit and as a consequence its line of apses will rotate. In general, one synodic period of M_2 and m later, the velocity vector of m will be no longer perpendicular to the radius vector of m. But if the initial distances of m and M_2 have been chosen correctly, M_2's perturbation of m may be such that the time of revolution of m from one pericentron to some future pericentron is equal to one or more synodic periods of M_2 and m. Then this new configuration, at t_1 say, satisfies the mirror condition again. Therefore the behaviour of m after t_1 is a mirror image in the conjunction line M_1mM_2 of its behaviour before t_1. Thus m will move in a disturbed periodic orbit, the perturbations having a period of $2(t_1-t_0)$. It seems likely that if mM_2 is such, initially, that the mirror configuration after t_0 does not arise to "reverse" the perturbations within a sufficiently short time, the disturbances might build up sufficiently to cause m to be captured by M_2, or to collide with it or M_1. Again, if the second mirror configuration does not arise sufficiently quickly after t_0, the strongly perturbed periodic orbit of m might take it close to M_2, when the perturbation of some fourth body might cause it to be captured by M_2 or sent to infinity.

The periodic orbits studied by G. H. Darwin (3) are of interest, and may be said to be examples of the mirror theorem. Darwin considers the classical coplanar three-body problem, and confines his attention to simple periodic orbits which are periodic after one synodic period only, although the possibility of more complex periodic orbits with periods which are multiples of the synodic period is mentioned. In addition to isolating certain periodic orbits which correspond to the orbits discussed in this section, Darwin examines the stability of such periodic orbits. His results (in particular those relating to his "Family A of planets") to some extent corroborate the suggestions made in this section.*

Hence the reasonable hypothesis is made that in this restricted three-body case, the periodic orbit of m will be the more stable against external perturbations the shorter the time-interval between the occurrence of mirror configurations.

7. *The effect of near-commensurability of mean motions of two bodies.*—One major effect of near-commensurability is the occurrence of large inequalities, since the amplitudes of long-period terms involve the factor

$$(A_2n_1 - A_1n_2)^{-1}.$$

* We are indebted to the Referee for drawing our attention to this fact.

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Let us consider, after Newcomb (5), the mean motions n_1 and n_2 of two bodies orbiting about a primary in the same plane, and so related that

$$A_2n_1 - A_1n_2 = 0, \tag{7.01}$$

A_1 and A_2 being integers, and $n_2 < n_1$. Then A_2 revolutions of the first body will require the same time as A_1 revolutions of the second body, and at the end of this period they will have returned to their original positions, having been in conjunction $(A_1 - A_2)$ times at the same number of equidistant points. Calling these points "conjunction points", we put

$$\nu = A_1 - A_2. \tag{7.02}$$

But if the commensurability (7.01) is not exact, so that

$$A_2n_1 - A_1n_2 = \kappa,$$

then the mean conjunction points will revolve about the origin with velocity k given by

$$k = \frac{A_2n_1 - A_1n_2}{\nu}, \tag{7.03}$$

k approaching zero as κ approaches zero.

In the cases which we shall discuss in detail in this paper, $A_1 = A_2 + 1$, and therefore $\nu = 1$, so that we are concerned with only one mean conjunction point revolving uniformly with velocity

$$k = A_2n_1 - A_1n_2. \tag{7.04}$$

By adjusting the mean motions, k can be given any value.

It was seen in Section 5 that, if the mirror condition is satisfied twice, the orbits are periodic, and the reasonable hypothesis was made that the periodic orbit of m would be the more stable the shorter the time-interval between the two satisfactions of the mirror condition. The shortest possible time between mirror configurations is obtained if one synodic period of m and M_2 equals the time taken by m's pericentron to move from one conjunction to the next. But, by (7.04), if the mean motions are not exactly commensurable, the line of conjunction revolves with a uniform velocity k, and a mirror configuration would occur at every conjunction if the mean motions of the bodies were such that

$$k = \dot{\omega},$$

where $\dot{\omega}$ is the rate at which m's pericentron revolves.

