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ON THE ANALOGY BETWEEN NEUTRON STAR MODELS AND ISOTHERMAL GAS SPHERES AND THEIR GENERAL RELATIVISTIC INSTABILITY

S. Yabushita

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SUMMARY

Investigation is made of the structure of gaseous spheres such that their equation of state is $p = q\rho$, where p and ρ are pressure and energy-density, respectively and q is a constant, and the relation is obtained between the central density and the mass of the gaseous spheres such that they exert the same pressure at the boundary. It is shown that this relation is very similar to the corresponding one for neutron star models, and in particular there exist mass-peaks in the diagram.

Assuming that the adiabatic index γ is related to q by the relation $\gamma = 1+q$, equation for radial oscillation about equilibrium is derived. It is shown that the neutral mode of the oscillation can be expressed in terms of the Emden function. By making use of the neutral mode solution, the degee of compression needed to make the sphere dynamically unstable is calculated. It can be shown that if $q \neq 0$ the sphere becomes unstable before the first mass-peak is reached in the mass-central density diagram, but if $q = 0$, the instability sets in precisely at the mass-peak.

1. INTRODUCTION

The equation of state $p = q\rho$, where p and ρ are pressure and energy–density respectively and q is a constant, is sometimes used as a physical property of the core of a neutron or hyperon star. The value of q is 1/3 for neutron star models, but it may be close to 1/13 when baryons are present (Ambartsumian & Saakyan 1961). Recently Chandrasekhar (1972) has shown that the structure of the gas sphere with the equation of state $p = q\rho$ calculated within the framework of general relativity is very similar to the structure of isothermal gas spheres. Of particular interest is the existence of the maximum pressure in the relation between the volume of a gas sphere with a given total mass and the pressure which it exerts at the boundary (Yabushita 1973a). On the other hand, the relation between the central density and the mass of gas spheres with a given value of pressure at the boundary will be shown to exhibit mass peaks very similar to neutron star models. In this paper it will be shown that these two features stem from the same property of the Emden equation calculated in general relativity.

The stability of isothermal gas spheres was calculated in Newtonian theory (Yabushita 1968): it has been shown that when γ (ratio of the specific heats) = 1, those spheres that lie right to the point of maximum pressure are stable, but the stability is lost precisely at the point of maximum pressure, or at the mass peak in the mass–central density diagram. On the other hand, the statement is sometimes made (Misner & Zapolsky 1964) that in neutron star models which exhibit similar mass peaks, general relativistic instability sets in at the first mass peak. The object of the present paper is to investigate whether this is the case for gas spheres such that $p = q\rho$ is valid, and are surrounded by an envelope which exerts a constant pressure. In a recent paper (Yabushita 1973b), the equation for radial pulsation has been derived for such gaseous spheres by assuming that the adiabatic index γ and q are two independent parameters, and the configurations that are marginally stable have been calculated. In the relativistic region, however, γ and q should be regarded as connected by the relation $\gamma = 1+q$ (see Misner & Zapolsky). When this relation is used, solutions corresponding to marginal stability can be obtained in terms of the relativistic Emden function as will be shown. And this solution enables one to decide when the stability is lost without numerically solving the equation of radial oscillation. It will be shown that the general relativistic instability sets in before the configuration of maximum pressure (or maximum mass) is reached.

2. THE PRESSURE–VOLUME AND MASS–RADIUS RELATIONS

Let us adopt the form of the metric used by Chandrasekhar and write

$$r = \alpha\xi, \qquad \alpha^2 = c^4 q[4\pi G\rho_c(1+q)]^{-1}$$

$$\rho = \rho_c\, e^{-\psi}, \qquad M(r) = (4\pi/c^2)\int_0^r \rho r^2\, dr = (4\pi\alpha^3\rho_c/c^2)\, M(\xi), \tag{2.1}$$

where r is the coordinate radius and ρ_c is the energy-density at the centre. The condition that the system is in hydrostatic equilibrium gives that the functions ψ and M satisfy the relativistic Emden equation

