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On the function describing the stellar initial mass function

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ABSTRACT

We propose a functional form for the initial mass function (IMF), the L_3 IMF, which is a natural heavy-tailed approximation to the log-normal distribution. It is composed of a low-mass power law and a high-mass power law which are smoothly joined together. Three parameters are needed to achieve this. The standard IMFs of Kroupa (2001, 2002) and Chabrier (2003a) (single stars or systems) are essentially indistinguishable from this form. Compared to other three-parameter functions of the IMF, the L_3 IMF has the advantage that the cumulative distribution function and many other characteristic quantities have a closed form, the mass generating function, for example, can be written down explicitly.

Key words: methods: data analysis – methods: statistical – stars: luminosity function, mass function.

1 INTRODUCTION

The initial mass function (IMF) of stars, the spectrum of stellar masses at their birth, is of fundamental importance in many fields of Astronomy. Since the seminal work of Salpeter (1955), who investigated the power-law part of the massive stars, a huge observational and theoretical effort has been made to constrain this distribution. Towards the lesser masses the IMF deviates from a power law and follows more a log-normal shape (Miller & Scalo 1979). At present, the whole shape of the IMF is usually described by power-law segments (Kroupa 2001, 2002) or by a log-normal segment plus a power-law segment (Chabrier 2003a,b, 2005). The aim of this paper is to provide an alternative, practical functional form for the IMF together with all its characteristic quantities (see Table 1 for the formulae and Figs 3 and 6).[1] More observational and theoretical aspects of the IMF can be found in recent reviews (e.g. Scalo 1986; Chabrier 2003a; Zinnecker & Yorke 2007; Elmegreen 2009; Bastian, Covey & Meyer 2010; Kroupa et al. 2011).

The IMF is usually believed to be a smooth function over the whole mass range, from brown dwarfs to O stars. However, Thies & Kroupa (2007) and Thies & Kroupa (2008) argued that a sudden change in binarity properties around the hydrogen burning limit introduces a discontinuity in the single-star IMF as well. This discontinuity in the single-star IMF can still lead to a system IMF without discontinuities over the whole mass range (Thies & Kroupa 2007; Kroupa et al. 2011). In view of the simplicity aspect of our proposed IMF form we neglect any discontinuity.

The proposed functional form, the L_3 IMF, fulfils several demands on the form of the IMF: it describes the whole (system) mass range with a single function. This has been achieved by several other functional forms as well (Larson 1998; Paresce & de Marchi 2000; Chabrier 2001; Parravano, McKee & Hollenbach 2011; Cartwright & Whitworth 2012). However, compared to these forms, the L_3 IMF has the advantage that its cumulative distribution function is invertible, so that sampling from the L_3 IMF is very easy. No special functions (e.g. the error function) are involved to normalize the L_3 IMF as a probability. Beyond that the analytical form allows also for simple, closed forms of characteristic quantities, such as the peak or the 'breaks', the masses from which on the power laws reign. Furthermore, with three parameters, two controlling the power-law behaviour at low and high masses and one location parameter, the number of parameters is as small as possible.

The motivation for the L_3 IMF is of purely pragmatic nature, it is a functional form that describes the data in a very practical way. It would be pleasing if the L_3 IMF could be more 'theoretically' motivated. One could try to find a connection to some generalized log-logistic growth processes, in analogy to logistic growth, as the L_3 is related to the log-logistic distribution. However, it remains questionable whether such a (non-stochastic) growth theory would be capturing the star formation process in its entirety (cf. the discussion about logistic growth in Feller 1968, p. 52). Where would be the place of, for example, feedback or stellar dynamics in shaping the IMF if growth alone gives all parameters of the IMF? Thus, it seems futile to follow such thoughts and we do not attempt to find any reasons for our proposed functional form, other than its utmost simplicity and practicality.

The organization of this paper is the following. After some general definitions we discuss in Section 2 established functional forms and required parameters of the IMF. The L_3 and B_4 IMFs are motivated and defined in Section 3 as heavy-tailed approximations and extensions to the log-normal distribution. This is followed by a detailed description of the L_3 IMF and its characteristic quantities in Section 4, the B_4 IMF is discussed in Appendix A. Section 5 gives the 'canonical' parameters for the L_3 IMF, matching it to the Kroupa

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[1] R code for the functions given in this paper is available as online material.

Table 1. Collection of formulae for the L_3 form of the IMF. The values given for the parameters and characteristic quantities are to match the 'canonical' single-star IMF (Kroupa 2001, 2002; Chabrier 2003a), values in parentheses for the 'canonical' system (binary star) IMF (Chabrier 2003a). $B(t; p, q)$ is the incomplete beta function. For the limits, we adopt the fiducial values $m_l = 0.01\,\mathrm{M}_\odot$ and $m_u = 150\,\mathrm{M}_\odot$, which are only needed for the normalization.

(1)	Auxilliary function:	$G(m) = \left(1 + \left(\frac{m}{\mu}\right)^{1-\alpha}\right)^{1-\beta}$	
	Quantity	Formula	Definition
	Functional form		
(2)	Cumulative distribution function (CDF)	$P_{L3}(m) = \frac{G(m) - G(m_l)}{G(m_u) - G(m_l)}$	$P(m_l) = 0$ and $P(m_u) = 1$
(3)	Probability density function (pdf)	$p_{L3}(m) = A\left(\frac{m}{\mu}\right)^{-\alpha}\left(1 + \left(\frac{m}{\mu}\right)^{1-\alpha}\right)^{-\beta}$	$p(m) = \frac{\mathrm{d}}{\mathrm{d}m}P(m)$
		$A = \frac{(1-\alpha)(1-\beta)}{\mu}\frac{1}{G(m_u) - G(m_l)}$	
(4)	Quantile function	$m(u) = \mu\left(\left[u\left(G(m_u) - G(m_l)\right) + G(m_l)\right]^{\frac{1}{1-\beta}} - 1\right)^{\frac{1}{1-\alpha}}$	$m(u) = P^{-1}(u)$, $u \in [0, 1]$
	Parameters		
	High-mass exponent	$\alpha = 2.3\,(2.3)$	$\alpha \neq 1$ (typically $\alpha > 0$)
	Low-mass exponent	$\beta = 1.4\,(2.0)$	$\beta \neq 1$ (typically $\beta > 0$)
	Scale parameter	$\mu = 0.2\,(0.2)\,\mathrm{M}_\odot$	$\mu > 0$
	Lower mass limit	$m_l = 0.01\,\mathrm{M}_\odot$	$m_l > 0$
	Upper mass limit	$m_u = 150\,\mathrm{M}_\odot$	$m_u > 0$
	Shape characterizing quantities		
	Effective high-mass exponent	$\alpha = 2.3\,(2.3)$	$\lim_{m\to\infty} p(m) \propto m^{-\alpha}$
(5)	Effective low-mass exponent	$\gamma = \alpha + \beta(1-\alpha) = 0.48\,(-0.3)$	$\lim_{m\to 0} p(m) \propto m^{-\gamma}$
(6)	Lower power-law mass limit	$m_\gamma = \mu \mathrm{e}^{\frac{2}{1-\alpha}} = 0.043\ (0.043)\ \mathrm{M}_\odot$	$p(m \in [m_l \ldots m_\gamma]) \approx m^{-\gamma}$
(7)	Upper power-law mass limit	$m_\alpha = \mu \mathrm{e}^{\frac{2}{\alpha-1}} = 0.93\ (0.93)\ \mathrm{M}_\odot$	$p(m \in [m_\alpha \ldots m_u]) \approx m^{-\alpha}$
(8)	Exponent [N.B. $S(\infty) = \alpha\ (=2.3)$]	$S(m) = \alpha + \beta(1-\alpha)\left(1 + \left(\frac{m}{\mu}\right)^{1-\alpha}\right)^{-1}\left(\frac{m}{\mu}\right)^{1-\alpha}$	$S(m) = -\frac{\mathrm{d}\log p(m)}{\mathrm{d}\log m}$
	Scale characterizing quantities		
	Mean mass (expectation value)	$E(m)\,(=\overline{m})$ = expressed by Beta function, see equation (25) $= 0.36\,(0.62)\,\mathrm{M}_\odot$	$E(m) = \int_{m_l}^{m_u} mp(m)\mathrm{d}m$
(9)	Median mass	$\widetilde{m} = \mu\left(\left[\frac{1}{2}\left(G(m_u) - G(m_l)\right) + G(m_l)\right]^{\frac{1}{1-\beta}} - 1\right)^{\frac{1}{1-\alpha}} = 0.10\,(0.21)\,\mathrm{M}_\odot$	$P(\widetilde{m}) = \frac{1}{2}$
(10)	Mode (most probable mass)	$\widehat{m} = \begin{cases} \mu\left(\frac{\beta(\alpha-1)}{\alpha} - 1\right)^{\frac{1}{(\alpha-1)}} & \gamma < 0 \\ m_l & \gamma > 0 \end{cases} = 0.01\ (0.04)\ \mathrm{M}_\odot$	$\widehat{m} = \arg\max_m p(m)$
(11)	'Peak' (maximum in log–log)	$m_P = \mu(\beta - 1)^{\frac{1}{\alpha-1}} = 0.10\ (0.20)\ \mathrm{M}_\odot$	$\frac{\mathrm{d}\log(mp(m))}{\mathrm{d}\log m} = 0$

