Resonant oscillations of GeV–TeV neutrinos in internal shocks from gamma-ray burst jets inside stars

Nissim Fraija⋆†

Instituto de Astronomía, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., A. Postal 70-264, 04510 México D.F., Mexico

Accepted 2015 April 1. Received 2015 March 4; in original form 2014 July 13

ABSTRACT

High-energy neutrinos generated in collimated jets inside the progenitors of gamma-ray bursts (GRBs) have been related to the events detected by IceCube. These neutrinos, produced by hadronic interactions of Fermi-accelerated protons with thermal photons and hadrons in internal shocks, are the only signature when the jet has not broken out or failed. Taking into account that the photon field is thermalized at keV energies and the standard assumption that the magnetic field maintains a steady value throughout the shock region (with a width of $10^{10}–10^{11}$ cm in the observed frame), we study the effect of thermal and magnetized plasma generated in internal shocks on the neutrino oscillations. We calculate the neutrino effective potential generated by this plasma, the effects of the envelope of the star, and the vacuum on the path to Earth. By considering these three effects, the two (solar, atmospheric and accelerator parameters) and three neutrino mixing, we show that although GeV–TeV neutrinos can oscillate resonantly from one flavour to another, a non-significant deviation of the standard flavour ratio (1:1:1) could be expected on Earth.

Key words: magnetic fields – neutrinos – plasmas – gamma-ray burst: general – stars: interiors – stars: jets.

1 INTRODUCTION

Long gamma-ray bursts (lGRBs) have been associated with the core collapse of massive stars leading to supernovae (CCSNe) of Type Ib, Ic and II. Type Ic SNe are believed to be He stars with radius $R_c \approx 10^{11}$ cm, and Type II and Ib SNe are thought to have a radius of $R_c \approx 3 \times 10^{12}$ cm. Depending on the luminosities and durations, successful lGRBs have revealed a variety of GRB populations: low-luminosity (ll), ultralong (ul) and high-luminosity (hl) GRBs (Mészáros & Waxman 2001; Liang et al. 2007; Gendre et al. 2013). While lGRBs and ulGRBs have a typical duration of $(\sim 10^4–10^5)$ s, hlGRBs have a duration of tens to hundreds of seconds. Another important population associated with CCSNe, although unobservable in photons, are failed GRBs, which could be much more frequent than successful GRBs, limited only by the ratio of Type Ib/c and Type II SNe to GRB rates. This population has been characterized by having high luminosities, mildly relativistic jets and durations from several to tens of seconds (Mészáros & Waxman 2001; Huang, Dai & Lu 2002; Soderberg et al. 2006, 2010).

Neutrinos are useful for studying the insides of stars, especially where photons cannot be observed either because the jet fails or has not broken out yet, so in this case, they could be the only signature that would display the dynamics of the star. High-energy (HE) neutrinos from this population of stars have been found to contribute significantly to the extragalactic neutrino background (ENB; Taboada 2010; Hümmer, Baerwald & Winter 2012; Murase & Ioka 2013; Murase, Kashiyama & Mészáros 2013; Razzaque 2013; Waxman 2013; Fraija 2014a; Murase, Inoue & Dermer 2014)and to explain the recent detections of TeV–PeV neutrinos by IceCube (Aartsen et al. 2013, 2014).

Measurements of HE neutrino properties such as flavour content would be involved with new physics if a deviation of the standard flavour ratio were observed (Learned & Pakvasa 1995; Athar, Jeżabek & Yasuda 2000; Kashti & Waxman 2005; Mena, Palomares-Ruiz & Vincent 2014). The neutrino flavour ratio is expected to be at the source, $\phi_{\nu_e}^{0}:\phi_{\nu_{\mu}}^{0}:\phi_{\nu_{\tau}}^{0} = 1:2:0$ and on Earth (due to neutrino oscillations between the source and Earth) $\phi_{\nu_e}^{0} : \phi_{\nu_{\mu}}^{0} : \phi_{\nu_{\tau}}^{0} = 1:1:1$ and $\phi_{\nu_e}^{0} : \phi_{\nu_{\mu}}^{0} : \phi_{\nu_{\tau}}^{0} = 1:8:1.8$ for neutrino energies less than and greater than 300 TeV, respectively (Kashti & Waxman 2005). Also, the measurement of a non-zero $\theta_{13}$ mixing angle coming from astrophysical sources could be relevant to clarify the neutrino mass hierarchy as well as CP violation searches in neutrino oscillations (Nunokawa, Parke & Valle 2008; Bandyopadhyay et al. 2009; Forero, Tórtola & Valle 2012).