We can now relax the conditions of the classical system considered hitherto, and consider two bodies moving in non-coplanar, non-circular orbits about a primary, and perturbing each other. We discuss two major cases:—

(*a*) If the inclinations are small, so that the orbits are approximately coplanar, and one eccentricity is much larger than the other, a first approximation to the frequent occurrence of mirror configurations is for the mean motions of the two bodies to be so nearly commensurable that the rate of revolution of the line of apses of the orbit of greater eccentricity equals the rate of rotation of the conjunction line of the two bodies. A perfect mirror configuration would occur only when

$$\omega_1 = \omega_2, \quad \text{or} \quad \omega_1 = \omega_2 + 180°,$$

where ω_1 and ω_2 are the longitudes of the pericentrons. Hence a libration will ensue, whose period is the synodic period of the pericentrons.

(*b*) If the inclinations are large, and the nodes of the orbits precess about the fundamental plane of the system at different rates, then by Section 4, a mirror configuration can occur only if conjunctions take place at either (i) the common node of the two orbits, or (ii) 90° from the common node, with the bodies on the apse lines of their orbits in each case. In this case, frequent occurrences of mirror configurations can occur if the mean motions are such that conjunctions always take place according to (i) or (ii).

It is of interest to make an estimate of the relative importance of (*a*) and (*b*) in an actual case. If the two orbits are coplanar, and one is circular while the other has a small eccentricity e, then it can be shown that, if a conjunction (or opposition) occurs at a small angle ξ from the apse-line of the eccentric orbit, then the angle between the velocity vectors at conjunction (or opposition) is of the order of $e\xi$. On the other hand, if the two orbits are circular, and inclined at a small angle i to each other, then, if a conjunction (or opposition) (measured along one of the orbits) occurs at a small angle ξ from the common node of the two orbits (or at an angle $90° \pm \xi$ from the common node), then the angle between the radii vectores (or between the velocity vectors) is of the order of $\xi \sin i$. If the two orbits have small inclinations i_1 and i_2 ($i_2 < i_1$) to the fundamental plane of the system about which their nodes precess, then the mutual inclination of the two orbits will vary between the limits $(i_1 + i_2)$ and $(i_1 - i_2)$. Therefore, in a semi-quantitative way, we would expect that if

$$e > \sin i_1$$

case (*a*) would be the more important, while if

$$e < \sin i_1$$

case (*b*) would be the more important.

In Part III we inspect certain systems in the solar system with outstanding commensurabilities, and see how far they support the above arguments.

Part III

8. *Saturn's system*

(*a*) *Hyperion and Titan.*—From Table II of Paper I, $(n_H/n_T) = \frac{3}{4} - 0{\cdot}000\,57$. The orbits are very nearly coplanar, and the eccentricities of Titan and Hyperion are 0·029 and 0·104 respectively. Hyperion is very strongly disturbed by Titan. On the basis of the arguments of Part II, we would expect that the mean motions of the two bodies would be such that the rate of revolution of the conjunction point would be the same as that of Hyperion's perisaturnium. From (7.04), with $n_T = 22°{\cdot}577\,009$ per day (**7**), the point of conjunction retrogresses at a rate of 18°·66(9) per year. The rate of retrogression of the perisaturnium of Hyperion is 18°·66(3) per year (**7**). It is, indeed, well known (**9**) that Hyperion and Titan come into conjunction with each other only at or near the aposaturnium of the former. More accurate elements derived since Newcomb's time (**6, 7, 8**) confirm this result.

If Titan had a circular orbit, a mirror configuration would occur at every conjunction and opposition. However, due to the eccentricity of Titan's orbit, a perfect mirror configuration would occur only when Titan was also at its aposaturnium or perisaturnium. At other conjunctions, the satisfaction of the mirror condition is only approximate, and hence the reversal of perturbations at such conjunctions or oppositions is only approximate. A libration is therefore to be expected, whose period is the synodic period of the apse lines of Titan and Hyperion, i.e. 18·75 years (**7**), and a major libration of this period, of amplitude 14°·0 (**7**) is present in the motion of Hyperion. The system of Hyperion and Titan is thus in excellent agreement with the arguments of Part II.

(*b*) *Dione and Enceladus*.—The orbits are equally inclined to the fundamental plane (the plane of Saturn's equator), and retrogress at different rates. However, the inclinations are very small ($\sim 1'$). The eccentricities of Dione and Enceladus are 0·0022 and 0·0044 respectively. Enceladus is the less massive body, and suffers the greater perturbations.