(2001, 2002) and Chabrier (2003a,b) IMFs. Section 6 contains the conclusions of this article.

2 PROPERTIES OF THE IMF

2.1 Definitions

We normalize the IMF as a *probability density function* (pdf), the IMF tells us about the relative frequencies of stars of various masses in linear mass space. This allows us to use common statistical techniques, e.g. to estimate the parameters. For functions normalized as pdf we use the symbol $p(m)$, for their integrals, the cumulative distribution function, the symbol $P(m)$. The cumulative distribution function is related to the observed number frequency, $N(m)$, by $P(m) = \frac{1}{n_{\mathrm{tot}}}N(m)$, where n_{tot} is the total number of observed stars. The standard normalization condition for a probability is

$$1 = \int_{m_l}^{m_u} p(m)\mathrm{d}m, \tag{1}$$

where m_l and m_u are the lower and upper mass limit, respectively.

Historically there exist two alternative descriptions of the IMF, in linear or in logarithmic space, the small-α and the big-Γ notation. The use of the IMF as probability of m leads naturally to the linear (small-α) description, the IMF is fulfils

$$p(m) = \frac{\mathrm{d}P(m)}{\mathrm{d}m}\left[= \frac{1}{n_{\mathrm{tot}}}\frac{\mathrm{d}N(m)}{\mathrm{d}m}\right]. \tag{2}$$

A power-law IMF has then the exponent $-\alpha$, $p(m) \propto m^{-\alpha}$. In the logarithmic description, the IMF is normalized as probability of $\log m$, not m,

$$p_{\log}(\log m) = \frac{\mathrm{d}P_{\log}(\log m)}{\mathrm{d}\log m}\left[= \frac{1}{n_{\mathrm{tot}}}\frac{\mathrm{d}N_{\log}(\log m)}{\mathrm{d}\log m}\right]. \tag{3}$$

$p_{\log}(\log m)$ is connected to the linear pdf via

$$p(m) = \frac{\mathrm{d}P_{\log}(\log m)}{\mathrm{d}\log m}\frac{\mathrm{d}\log m}{\mathrm{d}m} = \frac{1}{m}p_{\log}(m). \tag{4}$$

Thus, a power-law pdf in m, $p(m) \propto m^{-\alpha}$, transforms into $p_{\log}(m) \propto m^{-(\alpha-1)}$ or $p_{\log}(m) \propto m^{-\Gamma}$, where $\Gamma = \alpha - 1$.

We define the exponent (sometimes referred to as 'slope', but that should be reserved for the logarithmic description), as a function of mass via

$$S(m) = -\frac{\mathrm{d}\log p(m)}{\mathrm{d}\log m} = -m\frac{\mathrm{d}\log p(m)}{\mathrm{d}m}. \tag{5}$$

A power-law IMF can then be written as

$$p(m) \propto m^{-S(m)}. \tag{6}$$

We follow the convention that the negative sign is not included in the exponent. Thus, in our notation the Salpeter (1955) exponent is positive, $\alpha = +2.35$.

2.2 The standard IMFs and other functional forms

The Kroupa (2001, 2002) single-star IMF consists only of power-law segments,

$$p_{\mathrm{Kroupa}}(m) = \begin{cases} Ak_0 m^{-0.3} & 0.01 < m < 0.08\,\mathrm{M}_{\odot} \\ Ak_1 m^{-1.3} & 0.08 < m < 0.5\,\mathrm{M}_{\odot} \\ Ak_2 m^{-2.3} & 0.5\ < m < 1\,\mathrm{M}_{\odot} \\ Ak_3 m^{-2.3} & 1\quad < m\ \ (<150\,\mathrm{M}_{\odot}) \end{cases}, \tag{7}$$

with $k_0 = 1$, $k_1 = k_0 m_1^{-0.3+1.3}$, $k_2 = k_1 m_2^{-1.3+2.3}$ and $k_3 = k_2 m_3^{-2.3+2.3} (= k_2)$, where $m_1 = 0.08\,\mathrm{M}_{\odot}$, $m_2 = 0.5\,\mathrm{M}_{\odot}$ and $m_3 = 1\,\mathrm{M}_{\odot}$ (a practical algorithm for the calculation of the k_i is given by Pflamm-Altenburg & Kroupa 2006). A is some global normalization constant. This form is highly adaptable, which comes at the price of a large number of parameters. On the practical side, the Kroupa (2001, 2002) IMF has the advantage that many derived quantities can be calculated without involving special functions (cumulative distribution function, quantile function, mean mass etc.), but with several 'if' statements to specify the mass ranges.

Chabrier (2003a,b) combined for the single-star IMF a log-normal distribution at the low-mass end with a high-mass power law,

$$p_{\mathrm{Chabrier}}(m) = \begin{cases} Ak_1 \frac{1}{m} \mathrm{e}^{-\frac{1}{2}\left(\frac{\log_{10} m - \log_{10} 0.079}{0.69}\right)^2} & m < 1\,\mathrm{M}_{\odot} \\ Ak_2 m^{-2.3} & m > 1\,\mathrm{M}_{\odot} \end{cases}, \tag{8}$$

with $k_1 = 0.158$ and $k_2 = 0.0443$ and the global normalization constant A. The log-normal and the power-law part connect up more or less smoothly, without the 'kinks' of several power-law segments (although there is still the small kink at $1\,\mathrm{M}_{\odot}$). Calculating the cumulative distribution function involves the error function, but random variates can be created without any specialized algorithms from standard Gaussian distributed random numbers.