$$\left(1-\frac{2q}{1+q}\frac{M}{\xi}\right)\frac{d\psi}{d\xi} = M(\xi)/\xi^2+q\xi\, e^{-\psi}, \tag{2.2}$$

$$\frac{dM}{d\xi} = \xi^2\, e^{-\psi}.$$

The value of the pressure p at the boundary and the volume V of the gas sphere with a given amount $M(R)$ of total energy are respectively given by Yabushita (1973a)

$$V = \frac{4\pi}{3}\left[\frac{G(1+q)\,M(R)}{c^2 q}\right]^3 \frac{\xi^3}{M^3(\xi)},$$

$$p = \frac{c^4 q}{4\pi G(1+q)}\left[\frac{c^2 q}{M(R)\,G(1+q)}\right]^2 M^2(\xi)\exp\left[-\psi(\xi)\right]. \tag{2.3}$$

Now the value of ξ which corresponds to the maximum pressure (P) in the p–V diagram is given by the following consideration. At P, V decreases monotanically with increasing ξ. Hence the point P is characterized by $dp/d\xi = 0$, or

$$2\frac{dM}{d\xi}-\frac{d\psi}{d\xi}M(\xi) = 0. \tag{2.4}$$

On the other hand, if we consider a family of configurations with a given value of pressure (p_0) at the boundary, we have that

$$p_0\,(\text{regarded as given}) = \rho_c \exp\left[-\psi(\xi)\right], \tag{2.5}$$

which gives the relation between ρ_c and the value of ξ at the boundary. By inserting the relation (2.5) into (2.1), one finds that the central density ρ_c and the mass $M(R)$ are given by

$$\rho_c = p_0 \exp[\psi(\xi)],$$
$$M(R) = (4\pi/c^2 p_0^{1/2})[c^4 q/4\pi G(1+q)]^{3/2} M(\xi) \exp[-\tfrac{1}{2}\psi(\xi)]. \qquad (2.6)$$

These equations give the required relation between the density and total energy of gas spheres such that the pressure at the boundary has the same value. For the sake of illustration, the M–ρ_c relation is shown in Fig. 1 for the case $q = 1/3$.

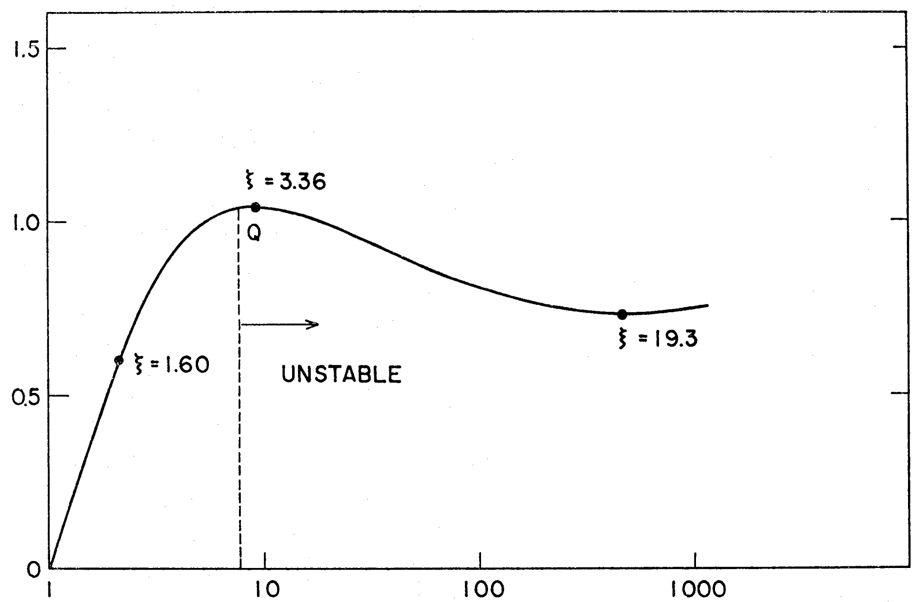

FIG. 1. *The relation between the mass and the central density. The abscissa gives exp* $[\psi(\xi)]$ *which is proportional to the central density and the ordinate gives* $M(\xi)$ *exp* $[-\psi(\xi)/2]$ *which is proportional to the mass.* $q = 1/3$. *The relation for* $q = 0$ *is similar to the case* $q = 1/3$.