For this system, a first approximation to the frequent occurrence of mirror configurations can be found by considering the orbits to be coplanar, and the orbit of Dione to be circular. Then, for a mirror configuration to occur at every conjunction, the rate of motion of the perisaturnium of Enceladus should equal the rate of motion of the point of conjunction. Now, from Table II of Paper I, $(n_D/n_E) = \frac{1}{2} + 0{\cdot}000\,643$. Therefore, from (7.04), with $n_E = 262^\circ{\cdot}731\,99$ per day (**7**), the point of conjunction advances by $123^\circ{\cdot}41$ per year. The rate of advance of the perisaturnium of Enceladus, $\dot{\omega}_E = 123^\circ{\cdot}43$ per year (**7**). Thus we have the relation noted by H. Struve (**10**) that

$$2n_D - n_E = \dot{\omega}_E$$

within the accuracy of the derived elements. Struve also noted that conjunctions of Enceladus and Dione occur always near the perisaturnium of Enceladus, about which point they oscillate.

A second approximation to the actual problem is to consider the orbits to be still coplanar, but to consider the eccentricity of Dione's orbit. Then mirror configurations at conjunction will be only approximate unless the conjunction occurs when Dione is also at its perisaturnium or aposaturnium. A libration is to be expected whose period is the synodic period of the perisaturnia of Enceladus and Dione. Now $\dot{\omega}_D = 30^\circ{\cdot}74$ per year (**6**), so that

$$4\dot{\omega}_D = \dot{\omega}_E$$

within the accuracy of the derived elements (**14**). We would thus expect a libration in the longitudes of Enceladus and Dione of period 3·88 years. According to the *Connaissance des Temps* (**7**), a libration of amplitude $11'{\cdot}24$ and period 3·89 years exists in the longitude of Enceladus, and a libration of amplitude $\sim 1'$ of the same period in the longitude of Dione.

A third approximation to the actual problem is to consider the inclinations of the orbits to the fundamental plane of the system. Exact mirror configurations can occur at conjunction only if the conjunction takes place at the common node or 90° from the common node of the two orbits. Owing to the fact that the two inclinations are sensibly the same, conjunctions which occur when the nodes of the two orbits on the fundamental plane are together will satisfy the mirror condition exactly, whatever the longitude of the conjunction point, since at this time the two orbital planes coincide. Now (**7**)

$$\theta_E = 328^\circ - 152^\circ{\cdot}7t$$

and

$$\theta_D = 276^\circ - 31^\circ{\cdot}0t,$$

where θ_E, θ_D are the longitudes of the nodes of Enceladus and Dione respectively, and t is measured in Julian years from the epoch 1889 April 0·5.

$\therefore$ the synodic period of the nodes $= 2{\cdot}96$ years $= B$, say.

But the synodic period of the apse lines $= 3{\cdot}88$ years $= A$, say.

$$\therefore 4B - 3A = 0^\circ{\cdot}20$$

or, within the accuracy of the derived elements (**14**),

$$4B = 3A.$$

A libration in the longitudes of Enceladus and Dione of period $3A = 11{\cdot}6$ years is to be expected. According to the *Connaissance des Temps*, there is a libration in the longitude of Enceladus of amplitude $20'{\cdot}0$ and period $12{\cdot}2$ years, while the longitude of Dione has a libration of the same period, and amplitude $\sim 2'$. According to the *Berliner Astronomisches Jahrbuch* (**6**), the amplitudes are $14'$ and $1'$ respectively, and the period is $11{\cdot}1$ years. It is seen that, within the accuracy of the elements, the libration period $= 3A$, as predicted.

As a further check, it was found that, according to the first set of elements (**7**), a conjunction of the nodes and a conjunction of the lines of apses occurred at $t = -5{\cdot}5$ years, thus giving an *exact* mirror configuration at conjunction at this epoch and at all epochs separated from it by multiples of $11{\cdot}6$ years. The agreement is less satisfactory for the second set of elements (**6**), but it should be noted that, although the rates of retrogression of the nodes may be reasonably well determined, owing to the small inclinations the position of either node at a particular epoch is poorly determined.

We may conclude that, within the limits of accuracy of the derived elements, the system of Dione and Enceladus conforms to the pattern of the arguments of Part II.