A piece-wise functional form of the IMF is somewhat unsatisfying, and several alternatives covering the whole mass range have been proposed in the literature. There are, for example, the functional forms of Larson (1998)

$$p_{\mathrm{Larson\ a}}(m) \propto \frac{1}{m}\left(1 + \frac{m}{\mu}\right)^{-(\alpha+1)}, \tag{9}$$

and

$$p_{\mathrm{Larson\ b}}(m) \propto m^{-\alpha} \mathrm{e}^{-\left(\frac{m}{\mu}\right)^{-1}}, \tag{10}$$

form 3 of Chabrier (2001),

$$p_{\mathrm{Chabrier\ 3}}(m) \propto m^{-\alpha} \mathrm{e}^{-\left(\frac{m}{\mu}\right)^{-\beta}}, \tag{11}$$

or the tapered power-law form of Paresce & de Marchi (2000), De Marchi, Paresce & Portegies Zwart (2010), Hollenbach, Parravano & McKee (2005) and Parravano et al. (2011),

$$p_{\mathrm{TaperedPL}}(m) \propto m^{-\alpha}\left(1 - \mathrm{e}^{-\left(\frac{m}{\mu}\right)^{-\beta}}\right). \tag{12}$$

The IMF forms of equations (9), (10), (11) and (12) are very similar to our proposed form of the IMF, but their integrals contain the incomplete gamma function or the hypergeometric function. A cumulative distribution function without closed form is hard to invert, so special algorithms are necessary for random variates from these distributions.

Recently, Cartwright & Whitworth (2012) proposed a completely different class of distribution functions for the IMF description, stable distributions. Stable distribution (e.g. the Gaussian distribution) arises naturally in the context of stochastic processes, of which the star formation process is one example. Related to stable distributions, and also the outcome of stochastic processes is the class of infinitely divisible distributions, such as the log-normal distribution (e.g. Elmegreen & Mathieu 1983; Zinnecker 1984; and, for infinite divisibility, Thorin 1977). The choice of stable distributions is motivated by their relation to stochastic processes; however, they are also used only as a fitting function, as the exact stochastic process describing star formation has not yet been formalized. Also, typically they do not have a closed form for the distribution function itself, which is an important practical aspect.

2.3 How many parameters for the IMF

The IMF seems to have a log-normal body with a power-law tail on both the high-mass and the low-mass side. In order to describe this behaviour, four parameters appear to be required: a location parameter (which is not necessarily the 'peak' or the mean), a scale or width parameter (which is not necessarily the variance), the low-mass and high-mass power-law exponents. There are no stars of zero or infinite mass, so that additionally an upper and a lower mass limit has to be introduced, so the total number of parameters is 4+2. This is two parameters less than in the schematic IMF of Bastian et al. (2010), where additionally two 'mass breaks' are introduced, i.e. 6+2 parameters. However, if one requires that the log-normal part merges smoothly into the power-law tails, then the scale parameter sets the width of the IMF and consequently the mass breaks. The mass 'breaks' are then not parameters any more, but derived quantities. 4+2 seem therefore to be the necessary number of parameters to describe the IMF. The B_4 IMF discussed later is a smooth function over all masses and has the mentioned 4+2 parameters.

The number of parameters of the IMF can be reduced by one, because it is not necessary to explicitly include a scale parameter to fit the 'canonical' IMF. Only a location parameter and the two exponents suffice to achieve this. Several 3+2 IMFs have been suggested in the literature (equation 11, IMF 3 of Chabrier 2001; equation 12, Paresce & de Marchi 2000; Hollenbach et al. 2005; De Marchi et al. 2010; Parravano et al. 2011). Our proposed L_3 IMF also has only 3+2 parameters.

2+2 parameter functional forms (equation 9 and 10) have been given by Larson (1998), with a location parameter and only a high-mass exponent. With only 2+2 parameters it is difficult to fit the low-mass end of the IMF.

For comparison, the Kroupa (2001, 2002) has 5+2 parameters (three exponents, two thresholds, two limits) and the Chabrier

(2003a,b) IMF has 4+2 parameters (mean, variance, one exponent, one threshold, two limits).

3 HEAVY-TAILED APPROXIMATIONS TO THE LOG-NORMAL DISTRIBUTION

Starting point for the search of a functional form for the IMF is the relation between the normal distribution and the logistic distribution (see e.g. Johnson, Kotz & Balakrishnan 1994, 1995). The normal distribution,

$$p_{\mathcal{N}}(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \tag{13}$$

can be approximated in the central region for $\sigma = 1$ by the logistic distribution,

$$p_L(x) = \frac{1}{\sigma'} \frac{\mathrm{e}^{-\frac{x-\mu}{\sigma'}}}{\left(1 + \mathrm{e}^{-\frac{x-\mu}{\sigma'}}\right)^2}, \tag{14}$$

where $\sigma' = \mathrm{e}^{-\frac{1}{2}}$. The ratio of the two probability densities is close to unity between -2σ and $+2\sigma$, but drops off strongly outside. This behaviour is evident in a logarithmic plot of both densities (Fig. 1), the tails of the logistic distribution are much heavier than the normal distribution, with fixed exponents.

In order to translate the relation of normal and logistic distribution to the log-normal distribution,

$$p_{\log\mathcal{N}}(x) \propto \frac{1}{x} \mathrm{e}^{-\frac{1}{2}\left(\frac{\ln x - \ln\mu}{\sigma}\right)^2}, \tag{15}$$

we rewrite the log-normal density function as

$$p_{\log\mathcal{N}}(x) \propto \frac{1}{x} \mathrm{e}^{-\frac{1}{2}\left(\ln\left[\left(\frac{x}{\mu}\right)^{\frac{1}{\sigma}}\right]\right)^2}. \tag{16}$$

Inserting $\ln\left[\left(\frac{x}{\mu}\right)^{\frac{1}{\sigma'}}\right]$ for $\frac{x-\mu}{\sigma'}$ into the logistic cumulative distribution function,

$$P_{\log L}(x) = \frac{1}{1 + \mathrm{e}^{-\frac{x-\mu}{\sigma'}}}, \tag{17}$$

and taking the derivative gives the log-logistic density,

$$p_{\log L}(x) \propto \frac{\left(\frac{x}{\mu}\right)^{-\frac{1}{\sigma'}-1}}{\left(1 + \left(\frac{x}{\mu}\right)^{-\frac{1}{\sigma'}}\right)^2}. \tag{18}$$

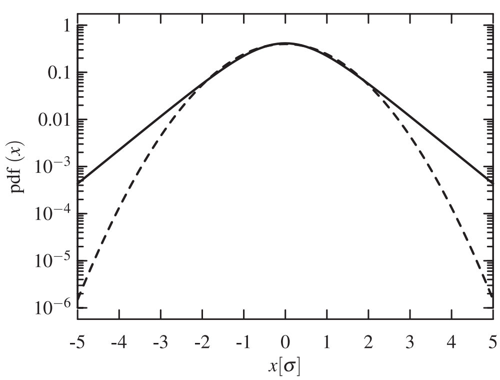

Figure 1. Comparison of the normal (dashed line) and logistic (solid line) probability density, with a logarithmic *y*-axis. The logistic distribution has heavier tails.