One may note the similarity of the figure to Fig. 1 of Misner & Zapolsky for neutron star models. Of particular interest is the occurrence of maximum mass at the point denoted by Q in the figure. Its position is given by

$$\frac{dM(R)}{d\rho_c} = \frac{dM(R)}{d\xi}\frac{d\xi}{d\rho_c} = 0$$

and since $d\xi/d\rho_c \neq 0$, we must have that

$$2\frac{dM(\xi)}{d\xi} - \frac{d\psi}{d\xi} M(\xi) = 0,$$

which is the same as equation (2.4). This shows that the occurrence of the maximum pressure in the p–V relation and that of the maximum mass in the density–mass relation stem from the same property of the Emden equation in both Newtonian and general relativistic theory. There exist infinitely many values of ξ for which equation (2.4) is satisfied. The first one (ξ_1) corresponds to the first mass-peak (or pressure-peak) and the second one (ξ_2) corresponds to the first mass-valley and so on.

3. THE PARTICULAR SOLUTION FOR RADIAL MOTION

In a previous paper (Yabushita 1973b) the equation for radial motion of a gaseous sphere has been derived within the framework of general relativity by assuming that the motion about equilibrium takes place adiabatically with index γ; the unperturbed equilibrium configuration is such that the equation of state is of the form $p = q\rho$. In relativistic scheme, it is natural to define γ by the relation (Wheeler 1966)

$$\gamma = \frac{p+\rho}{p}\left(\frac{dp}{d\rho}\right)$$

as calculated by equation of state, or

$$\gamma = 1+q,$$

instead of treating γ and q as two independent parameters. When γ is connected to q by this relation, the equation for radial oscillation reads

$$\frac{d^2f}{d\xi^2}+\left(-\frac{2}{\xi}+\frac{d\psi}{d\xi}+q\xi\,e^{\lambda-\psi}\right)\frac{df}{d\xi}+e^{\lambda-\psi}\left[1+\frac{2q\xi}{1+q}\frac{d\psi}{d\xi}\right]f = \frac{\sigma^2c^4}{4\pi G\rho_c(1+q)}\,e^{\lambda-\nu}f, \qquad (3.1)$$

where the energy-density perturbation $\delta\rho$ is given by

$$\delta\rho = \frac{1}{\alpha^3\xi^2}\frac{df}{d\xi}\exp(\sigma ct)$$

and where $e^{-\lambda} = 1-2qM(\xi)/(1+q)\,\xi$, $\nu = 2q\psi(\xi)/(1+q)$. In order to discuss the stability, a solution to equation (3.1) with $\sigma^2 = 0$ is important since it corresponds to marginal stability. We show below that such a solution can be constructed from the Emden functions $M(\xi)$ and $\psi(\xi)$.

For this purpose, denote the differential operator on the left of the above equation by **L**. By differentiating equation (2.1), one finds that

$$\frac{d^2M}{d\xi^2} = 2\xi\,e^{-\psi}-\xi^2\frac{d\psi}{d\xi}\,e^{-\psi}$$

$$= \frac{2}{\xi}\frac{dM}{d\xi}-\xi^2\,e^{-\psi}\left(\frac{M}{\xi^2}+q\xi\,e^{-\psi}\right)\left(1-\frac{2q}{1+q}\frac{M}{\xi}\right)^{-1}$$

or

$$\frac{d^2M}{d\xi^2}-\frac{2}{\xi}\frac{dM}{d\xi}+q\xi\,e^{-\psi+\lambda}\frac{dM}{d\xi}+e^{-\psi+\lambda}M = 0,$$

whence follows the relation

$$\mathbf{L}(M) = \frac{d\psi}{d\xi}\frac{dM}{d\xi}+\frac{2q\xi}{1+q}\frac{d\psi}{d\xi}M\,e^{-\psi+\lambda}. \qquad (3.2)$$