(*c*) *Mimas and Tethys*.—The following elements have been given (**7**):—

	Mean Motion	Inclin.	Eccen.	Mass/primary
Mimas	$381^\circ{\cdot}9945$	$1^\circ\ 36'{\cdot}5$	$0{\cdot}0190$	$(16\,340\,000)^{-1}$
Tethys	$190^\circ{\cdot}69795$	$1^\circ\ 4'{\cdot}36$	$0{\cdot}0000$	$(\ \ 921\,500)^{-1}$

In this case, therefore, we have $\sin i_1 > e$, and we would expect the mirror configurations to be related primarily to the position of the common node of the two orbits.

For two orbits whose inclinations to the fundamental plane of the system are i_1 and i_2 $(i_1 > i_2)$, and whose nodes on the fundamental plane of the system have longitudes θ_1 and θ_2 respectively, it can be shown that the longitude of the common node θ_j is given by

$$\theta_j = \theta_1 + \zeta,$$

where

$$\tan \zeta = \frac{\sin(\theta_1 - \theta_2)}{\tan i_1 \, . \, \cot i_2 - \cos(\theta_1 - \theta_2)}, \qquad (8.01)$$

θ_j being measured along the fundamental plane of the system.

As a first approximation to the actual situation of Mimas and Tethys, we will consider that Tethys moves undisturbed by Mimas, that the two orbits are equally inclined to the fundamental plane, and that both orbits have zero eccentricity. Then, from (8.01),

$$\zeta = 90^\circ - \frac{(\theta_1 - \theta_2)}{2}.$$

If, therefore, conjunctions of Mimas and Tethys follow case (ii) of Section 4, the longitude of conjunction must always be $(\theta_M - \theta_T)/2$. If $\dot{\theta}_M, \dot{\theta}_T$ are the rates of revolution of the nodes of Mimas and Tethys on the fundamental plane, then, for this condition to hold, it is necessary that

$$2n_T - n_M = (\dot{\theta}_M + \dot{\theta}_T)/2$$

or

$$4n_T - 2n_M - \dot{\theta}_M - \dot{\theta}_T = E = 0. \qquad (8.02)$$

But this is precisely the situation for these two bodies, as noted by H. Struve (**11**). Struve also noted that, if the mean longitudes (i.e. ignoring libration terms) of the satellites are l_M and l_T, then

$$4l_T - 2l_M - \theta_T - \theta_M = 0. \tag{8.03}$$

Struve was thus led to enunciate the theorem that, ignoring libration, conjunctions of Mimas and Tethys occur always at a longitude which is the mean of the longitudes of the two bodies. Within the framework of the approximation considered here, this statement is equivalent to the statement that, ignoring libration, conjunctions occur always 90° from the common node of the two orbits. The mirror condition would, therefore, be satisfied at every conjunction.

In passing, it should be noted that the improvement of the determination of the elements of the two orbits with time has rendered (8.02) more exact, viz.

	Tisserand	*C. des T.* (7)	*B.A.J.* (6)
$E=$	$+3°$	$+0°{\cdot}532$	$+0°{\cdot}254$

A second approximation to the actual problem is obtained by considering both orbits to be circular, but of different inclinations to the fundamental plane. Then the mean conjunction point is only *exactly* 90° from the common node if $\theta_M = 180° + \theta_T$. Allowing for the different inclinations, we would expect a libration in mean longitude of both bodies of a period equal to an integral number of synodic periods of the nodes. A libration of amplitude $\sim 45°$ and period ~ 70 years is present in the longitude of Mimas, and a small libration of the same period in the longitude of Tethys. For the satisfaction of the mirror condition at each phase of zero libration, on the present approximation, the period of libration, P, must be such that

$$(\dot{\theta}_M - \dot{\theta}_T)\, P/2 = 360° m_1$$

or

$$P = 720° m_1 / (\dot{\theta}_M - \dot{\theta}_T), \tag{8.04}$$

where m_1 is an integer. At the same time, the phase of the libration must be such that zero libration corresponds to an opposition of the nodes.

With the elements of the *Connaissance des Temps* (7), an adjustment of 0·37 years in the libration period, and of 0·14 years in the epoch of zero libration would suffice to bring agreement with (8·04) with $m_1 = 29$. A comparison of the various sets of elements suggests that, owing to the long period of the libration, such adjustments are tolerable, but at the same time, the observational errors of both period and epoch being of the order of the synodic period of the nodes, little significance can be attached to such an agreement.