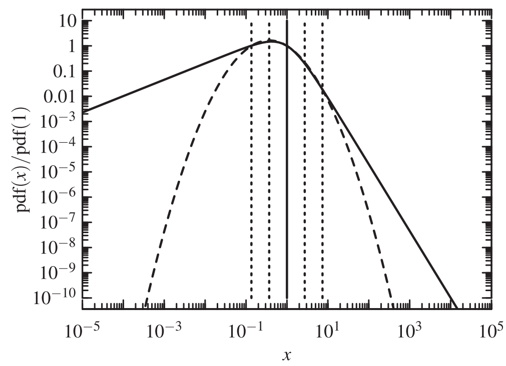

Figure 2. Comparison of the log-normal (dashed line) and log-logistic (solid line) distribution, scaled to 1 at $x = 1$. The dotted lines are at e^{-2}, e^{-1}, e^{1} and e^{2}. The tails of the log-logistic distribution are asymmetric.

Fig. 2 shows $p_{\log\mathcal{N}}$ and $p_{\log L}$, again with $\sigma = 1$ and $\sigma' = \mathrm{e}^{-\frac{1}{2}}$. The log-logistic distribution follows the log-normal distribution over about two orders of magnitude and deviates with asymmetric tails.

The log-logistic distribution of Fig. 2 already looks very much like the IMF. Only the high-mass and low-mass exponents are still fixed. In fact, this is not quite correct, because the meaning of σ' has been changed from the width of the distribution (i.e. a scale parameter) to determining the low-mass exponent (i.e. a shape parameter). Arbitrary exponents for the low-mass and the high-mass tail can be introduced by writing

$$p_{L3}(m) \propto \frac{\left(\frac{m}{\mu}\right)^{-\alpha}}{\left(1 + \left(\frac{m}{\mu}\right)^{1-\alpha}\right)^{\beta}}. \tag{19}$$

Unfortunately, β is *not* the exponent at low masses, which is the price paid for equation (19) having a very simple cumulative distribution function. Probability densities similar to equation 19 (two exponents and μ) are known under several other names, particularly in economics. We will refer to it as generalized log-logistic distribution, or in short 'L_3 IMF', because it has three (shape) parameters.

A parameter that changes the width of the IMF can be introduced by writing

$$p_{B4}(m) = \frac{\left(\frac{m}{\mu}\right)^{\beta}}{\left(1 + \left(\frac{m}{\mu}\right)^{\frac{1}{\sigma}}\right)^{\sigma(\alpha+\beta)}}. \tag{20}$$

σ is now the scale parameter, and α and β the exponents of the power-law tails. The integral of equation (20) does not have a closed form, but can be transformed into the incomplete beta function. Therefore, probability densities of the type of equation (20) are known as (generalized) beta distributions. Because of the four parameters we will refer to it as B_4 IMF.

The following sections will show that the 'canonical' IMF (Kroupa 2001, 2002; Chabrier 2003a) can be very satisfyingly described by the L_3 IMF. The introduction of σ as an additional scale parameter seems not to be necessary. Therefore, we consider in the following only the L_3 IMF and give the corresponding equations and parameter values for the B_4 IMF in Appendix A.

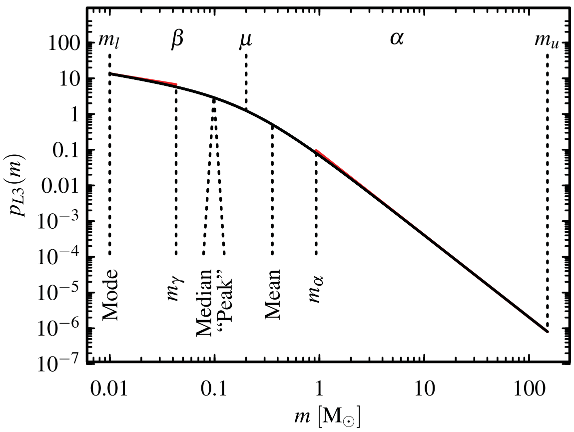

Figure 3. Probability density function for the L_3 functional form of the IMF (Table 1, equation 3) with the 'canonical' parameters, $\alpha = 2.3$, $\beta = 1.4$, $\mu = 0.2$ and the limits $m_l = 0.01\,\mathrm{M}_\odot$ and $m_u = 150\,\mathrm{M}_\odot$. It follows approximately $m^{-\alpha}$ for $m > m_\alpha(\approx 0.9\,\mathrm{M}_\odot)$ and $m^{-\gamma} = m^{-(\alpha+\beta(1-\alpha))}$ for $m < m_\gamma(\approx 0.04\,\mathrm{M}_\odot)$ with $\gamma = 0.48$. Also shown are the locations of mean, median and mode, which are all different because of the skewed distribution. The infamous 'peak' (maximum in log–log) *is not* the location at which the two power laws cross over. This happens at the scale parameter μ.

4 THE L_3 IMF

4.1 Functional form

The probability density of the L_3 IMF is given in equation (19), or, with the normalization constant, in Table 1 equation 3. Table 1 collects all formulae for the L_3 IMF. Fig. 3 shows the L_3 IMF with its characteristic quantities for the 'canonical parameters' of the single-star IMF. The particular advantage of the L_3 IMF is that the integral of the probability density is very simple,

$$\int \frac{\left(\frac{m}{\mu}\right)^{-\alpha}}{\left(1+\left(\frac{m}{\mu}\right)^{1-\alpha}\right)^{\beta}}\mathrm{d}m \propto \left(1+\left(\frac{m}{\mu}\right)^{1-\alpha}\right)^{1-\beta} =: G(m). \tag{21}$$

The full cumulative distribution function, including the upper and lower limits (m_l and m_u), is then

$$P(m) = \frac{G(m) - G(m_l)}{G(m_u) - G(m_l)} \tag{22}$$

(also equation 2, Table 1). Equation (22) can be readily inverted to give the quantile function (equation 4, Table 1). Generating a random mass from the L_3 IMF (i.e. inserting a uniform random number u in the quantile function) can then essentially be done in a single line of code.

The two shape parameters have different meanings for the L_3 IMF. For large masses $\lim_{m\to\infty} p(m) \propto m^{-\alpha}$, i.e. α is the high-mass exponent. In order that the L_3 IMF is defined, $\alpha \neq 1$ is required, typically will be $\alpha > 1$. For small masses the limiting case is $\lim_{m\to 0} p(m) \propto m^{-\gamma}$ with $\gamma = \alpha + \beta(1-\alpha)$. Therefore, the parameter β *is not* the low-mass exponent. This inconvenience of β and γ is the trade-off for the very simple cumulative distribution, Again, in order for the L_3 IMF to be defined $\beta \neq 1$ is required, typically will be $\beta > 1$. For $\alpha > 1$ and $\beta > 1$, the largest value that γ can take is $+1$, i.e. $p(m) \propto m^{-1}$. γ will be negative for $\beta > \frac{\alpha}{\alpha-1}$. A graphical representation of the relation between the exponents is given in the '$\alpha\beta\gamma$ plot', Fig. 4, where the value of γ for given α and β can easily be read off.