In order to show that a similar relation exists for $\xi^3\,e^{-\psi}$, we first note the relation

$$\frac{d^2\psi}{d\xi^2}-e^{\lambda-\psi} = \left[-\frac{2M}{\xi^3}+q\,e^{-\psi}+\frac{d\psi}{d\xi}\left(\frac{2q}{1+q}\,\xi\,e^{-\psi}-\frac{2q}{1+q}\frac{M}{\xi^2}-q\xi\,e^{-\psi}\right)\right]e^{\lambda} \qquad (3.3)$$

which follows from equation (2.2). We also have that

$$\frac{d^2}{d\xi^2}(\xi^3\,\mathrm{e}^{-\psi})+\left(-\frac{2}{\xi}+\frac{d\psi}{d\xi}+q\xi\,\mathrm{e}^{-\psi+\lambda}\right)\frac{d}{d\xi}(\xi^3\,\mathrm{e}^{-\psi})$$

$$=\mathrm{e}^{-\psi}\left\{-\xi^3\frac{d^2\psi}{d\xi^2}-\xi^2\frac{d\psi}{d\xi}+q\xi\,\mathrm{e}^{-\psi+\lambda}\left(3\xi^2-\xi^3\frac{d\psi}{d\xi}\right)\right\}. \quad (3.4)$$

By adding $\xi^3\,\mathrm{e}^{-2\psi+\lambda}\,(d\psi/d\xi)\,2q\xi/(1+q)$ to the above equation and by making use of (3.3), one readily finds that

$$\mathbf{L}(\xi^3\,\mathrm{e}^{-\psi})=\frac{d\psi}{d\xi}\frac{dM}{d\xi}+\frac{2q\xi}{1+q}\,\mathrm{e}^{\lambda-\psi}\,M\frac{d\psi}{d\xi}. \quad (3.5)$$

Equations (3.5) and (3.2) show that the function

$$g\equiv M-\xi^3\,\mathrm{e}^{-\psi} \quad (3.6)$$

is a particular solution of the pulsational equation (3.1) with σ^2 put equal to zero. Moreover, when $\xi\ll 1$, $g\propto\xi^3$ so that $\delta\rho$ remains finite at $\xi=0$, and it follows that for g to be the solution of the eigenvalue problem, it has to satisfy the condition imposed at the boundary. It should be pointed out that in the limit $q\to 0$, equation (3.1) reduces to the corresponding one in Newtonian theory with $\gamma=1$ (Yabushita 1968), so that the particular solution (3.6) is valid both in Newtonian and relativistic theory.

4. MARGINALLY STABLE CONFIGURATIONS

Let us contend that during the motion about an equilibrium the outer medium exerts a constant pressure at the boundary so that the Lagrangian change of the pressure vanishes at the boundary. This condition is more physically admissible than the one adopted in the previous paper, namely that the Eulerian change of pressure be zero at the boundary. The vanishing of the Lagrangian change of pressure (see equation (2.16) of Yabushita 1973b) gives the condition

$$\frac{df}{d\xi}+\frac{d\psi}{d\xi}\frac{f}{1+q}=0 \qquad \text{at the boundary.} \quad (4.1)$$