It should be noted that, if the epochs of zero libration correspond to oppositions of the two nodes, then the epochs of maximum and minimum libration correspond to conjunctions of the nodes. At such epochs, the mutual inclination of the orbits is at its smallest value of $(i_1 - i_2)$, and whatever the longitude of conjunction of the two bodies, a reasonable satisfaction of the mirror condition would occur.

A further approximation to the actual case of Mimas and Tethys is obtained by considering the eccentricity of Mimas' orbit. In this case, it is necessary that the epochs of zero libration correspond to conjunctions of the bodies when Mimas is at either its perisaturnium or its aposaturnium. With the mean motions used in Paper I, Table II, the rate of retrogression of the conjunction point of the two bodies is 218°·77 per year. Therefore, if $\dot{\omega}_M$ is the rate of advance of the apse line of the orbit of Mimas, it is necessary that

$$(\dot{\omega} + 218°{\cdot}77)P = 180° m_2, \tag{8.05}$$

where m_2 is an integer. At the same time it is necessary that the position of perisaturnium be such that $\omega_M = \theta_M \pm 90°$ at the epochs of zero libration, where ω_M is the longitude of the perisaturnium of Mimas.

While it is possible to derive a set of values for θ_M, $\dot{\theta}_T$, P and $\dot{\omega}_M$ which satisfy (8.02), (8.04) and (8.05), and which are probably consistent with the observational accuracy of the determined values of these elements, it does not seem possible for ω_M to have the required value, unless there is an error $\sim 30°$ in the position of perisaturnium at the epoch of the published elements.

It may be concluded that the system of Mimas and Tethys agrees with the predictions of the arguments of Part II of the present paper to the first, and major, approximation, but it is uncertain whether the agreement is exact.

9. *Enforced eccentricities.*—As is well known, the occurrence of near-commensurability of mean motions of two bodies produces large inequalities, which may take the form of enforced eccentricities. The usual viewpoint is that the nature of the enforced eccentricities, in particular the rates of revolution of the apse lines, are consequences of the magnitudes of the mean motions. The viewpoint of the present paper is in no way in conflict with this traditional view, but it suggests that the mean motions have their observed values because, in such orbits, the enforced eccentricities will be such as to give the frequent satisfactions of the mirror condition. As is seen from the theorem of Newcomb (**15, 16**):

"In the case of two orbits, originally circular, if the movement of the point of conjunction, supposed unique, is small compared with the movements of the two satellites, the perturbations caused by one of the bodies engenders in the other an eccentricity, and the apse line of each orbit passes always through the point of conjunction."

The theorem thus states that, for near-commensurable mean motions, the enforced eccentricities will provide frequent satisfactions of the mirror condition.

10. *Jupiter's system.*—In this system, there are two near-commensurabilities where the ratio of integers is of the form $(A_2+1)/A_2$, namely J II/J I and J III/J II. From Table III of Paper I, we have

$$n_2/n_1 = \tfrac{1}{2} - 0{\cdot}001\,82$$

and

$$n_3/n_2 = \tfrac{1}{2} - 0{\cdot}003\,65.$$

In addition, there are the Laplace relations

$$n_1 - 3n_2 + 2n_3 = 0, \qquad (10.01)$$

$$l_1 - 3l_2 + 2l_3 = 180° \qquad (10.02)$$

where l denotes the mean longitude (**12**). The orbits have small inclinations and small but variable eccentricities.

The relations (10.01) and (10.02) mean that when J II and J III are in conjunction, J I is in opposition to them, giving a close approximation to the mirror condition which is repeated every 7·0509 days, which is less than a single period of J III.*

11. *Uranus' system.*—The satellite system of Uranus is unique among the satellite systems of the planets in that the orbits of the satellites are sensibly circular and coplanar, as is shown by the extensive investigation of Newcomb (**13**), who, in giving the elements, remarked that there is but slight evidence

* According to Griffin (**17**), whenever J II is in conjunction with J I or J III, the inner of the pair is near perijove and the outer of the pair near apojove, which would give an even closer approximation to mirror conditions.

of any real eccentricity of the orbits, and no evidence of any mutual inclinations. Thus, within the framework of the arguments of this paper, *for pairs of satellites* the system of Uranus presents a completely degenerate case, in that any conjunction of a pair of bodies therein will satisfy the mirror condition to a high degree of approximation. We would thus expect to find *no* preference for near-commensurability of mean motion among *pairs* of satellites of Uranus. Reference to Table IV of Paper I shows that this expectation is, indeed, confirmed. This may be considered evidence in support of the given interpretation of the preference for near-commensurability of mean motion found in the other satellite systems and the planetary system.