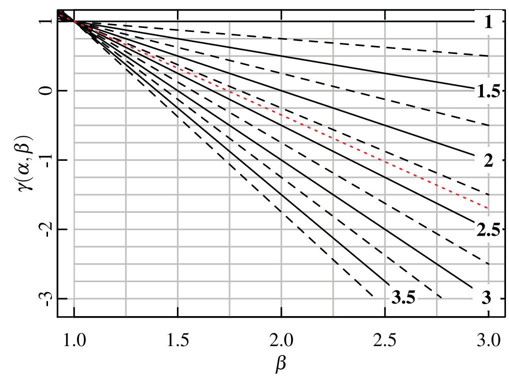

Figure 4. $\alpha\beta\gamma$ plot, showing the value of the low-mass exponent ($p(m) \propto m^{-\gamma}$, Table 1, equation 5) as a function of α and β. The lines solid for integer α and $\alpha + 1/2$ and dashed for $\alpha + 1/4$ and $\alpha + 3/4$. The red dotted line is for $\alpha = 2.35$.

4.2 Breakpoints

Related to the low- and high-mass exponents is the question of the 'breakpoints' in the IMF. As for the L_3 (and the B_4) IMF there is a smooth transition between the exponents, proper breakpoints do not exist. Nevertheless, it is useful to know from where the L_3 IMF can be approximated by a power law. Our approach to find the breakpoints is via the exponent as a function of mass (defined in equation (5), given for the L_3 IMF in equation 8, Table 1) For the L_3 IMF the curve of the exponent versus $\log m$ is 'S'-shaped, see the black solid line in Fig. 5. This 'S' shape can be approximated by three straight lines (red in Fig. 5), of which two are horizontal at γ and α. The intermediate, increasing part follows

$$g(m) = \left.\frac{\mathrm{d}S(m)}{\mathrm{d}\log m}\right|_{\mu} \log\left(\frac{m}{\mu}\right) + S(\mu), \tag{23}$$

a straight line in $\log m$. We define now the breakpoints, m_γ and m_α, as the points where $g(m_\gamma) = \gamma$ and $g(m_\alpha) = \alpha$. Formulae are given in Table 1, equations 6 and 7. The agreement of the L_3 IMF and the power-law segments below m_γ and above m_α is good, as can be seen in Fig. 3, where the power-law segments are shown as red lines, which are in fact barely visible.

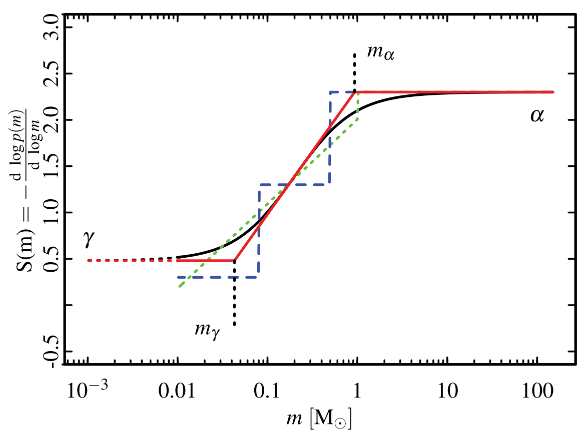

Figure 5. ('Alpha plot') Exponent of the L_3 IMF (black solid curve) and its approximation in log space (red solid lines). The points m_α and m_γ are defined as the intersection of the straight line approximation of the exponent at μ with the limiting exponents γ and α. For comparison the Kroupa (2001, 2002) IMF (blue dashed line) and the Chabrier (2003a) IMF (green dotted line) are given as well.

4.3 Characteristic masses

Characteristic mass scales of the L_3 IMF are also shown in Fig. 3 and given in Table 1. Because the IMF is skewed, the mean, median (equation 9, Table 1) and mode (most probable value, equation 10, Table 1) are all different. Also, note that μ is not directly related to any of them, it is the inflexion point of the exponent. Calculating the mean of the L_3 IMF involves incomplete beta functions[2] ($B(x; p, q) = \int_0^x t^{p-1}(1-t)^{q-1}\mathrm{d}t$). Using the transformation

$$t(m) = \frac{\left(\frac{m}{\mu}\right)^{1-\alpha}}{1 + \left(\frac{m}{\mu}\right)^{1-\alpha}} \tag{24}$$

the mean can be expressed as

$$E(m) = \mu(1-\beta)\frac{B\left(t(m_u); a, b\right) - B\left(t(m_l); a, b\right)}{G(m_u) - G(m_l)}, \tag{25}$$

where $a = \frac{2-\alpha}{1-\alpha}$ and $b = \beta - \frac{2-\alpha}{1-\alpha}$ and $G(m)$ is the auxiliary function given in Table 1, equation 1.

The 'peak' of the IMF refers to the maximum in the logarithmic description. The also very simple formula for m_{P} is given in equation 11, Table 1).

5 'CANONICAL' PARAMETERS FOR THE L_3 IMF

Observationally, the shape of the IMF is constrained mainly by the number ratios of different mass ranges to each other, for example the ratio of high-mass to low-mass stars. Thus, a first approach to find the 'canonical' parameters for the L_3 IMF could be a fit to the cumulative distributions of the Kroupa or Chabrier IMF. This could be done in some objective way, for example by matching histograms of L_3 to Kroupa or Chabrier. However, there are more properties that a 'canonically' parametrized IMF should fulfil: not only the number ratios, but also the mass ratios, the shape and the exponent should agree with each other. We could not find an 'objective' procedure that would fit these constraints such that for all of them the fit is good, the high-mass power-law tail leads to problems. Therefore, we choose the parameters 'by hand' for an optimal agreement of the L_3 with Kroupa and Chabrier in all the criteria.

For observational data objective fits are, of course, possible, for example with the maximum likelihood method. There not only the upper mass exponent and the lower mass exponent, but also the scale parameter μ can be estimated. This is an advantage compared to the piece-wise-defined IMFs, where typically the 'breakpoints' are not estimated. It is also possible to estimate the limits, in particular m_u, which can also vary between star-forming regions (cf. e.g. Weidner & Kroupa 2006, Maschberger & Clarke 2008, or Weidner, Kroupa & Bonnell 2010 for an observational perspective and Maschberger et al. 2010 for a varying m_u in simulations).

In order to normalize the IMFs to be able to find the 'canonical' we choose $m_l = 0.01\,\mathrm{M}_\odot$, near the deuterium burning limit. We set $m_u = 150\,\mathrm{M}_\odot$, as this is commonly assumed (cf. Weidner & Kroupa 2004; Figer 2005; Oey & Clarke 2005), but are aware that in some star-forming regions m_u can be at much higher masses (Crowther et al. 2010). As m_u lies well in the power-law tail, the exact value of it does not affect the parameter determination. α, β and μ are mainly constrained by the behaviour of the IMF below m_α.