For the g defined by (3.6), we now calculate the value of

$$\frac{dg}{d\psi}+\frac{d\psi}{d\xi}\frac{g}{1+q}$$

at the values of ξ(ξ_n, say) for which the equation (2.20) holds, that is $dp/dV=0$ or $dM/d\rho_c=0$. It is easy to calculate that

$$\left.\frac{dg}{d\xi}+\frac{d\psi}{d\xi}\frac{g}{1+q}\right|_{\xi=\xi_n}=\frac{q}{1+q}\left\{\frac{d\psi}{d\xi}(\xi^3\,\mathrm{e}^{-\psi}-M)\right\}_{\xi=\xi_n} \quad (4.2)$$

and when $q=0$, this expression vanishes. When q is non-zero, this is not generally the case. We have thus established that in Newtonian theory ($q=0$) the gas sphere becomes dynamically unstable when it is compressed to the configuration P in the p–V diagram and at this point, the first eigenvalue σ^2 changes from negative to positive value. This conclusion had been reached earlier (Yabushita 1968) by numerical means. Since the curve left to the maximum mass in the M–ρ_c diagram corresponds to the portion of the curve right to the point P in the p–V diagram, this is equivalent to the statement that configurations left to the point Q in the M–ρ_c diagram are dynamically stable. Again, since there are infinitely many values of ξ for which equation (2.4) is satisfied, it follows that when the gas sphere is compressed to the second value ξ_2 of (2.4), the second eigenvalue σ^2 changes from negative to positive value, and so on.

The above statement is valid only when $q = 0$, namely in the Newtonian theory. For non-zero values of q, the function $2dM/d\xi - M\, d\psi/d\xi$ has been calculated to obtain the values of ξ for which the function vanishes. In Table I below, these values are tabulated, together with the values of $\xi^3\, e^{-\psi} - M$ at these points.

Table I

The values $\xi_1, \xi_2, \ldots$ for which the function $2dM/d\xi - M\, d\psi/d\xi$ vanishes. $q = 1/3$

ξ	ξ_1	ξ_2	ξ_3
	3·36	19·27	123·7
$\xi^3\, e^{-\psi} - M$	0·88	−1·04	1·78

Now $d\psi/d\xi$ and $e^{-\psi}$ are everywhere positive, so that from the above Table it follows that there is a value of ξ in $\xi_i < \xi < \xi_{i+1}$ ($i = 1, 2$) such that

$$dg/d\xi + g(1+q)^{-1}\, d\psi/d\xi$$

vanishes. That this function is negative when $\xi \ll 1$ follows from the behaviours of $M(\xi)$ and (ξ) namely

$$\psi(\xi) = (\tfrac{1}{3}+q)\, \xi^2/2 + \ldots,\ M(\xi) = \xi^3/3 + \ldots,$$

Hence, as ξ increases from zero, the function $dg/d\xi + g(1+q)^{-1}\, d\psi/d\xi$ vanishes at a value less than ξ_1, or in other words, the first eigenvalue σ^2 changes from negative to positive value before the point of maximum pressure is reached in the p–V diagram. In a similar manner, the second eigenvalue changes from negative to positive value before the point ξ_2 is reached and so on.

Although this argument is sufficient to show that the general relativistic instability sets in at a configuration less compressed than the one at P or Q, the first value of ξ where $dg/d\xi + g(1+q)^{-1}\, d\psi/d\xi$ vanishes has been computed for various values of q, and is shown in Table II. For the sake of comparison, we give

Table II

The value of ξ for which $dg/d\xi + (d\psi/d\xi)\, f/(1+q)$ vanishes. This value gives the gas sphere which is marginally stable. Compare the value with Table III. Note that each value is less than the value of ξ given by Yabushita (1973a) which corresponds to maximum pressure.

$q =$ 0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
$\xi =$ 4·50	3·68	3·23	2·94	2·75	2·61	2·50	2·42	2·34	2·29

in Table III the configurations of marginal stability calculated with the boundary condition (4.1) and by regarding q and γ as two independent parameters. These values have been computed by the same procedure as in the previous paper, where a somewhat artificial boundary condition has been adopted.