As known, neutrino properties are modified when they propagate in a thermal and magnetized medium. A massless neutrino acquires an effective mass and an effective potential. The resonant conversion of an active neutrino from one flavour to another ($\nu_e \leftrightarrow \nu_{\mu}$, $\nu_\tau$) due to thermal and magnetized medium has been explored in many...
astrophysical contexts and has had relevant consequences in their dynamics (Wolfenstein 1978a; Goodman, Dar & Nussinov 1987; Nötzold & Raffelt 1988; Enqvist, Kainulainen & Maalampi 1991; D’Olivo, Nieves & Torres 1992; D’Olivo & Nieves 1994, 1996a,b; Ruffert & Janka 1999; Erdas & Isola 2000; Volkas & Wong 2000; D’Olivo, Nieves & Sahu 2003; Dasgupta et al. 2008). For instance, Fraija (2014b) has shown that the effect of a magnetic field in the dynamics of the fireball evolution of GRBs was to decrease the proton-to-neutron ratio aside from the number of multi-GeV neutrinos expected in a neutrino detector.

Neutrino oscillations in vacuum and by matter effects in the failed GRB framework (along the jet and envelope of the star) have been examined by many authors (Mena, Mocioiu & Razzaque 2007; Razzaque & Smirnov 2010; Sahu & Zhang 2010; Osorio Oliveros, Sahu & Sanabria 2013; Fraija 2014a). Although these authors have studied the oscillations on the surface of the star due to its envelope, the effect of thermal and magnetic field plasma generated on internal shocks has not been explored. In this paper, we calculate the effect of the magnetized and thermal shocked plasma on neutrino oscillations and then we estimate the flavour ratio on Earth. The organization of the paper is as follows. In Section 2, we show a brief description of internal shocks. In Section 3, we derive, first, the neutrino effective potential for $m_W \lesssim E_\nu$, as a function of the magnetic field, temperature, angle (between the neutrino propagation and magnetic field) and chemical potential and, secondly, the neutrino effective potential produced by the envelope of the star. In Section 4, we derive the resonance condition, the flip probability for two- and three-neutrino mixing and the flavour ratio expected on Earth. In Section 5, we discuss our results.

Hereafter, we use $Q_t = Q/10^5$ in cgs units and $k = h = c = 1$ in natural units.

**2 DESCRIPTION OF INTERNAL SHOCKS**

One of the most prosperous theories to explain the prompt emission and the afterglow in successful GRBs is the fireball model (Zhang & Mészáros 2004; Mészáros 2006). A GRB is considered successful when the jet drills inside the progenitor and breaks through the stellar envelope; otherwise it is taken to be a failed GRB. When the jet encounters the stellar envelope, two shocks are involved: an outgoing, or forward, shock (Paczynski &Rhoads 1993; Rees & Mészáros 1994), and another that propagates back decelerating the ejecta, that is, the reverse shock (Mészáros & Rees 1994; Rees & Mészáros 1994). The jet dynamics is mainly dominated by the jet head, which is controlled by the ram pressure balance between the reverse and forward shocks. If the luminosity ($L_j$) is low enough and/or the density of the stellar envelope is high enough, then the hydrodynamic jet is collimated and internal shocks might occur inside the progenitor (Bromberg et al. 2011; Mizuta & Ioka 2013; Murase & Ioka 2013). In this model, inhomogeneities in the jet lead to internal shell collisions, higher shells ($\Gamma_{\mu}$) catching slower shells ($\Gamma_j$). The kinetic energy of ejecta is partially dissipated via these internal shocks, which take place at a distance of $r_j = 2\Gamma^2 t_j < R_c$, where $t_j$ is the variability time-scale of the central object, $\Gamma_j = \sqrt{\Gamma_j^2 \Gamma_{\mu}}$ is the bulk Lorentz factor of the propagating shock and $R_c$ is the radius of the progenitor’s stellar surface. The constraint $r_j < R_c$ gives rise to those shocks inside the star. The physical width of the internal shock is lower by a factor of $\Gamma_j$ (i.e. $\Delta r_j = \Gamma t_j$). These internal shocks are expected to be collisionless, so that particles may be accelerated. In internal shocks, the total energy density $U = 1/(8\pi m_p)\Gamma^{-1} L_j t_j^{-2}$ is equipartitioned to generate and/or amplify the magnetic field $\epsilon_B = U_{\mu}/U = (B^2/8\pi U)$ (Piran 2005) and to accelerate particles $\epsilon = U_{\mu}/U$, where $m_p$ is the proton mass. Then, the magnetic field generated at the shocks is written as