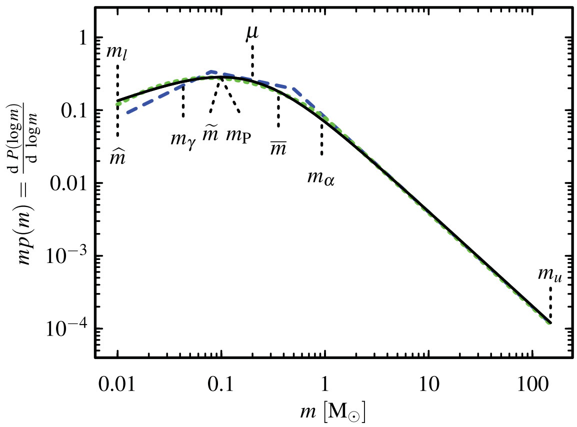

Figure 6. Probability density function for the L_3 IMF, like Fig. 3 but using the logarithmic description, shown together with the Kroupa (2001, 2002) IMF (dashed blue line) and the Chabrier (2003a) IMF (green dotted line).

5.1 L_3 single star IMF

In Fig. 6, we show in the logarithmic description the L_3 IMF with parameters chosen such that it fits the 'canonical' single-star IMF ($\alpha = 2.3$, $\beta = 1.4$ and $\mu = 0.2\,\mathrm{M}_\odot$). For comparison, we also show the Kroupa (2001, 2002) IMF and the Chabrier (2003a) IMF, both also normalized as probabilities. The difference between L_3 and Chabrier (2003a) is marginal, between L_3 and Kroupa (2001, 2002) equal to the difference between Kroupa (2001, 2002) and Chabrier (2003a). The effective low-mass exponent is $\gamma = 0.48$ for $m < m_\gamma = 0.042\,\mathrm{M}_\odot$. The high-mass break occurs at $m_\alpha = 0.93\,\mathrm{M}_\odot$, comparable to the start of the high-mass power law of Chabrier (2003a) at $1\,\mathrm{M}_\odot$. In the Kroupa (2001, 2002) IMF, the high-mass power law continues to $0.5\,\mathrm{M}_\odot$.

A comparison of the number fraction of stars in several mass bins is shown in the top panel of Fig. 7. The agreement between L_3 and Chabrier (2003a) is again very good. The fraction of stars in the mass range 0.6–$2\,\mathrm{M}_\odot$ is slightly smaller for L_3, because of the smooth transition to the high-mass power law. There are differences between L_3 and the Kroupa (2001, 2002) IMF at 0.3–1 and at $0.01 - 0.3\,\mathrm{M}_\odot$, caused by the segments in the Kroupa form. The lower panel of Fig. 7 shows the fraction of total mass in the mass bins,

$$\text{per cent } m = 100\frac{\int_{m_a}^{m_b} mp(m)\mathrm{d}m}{\int_{m_l}^{m_u} mp(m)\mathrm{d}m} \tag{26}$$

(m_a and m_b being the bin limits). L_3 again agrees very well with Chabrier (2003a) and well with Kroupa (2001, 2002).

As a last point, we compare the exponent of the L_3 IMF with Kroupa (2001, 2002) and Chabrier (2003a), see Fig. 5. Interestingly, although the pdf, the cumulative distribution function (fraction of stars, top panel of Fig. 7) and the mass distribution function (fraction of mass, bottom panel of Fig. 7) of the L_3 IMF agree more with a Chabrier (2003a), the exponent of the L_3 IMF follows more closely the Kroupa (2001, 2002) IMF.

[2] The incomplete beta function is available in many scripting languages for data processing [R (open source), IDL etc.] or via Numerical Recipes (Press et al. 2007). Sometimes what is called 'incomplete beta function' is actually the regularized incomplete beta function, $I_x(p, q) = B(x; p, q)/B(p, q)$. This is the case for the functions PBETA in R and IBETA in IDL.

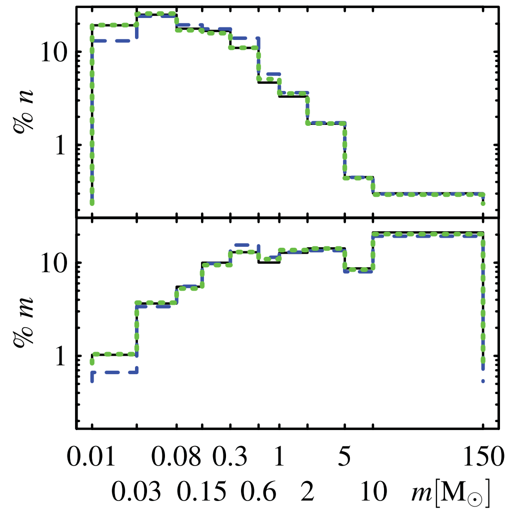

Figure 7. Comparison of the fractions of number (upper panel) and fractions of mass (lower panel) for the L_3 IMF (solid), Kroupa (2001, 2002) IMF (blue dashed) and Chabrier (2003a) IMF (green dotted). The L_3 IMF agrees better with the Chabrier (2003a) IMF, except for the mass range around 1 $\mathrm{M}_\odot$, where the power law is mounted on to the log-normal in the Chabrier (2003a) IMF.

5.2 L_3 system IMF

The system IMF for $m < 1\,\mathrm{M}_\odot$ has been given by Chabrier (2003a),

$$p_{\text{Chabrier 2003, System}}(m) = A0.086\frac{1}{m}\mathrm{e}^{-\frac{1}{2}\left(\frac{\log_{10}m-\log_{10}0.22}{0.57}\right)^2}, \tag{27}$$

and, with slightly modified parameters by Chabrier (2005),

$$p_{\text{Chabrier 2005, System}}(m) = A0.076\frac{1}{m}\mathrm{e}^{-\frac{1}{2}\left(\frac{\log_{10}m-\log_{10}0.25}{0.55}\right)^2}, \tag{28}$$

where A is a normalization constant. Above $1\,\mathrm{M}_\odot$ the system IMF follows a power law with exponent 2.35 both in Chabrier (2003a) and Chabrier (2005). We adopt the Chabrier (2003a) form for $m < 1\,\mathrm{M}_\odot$ and a power law with exponent 2.3

The best parameters for L_3 to fit the Chabrier (2003a) *system* IMF are $\alpha = 2.3$, $\beta = 2$ and $\mu = 0.2\,\mathrm{M}_\odot$, taking $m_l = 0.01\,\mathrm{M}_\odot$ and $m_u = 150\,\mathrm{M}_\odot$. A graph of both IMFs in the logarithmic description is given in Fig. 8, where very good agreement is achieved.

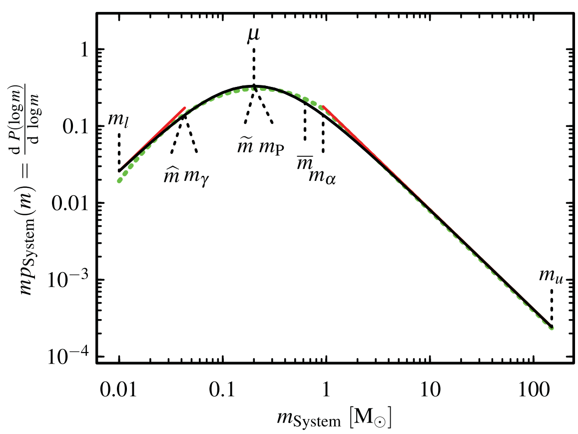

Figure 8. Comparison of the L_3 system mass function (solid) and the Chabrier (2003a) system mass function (green dotted) in the logarithmic description.