The configurations of marginal stability are marked on the mass-density diagrams of Figs 1 and 2. As one may easily see, these configurations lie left to the

TABLE III

The values of ξ_b such that gas spheres with boundary values of ξ greater than ξ_b are unstable. Boundary condition (4.1) is used and q and γ are regarded as two independent parameters. Note that the underlined value is less than the value that corresponds to the mass-peak in the mass–central density relation, or the pressure-peak in the pressure–volume relation

	q = 0·1	1/3	1·0
1·0	3·92	2·49	1·58
1·1	4·49	2·68	1·65
1·2	5·25	2·86	1·72
1·3	6·36	3·05	1·80
4/3		3·12	
1·4	8·25	3·26	1·87
1·5		3·50	1·94
1·6		3·76	2·01
1·7		4·06	2·08
1·8			2·15
1·9			2·22
2·0			2·29

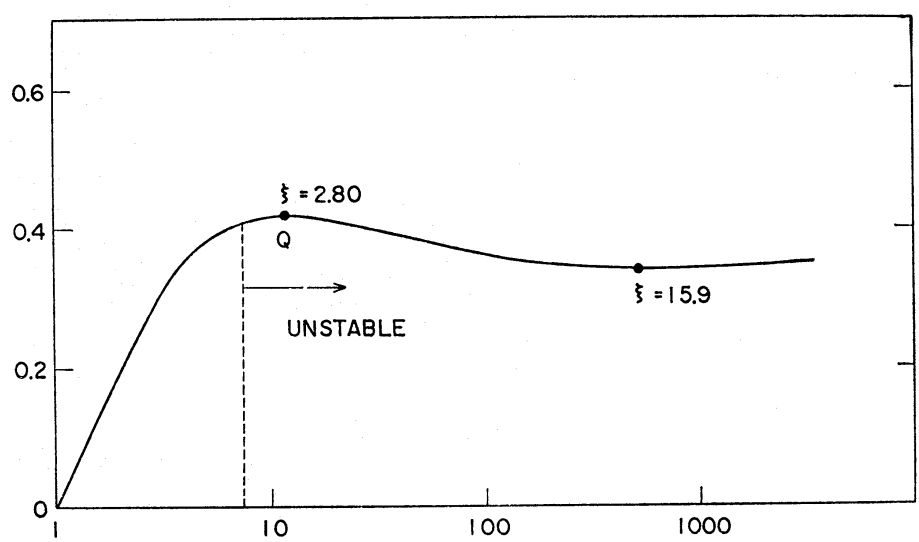

FIG. 2. *The mass–central density relation for q = 1·0.*

mass peak. Thus, general relativistic instability sets in before the mass peak (or pressure peak) is reached.

5. DISCUSSIONS

A core for which is the equation of state $p = q\rho$ s valid, surrounded by an envelope is sometimes used as a model of neutron stars. The model adopted by Misner & Zapolsky (1964) is one of them; it has a core in which $p = q\rho$ (q = 1, 1/3, 1/5, 1/13) and a relatively thin envelope. All such models show mass–density relations similar to the one shown in Fig. 1. Adopting Chandrasekhar's (1964) variational principle for investigating the stability of the neutron star model, Misner & Zapolsky claim to have found that the neutron star model becomes unstable precisely at the mass-peak, and this statement appears to be widely accepted in the literature (see, for instance, Wheeler 1966).

In the previous section we have shown that only in the Newtonian theory, the gas sphere loses stability at the mass peak and that the second eigenmode becomes unstable at the mass valley and the third one becomes unstable at the second mass peak and so on. But in general relativity we have found that this is not the case and indeed dynamical stability is lost before the first mass-peak is reached. This conclusion is different from Misner & Zapolsky's finding. The gas spheres investigated in the present paper have envelopes which exert constant pressure, whereas Misner & Zapolsky's neutron star model has a more realistic envelope in which the equation of state has a physically admissible form. To what extent the difference in the conclusions depends upon the difference in the models used is not easy to decide.

The critical mass which is marginally stable is only a few per cent less than the mass that corresponds to the mass peak and this amount of uncertainty is always involved in neutron star model calculations. However, that the marginally stable mass is less than the maximum mass is a genuine effect of relativity and as such its significance should be appreciated.

Department of Applied Mathematics and Physics, Kyoto University, Japan

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