12. *Griffin's analysis.*—After the above investigation had been completed, the attention of the authors was drawn to an investigation of periodic orbits by Griffin (17). For the restricted case of n finite satellites moving in almost-circular, coplanar orbits, Griffin deduced his "symmetry theorem":

> "If a symmetrical conjunction occurs at any instant $t=t_0$, then the orbit of each satellite before and after the conjunction is symmetrical, both with regard to geometric equality of figures and with regard to intervals of time",

where a "symmetrical conjunction" is the coplanar case of a "mirror configuration". Griffin's "symmetry theorem" is thus a special case of the general "mirror theorem" deduced above.

Griffin further deduced that commensurabilities of the type

$$n_1/n_2=(A_1+1)/A_1 \tag{12.01}$$

are *excluded* for periodic orbits. This is in conflict, not only with the detailed conclusions of the present paper, but also with the actual occurrence of commensurabilities in the planetary and satellite systems, as is seen at once on reference to Tables I–IV of Paper I.

The contradiction is resolved by an examination of the basis of Griffin's analysis. He begins with a system of infinitesimal satellites moving in circular orbits, the periodicity of configurations being assured by the introduction of commensurabilities into these original orbits. He then considers the orbits that are obtained when the masses of the satellites are increased from zero, and obtains a series of periodic, nearly-circular orbits which "grow out of"* the original circular orbits. He shows that in the special case (12.01), periodic orbits do *not* grow out of the original circular ones.

His conclusion is thus that there is no continuity, with varying satellite masses, between the finite and the infinitesimal cases when (12.01) holds. His analysis does *not* preclude the situation whereby the perturbations of finite satellites obeying (12.01) build up into a pair of periodic elliptical orbits, beyond the limits of his approximation. The theorem of Newcomb cited in Section 9 is an example of this process. Griffin specifically points out that any commensurabilities of the form (12.01) can only yield periodic orbits if the orbits are elliptical, and the orbits would suffer large perturbations.

As evidence of the non-existence of periodic orbits obeying (12.01), Griffin cites the asteroid gaps. This interpretation of the asteroid gaps has been criticized, after Brown and Shook's analysis, in Section 7 of Paper I.

* This term is Griffin's.

13. *Conclusions.*—The mirror theorem in its two aspects, together with the periodicity theorem, provides reasons why near-commensurabilities of the form $(A_2+1)/A_2$ should be preferred among the mean motions of the planets and satellites, except in the case of coplanar, circular orbits, when they should be absent, in agreement with observation. It appears that such orbits satisfy Poincaré's conditions of stability if only that for a considerable part of the system's lifetime the bodies will remain near these orbits and suffer only librations.

The frequent occurrence of mirror configurations is most likely to occur in a system of nearly-circular, nearly-coplanar orbits, when it is satisfied at every conjunction and opposition (for a pair of bodies only). An extension of the arguments of this paper suggests, therefore, that the most stable configuration of orbits for a gravitating system of point-masses (i.e. neglecting tidal friction) is a system of nearly-circular, nearly-coplanar orbits, and that any gravitating system of bodies will spend most of its lifetime in such a configuration, whatever the original distribution of orbital inclinations and eccentricities may have been. The preference in the solar system for orbits of small inclinations and small eccentricities may be further evidence for the arguments of this paper.

The authors realize that these arguments are by no means rigorous (excepting the proofs of the mirror theorem and the periodicity theorem given in Sections 3, 4 and 5, which are rigorous) but they hope that the support given to these arguments by the observational evidence will encourage more interest in this problem of celestial mechanics. It is hoped that future work will show whether or not the other cases of near-commensurability of mean motions in the solar system fall within the pattern of the arguments of the present paper.

Acknowledgments.—It is a pleasure to acknowledge the encouragement of Professor W. M. Smart, and our debt to Professor G. R. Goldsbrough for drawing our attention to Griffin's analysis and for making valuable comments on the first draft of this paper. One of us (A. E. R.) is indebted to the Royal Society of Edinburgh, Robert Cormack Bequest Committee, for the award of a Research Fellowship.

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1954 *November* 20.

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