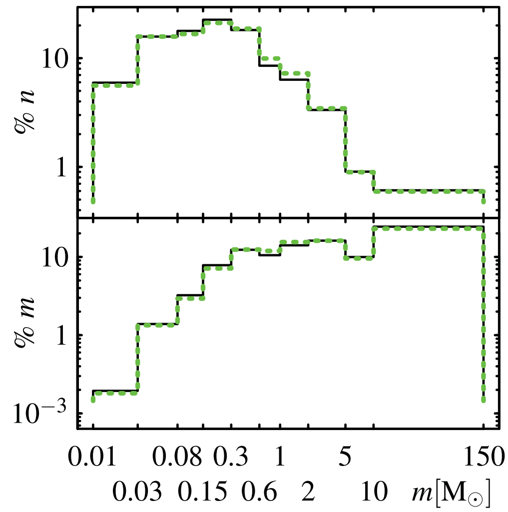

Figure 9. Comparison of the fractions of number (upper panel) and fractions of mass (lower panel) for the L_3 system IMF (solid) and Chabrier (2003a) system IMF (green dotted). As for the single-star IMF (Fig. 7) the log-normal-power law transition of the Chabrier (2003a) IMF around 1 $\mathrm{M}_\odot$ cannot be fitted exactly.

The effective low-mass exponent is then $\gamma = -0.3$ with breakpoint $m_\gamma = 0.043\,\mathrm{M}_\odot$ and high-mass breakpoint at $m_\alpha = 0.93\,\mathrm{M}_\odot$. The mean mass is $0.62\,\mathrm{M}_\odot$ which compares well with the $0.64\,\mathrm{M}_\odot$ for the Chabrier (2003a) system IMF. The mass for the median ($0.21\,\mathrm{M}_\odot$), the 'peak' ($0.20\,\mathrm{M}_\odot$) and the mass scale parameter ($\mu = 0.20\,\mathrm{M}_\odot$), by chance, coincide. Another coincidence is the near-equality of m_γ and the mode ($\widehat{m} = 0.042\,\mathrm{M}_\odot$).

As for the single-star IMF, the fraction of stars and the fractions of mass over the range of mass bins is very comparable for the L_3 system IMF and the Chabrier (2003a) system IMF (Fig. 9).

6 SUMMARY

The L_3 IMF, a functional form of the IMF generalizing the log-logistic distribution, describes the whole stellar mass range with a minimum number of parameters (three shape, two limits, see Table 1 that collects all formulae). It consists of a low-mass and a high-mass power law that are joined smoothly together. Due to its analytical simplicity many characteristic quantities (e.g. peak and mass breaks) can be given explicitly. The cumulative distribution function is analytically invertible, so that drawing random masses from the L_3 IMF is also very simple and does not involve a large programming effort.

We have determined the parameters that fit the L_3 IMF to the widely used single-star IMFs of Kroupa (2001, 2002) and Chabrier (2003a) and the system IMF of Chabrier (2003a). The L_3 IMF follows these IMFs very well, obtaining the same number and mass fractions of various mass ranges, so that it is a viable alternative functional form.

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REFERENCES

Bastian N., Covey K. R., Meyer M. R., 2010, ARA&A, 48, 339
Cartwright A., Whitworth A. P., 2012, MNRAS, 3039
Chabrier G., 2001, ApJ, 554, 1274
Chabrier G., 2003a, PASP, 115, 763
Chabrier G., 2003b, ApJ, 586, L133
Chabrier G., 2005, in Corbelli E., Palla F., Zinnecker H., eds, Astrophysics and Space Science Library, Vol. 327, The Initial Mass Function 50 Years Later. Springer, Dordrecht, p. 41
Crowther P. A., Schnurr O., Hirschi R., Yusof N., Parker R. J., Goodwin S. P., Kassim H. A., 2010, MNRAS, 408, 731
De Marchi G., Paresce F., Portegies Zwart S., 2010, ApJ, 718, 105
Elmegreen B. G., 2009, in Sheth K., Noriega-Crespo A., Ingalls J., Paladini R., eds, The Evolving ISM in the Milky Way and Nearby Galaxies. Available at: http://ssc.spitzer.caltech.edu/mtgs/ismevol
Elmegreen B. G., Mathieu R. D., 1983, MNRAS, 203, 305
Feller W., 1968, An Introduction to Probability Theory and Its Applications, Vol. 2, 3rd edn. Wiley, New York
Figer D. F., 2005, Nat, 434, 192
Hollenbach D., Parravano A., McKee C. F., 2005, in Corbelli E., Palla F., Zinnecker H., eds, Astrophysics and Space Science Library, Vol. 327, The Initial Mass Function 50 Years Later. Springer, Dordrecht, p. 417
Johnson N. L., Kotz S., Balakrishnan N., 1994, Continuous Univariate Distributions, Vol. 1, 2nd edn. Wiley, New York
Johnson N. L., Kotz S., Balakrishnan N., 1995, Continuous Univariate Distributions, Vol. 2, 2nd edn. Wiley, New York
Kroupa P., 2001, MNRAS, 322, 231
Kroupa P., 2002, Sci, 295, 82
Kroupa P., Weidner C., Pflamm-Altenburg J., Thies I., Dabringhausen J., Marks M., Maschberger T., 2011, in press (arXiv:1112.3340)
Larson R. B., 1998, MNRAS, 301, 569
Maschberger T., Clarke C. J., 2008, MNRAS, 391, 711
Maschberger T., Clarke C. J., Bonnell I. A., Kroupa P., 2010, MNRAS, 404, 1061
Miller G. E., Scalo J. M., 1979, ApJS, 41, 513
Oey M. S., Clarke C. J., 2005, ApJ, 620, L43
Paresce F., de Marchi G., 2000, in Favata F., Kaas A., Wilson A., eds, ESA SP-445: Star Formation from the Small to the Large Scale. ESA, Noordwijk, p. 279
Parravano A., McKee C. F., Hollenbach D. J., 2011, ApJ, 726, 27
Pflamm-Altenburg J., Kroupa P., 2006, MNRAS, 373, 295
Press W. H., Teukolsky S. A., Vetterling W. T., Flannery B. P., 2007, Numerical Recipes. Cambridge Univ. Press, Cambridge
Salpeter E. E., 1955, ApJ, 121, 161
Scalo J. M., 1986, Fundam. Cosm. Phys., 11, 1
Thies I., Kroupa P., 2007, ApJ, 671, 767
Thies I., Kroupa P., 2008, MNRAS, 390, 1200
Thorin O., 1977, Scand. Actuarial J., 1977, 121
Weidner C., Kroupa P., 2004, MNRAS, 348, 187
Weidner C., Kroupa P., 2006, MNRAS, 365, 1333
Weidner C., Kroupa P., Bonnell I. A. D., 2010, MNRAS, 401, 275
Zinnecker H., 1984, MNRAS, 210, 43
Zinnecker H., Yorke H. W., 2007, ARA&A, 45, 481

APPENDIX A: THE B_4 IMF

In some cases it might be necessary to include explicitly the width of the IMF in its functional form. For this an additional parameter has to be introduced, so that the total number of parameters is 4+2, two exponents, one location and one scale parameter, plus the obligatory upper and lower mass limit. The L_3 IMF can be extended to include this additional parameter, the cumulative distribution function then contains then beta functions, thus the name B_4 IMF. We give the formulae for the B_4 IMF in Table A1. The 'canonical' parameters were determined 'manually', as for the L_3 IMF, and the agreement with the IMFs for single stars (Kroupa 2001, 2002; Chabrier 2003a) and the IMF for systems (Chabrier 2003a) is comparably good. Figs A1 and A2 show this in the logarithmic description.