$$B' = \epsilon_B^{1/2} \Gamma^{-2} L_j^{1/2} t_j^{-1}. \quad (1)$$

It is important to say that the strength of the magnetic field falls out of the shocked region achieving some Gauss and, although its direction might be random, it is mostly transverse to the jet direction (Razzaque & Smirnov 2010). From the causality condition, the coherence length of such a magnetic field is only the order of $L_j \sim t_j$. However, electrons are accelerated up to ultrarelativistic energies and then are cooled down rapidly in the presence of the magnetic field, producing the prompt emission by synchrotron radiation. The opacity to Thomson scattering is $\tau_{th} = [\sigma_{th}/(4\pi m_p)] L_j t_j^{-1}$ and

**Figure 1.** W-exchange diagram of one-loop contribution to the neutrino self-energy in a magnetized medium. The dashed line represents the electron propagator $e^-(p)$, the solid line corresponds to the electron neutrino propagator $\nu_e(k)$ and the wiggly line is the W-boson propagator $W^+(q)$.

**Figure 2.** Contour lines of variability time-scale ($t_j$) and bulk Lorentz factor ($\Gamma_j$) as a function of the distance of the internal shocks ($x_j$) for which these shocks take place inside the progenitors. We have considered progenitors such as WR (left-hand panel) and BSG (right-hand panel) stars.
photons thermalize at a blackbody temperature with peak energy given by (Razzaque, Mészáros & Waxman 2004)

$$T'_\gamma \simeq \frac{1.2}{\pi} e^{1/4} L_j^{1/4} \Gamma^{-1} t_i^{-1/2},$$

where $\sigma_T$ is the Thompson cross-section.

Protons are also accelerated and cooled down in internal shocks via electromagnetic (synchrotron radiation and inverse Compton scattering) and hadronic (proton–photon and proton–proton interactions) channels. Proton–photon and proton–proton interactions take place when accelerated protons interact with thermal keV photons (equation 2) and proton density at the shock, $n_p = 1/(8\pi m_p) l^{-4} L / \Gamma_i$ (Mészáros & Waxman 2001). In both interactions, HE charged pions and kaons are produced: $p + \gamma \rightarrow X + \pi^\pm / K^\pm$, and subsequently neutrinos $\pi^+ \rightarrow \mu^+ + \nu_\mu \rightarrow e^+ + \bar{\nu}_e + \nu_\mu + \bar{\nu}_\mu$, and $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu \rightarrow e^- + \bar{\nu}_e + \nu_\mu + \bar{\nu}_\mu$. In this approach, the neutrino created by these processes will lie in the TeV–PeV energy range (Razzaque & Smirnov 2010; Murase & Ioka 2013; Fraija 2014a).

## 3 Neutrino Effective Potential

In this section, we compute the neutrino effective potential due to the magnetized and thermal shocked plasma, and the envelope of the star.

### 3.1 Magnetized and thermal plasma

Recently, Fraija (2014b) derived the neutrino self-energy and effective potential up to order $m_\nu^4$ at strong, moderate and weak magnetic field approximation as a function of temperature, chemical potential and neutrino energy for moving neutrinos along the magnetic field. In this subsection, we calculate the neutrino effective potential at the moderate and weak magnetic field limit for any direction of neutrino propagation. Therefore, following Fraija (2014b), we show the equations that are more relevant for deducing the neutrino effective potential.

The neutrino effective potential is calculated by means of the dispersion relation

$$V_{\text{eff}} = k_0 - |k|,$$

where $k$ is estimated through the neutrino field equation in a medium (Nötzold & Raffelt 1988; Enqvist et al. 1991)

$$[\not k - \Sigma(k)]\psi_L = 0,$$

and $\Sigma(k) = R(a_1 \not k + a_2 \not k_L + b \not q + c \not p)L$ is obtained from the real part of its self-energy diagram. Here, $k_0^L = (k_0^L, k_3^L)$ and $k_0^W = (k_0^W, k_3^W)$ are the momentum along and perpendicular to the magnetic field, respectively, $u^\nu$ stands for the four-velocity of the centre-of-mass of the medium given by $u^\nu = (1, 0)$, $R = (1/2)(1 + \gamma_5)$ and
\( \mathcal{L} = \frac{1}{2}(1 - \gamma) \) are the projection operators and \( a, b \) and \( c \) are the Lorentz scalars, which are functions of neutrino energy, momentum and magnetic field. These scalars are calculated from the neutrino self-energy due to interactions through neutral and charged currents of neutrino with the background particles. The effect of the magnetic field is introduced through the four-vector \( b^\mu \), which is given by \( b^\mu = (0, \hat{b}) \) (Fraija 2014b). Using the Dirac algebra and from the dispersion relation (equation 3), the neutrino effective potential can be written as