Table A1. Collection of formulae for the B_4 form of the IMF. The values given for the parameter are to match the 'canonical' single-star IMF (Kroupa 2001, 2002; Chabrier 2003a), values in parentheses for the 'canonical' system (binary star) IMF (Chabrier 2003a). $B(t; p, q)$ is the incomplete beta function.

(1)	Auxilliary function:	$\widetilde{B}_i(m) = B\left(t(m); \sigma(\beta+i), \sigma(\alpha-i)\right)$	$t(m) = \frac{\left(\frac{m}{\mu}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{m}{\mu}\right)^{\frac{1}{\sigma}}}$
	Quantity	Formula	Definition
	Functional form		
(2)	Cumulative distribution function (CDF)	$P_{B4}(m) = \frac{\widetilde{B}_1(m) - \widetilde{B}_1(m_l)}{\widetilde{B}_1(m_u) - \widetilde{B}_1(m_l)}$	
(3)	Probability density function (pdf)	$p_{B4}(m) = A \frac{\left(\frac{x}{\mu}\right)^{\beta}}{\left(1+\left(\frac{x}{\mu}\right)^{\frac{1}{\sigma}}\right)^{\sigma(\alpha+\beta)}}$	
		$A = \frac{1}{\sigma\mu} \frac{1}{\widetilde{B}_1(m_u) - \widetilde{B}_1(m_l)}$	
	Parameters		
	High-mass exponent	$\alpha = 2.3\,(2.3)$	$\alpha > 1$
	Low-mass exponent	$\beta = -0.15\ (0.4)$	$\beta > -1$
	Location Parameter	$\mu = 0.15\ (0.20)\,\mathrm{M}_{\odot}$	$\mu > 0$
	Scale parameter	$\sigma = 0.85\ (0.80)$	$\sigma > 0$
	Lower mass limit	$m_l = 0.01\,\mathrm{M}_{\odot}$	$m_l > 0$
	Upper mass limit	$m_u = 150\,\mathrm{M}_{\odot}$	$m_u > 0$
	Shape characterizing quantities		

Table A1 – *continued*

(4)	Lower power-law mass limit	$m_\beta = \mu \mathrm{e}^{-2\sigma}$	$p(m \in [m_l \dots m_\beta]) \approx m^\beta$
(5)	Upper power-law mass limit	$m_\alpha = \mu \mathrm{e}^{2\sigma}$	$p(m \in [m_\alpha \dots m_u]) \approx m^{-\alpha}$
(6)	Slope (NB: $S(\infty) = +\alpha$ (= 2.35))	$S(m) = -\beta + (\alpha + \beta)\dfrac{\left(\frac{x}{\mu}\right)^{\frac{1}{\sigma}}}{1 + \left(\frac{x}{\mu}\right)^{\frac{1}{\sigma}}}$	$S(m) = -\dfrac{\mathrm{d}\log p(m)}{\mathrm{d}\log m}$
	Scale characterizing quantities		
(7)	Mean mass (expectation value)	$E(m)\,(=\overline{m}) = \mu \dfrac{\widetilde{B}_2(m_u) - \widetilde{B}_2(m_l)}{\widetilde{B}_1(m_u) - \widetilde{B}_1(m_l)}$	$E(m) = \displaystyle\int_{m_l}^{m_u} m p(m) \mathrm{d}m$
(8)	Median mass	$\widetilde{m} = P^{-1}\left(\frac{1}{2}\right)$ (no closed form)	$P(\widetilde{m}) = \dfrac{1}{2}$
(9)	Mode (most probable mass)	$\widehat{m} = \mu \left(\frac{\beta}{\alpha}\right)^\sigma$ $(\beta > 0)$ or $\widehat{m} = m_l$ $(\beta < 0)$	$\widehat{m} = \arg\max_m \, p(m)$
(10)	'Peak' (maximum in log–log)	$m_P = \mu \left(\dfrac{\beta + 1}{\alpha - 1}\right)^\sigma$	$\dfrac{\mathrm{d}\log(m p(m))}{\mathrm{d}\log m} = 0$

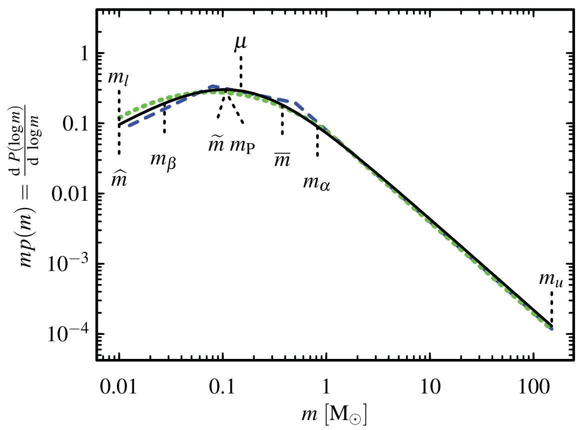

Figure A1. Logarithmic description of the B_4 form of the IMF, like Fig. 6 for L_3. The blue dashed curve is the Kroupa (2001, 2002) IMF and the green dotted curve is the Chabrier (2003a) IMF for comparison.

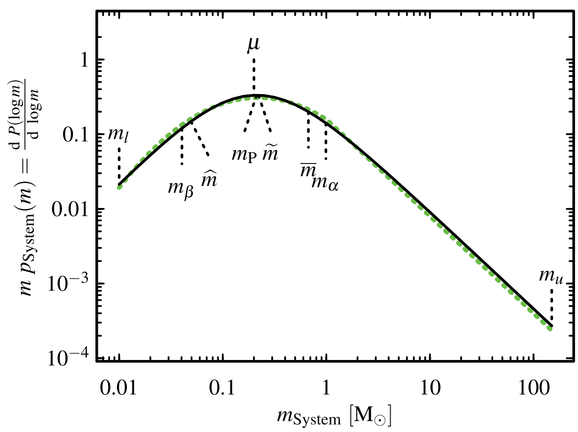

Figure A2. Comparison of the B_4 system mass function (solid) and the Chabrier (2003a) system mass function (green dashed) in the logarithmic description, like Fig. 8.

SUPPORTING INFORMATION

Additional Supporting Information may be found in the online version of this article:

R code for the functions given in this paper (http://mnras.oxfordjournals.org/lookup/suppl/doi:10.1093/mnras/sts479/-/DC1).

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