\[
V_{\text{eff}} = b - c \cos \varphi - a \frac{1}{|k|} \sin^2 \varphi, \tag{5}
\]

where \( \varphi \) is the angle between the neutrino momentum and the magnetic field vector. Moreover, the effective potential that is applicable to the neutrino oscillations in matter is \( V_{\text{eff}} = V_e - V_\mu \), which depends only on electron density (Wolfenstein 1978b; D’Olivo et al. 1992). For that reason, although the one-loop neutrino self-energy comes from three parts, the W-exchange, Z-exchange and tadpole (Babaev 2004; Erdas, Kim & Lee 1998; Sahu, Fraija & Kueum 2009a,b), we only consider the neutrino effective potential due to charged currents \( \Sigma(k) = \Sigma_W(k) \). We use the finite temperature field theory formalism and the Schwinger proper time method to include the magnetic field (Schwinger 1951). From the W-exchange diagram (see Fig. 1), the effective energy can be explicitly written as

\[
- \frac{i}{2} \mathcal{R} \left[ \frac{g^2}{2} \int \frac{d^4p}{(2\pi)^4} \Sigma_{\gamma}(p) \gamma_i W^{\mu\nu}(q) \right] \mathcal{L}, \tag{6}
\]

where \( g^2 = 4\sqrt{2} G_F m_W^2 \) is the weak coupling constant, \( W^{\mu\nu} \) is the W-boson propagator that in unitary gauge can be written as (Erdas et al. 1998; Sahu et al. 2009b)

\[
W^{\mu\nu}(q) = \frac{g^{\mu\nu}}{m_W^2} \left( 1 + \frac{q^2}{m_W^2} \right) - \frac{q^\mu q^\nu}{m_W^2} + \frac{3ie}{2m_W^2} F^{\mu\nu}. \tag{7}
\]

Here, \( m_W \) is the W-boson mass, \( G_F \) is the Fermi coupling constant, \( g^{\mu\nu} \) is the metric tensor and \( F^{\mu\nu} \) is the electromagnetic field tensor. From equation (6), \( S_i(p) \) is the charged lepton propagator that is split into two propagators: one in the presence of a uniform background magnetic field \( S^0_i(p) \), and the other in a magnetized medium \( S^\beta_i(p) \). Then, it can be written as

\[
S_i(p) = S^0_i(p) + S^\beta_i(p). \tag{8}
\]
We can express the charged lepton propagator in the presence of a uniform background magnetic field as

\[
iS'_\ell(p) = \int_0^\infty e^{i\Phi(p,s)} G(p,s) \, ds, \tag{9}
\]

where the functions \( \Phi(p,s) \) and \( G(p,s) \) are written as

\[
\Phi(p,s) = is(p_0^2 - m_0^2) - is \left[ p_L^2 + \frac{\tan z}{\tau} p_\perp^2 \right],
\]

\[
G(p,s) = \sec^2 z [A + i B y] + m_\ell (\cos^2 z - i \sigma^3 \sin z \cos z), \tag{10}
\]

Here, \( m_\ell \) is the mass of the charged lepton, \( p_0^2 = p_0^2 - p_\perp^2 \), \( p_L^2 \) and \( p_\perp^2 \) are the projections of the momentum on the magnetic field direction and \( z = eBs \), where \( e \) is the magnitude of the electron charge. Additionally, the covariant vectors are given as follows:

\[
A_\mu = p_\mu - \sin^2 z (p \cdot u u_\mu - p \cdot b b_\mu), \quad B_\mu = \sin z \cos z (p \cdot u b_\mu - p \cdot b u_\mu) \quad \text{and} \quad \sigma_3 = \gamma_5 \gamma_4. \]

The other term in equation (8) (due to magnetized medium) is given by \((D’Olive & Nieves 1996a)\)

\[
S'_\ell(p) = i\eta(p \cdot u) \int_{-\infty}^\infty e^{i\Phi(p,s)} G(p,s) \, ds, \tag{11}
\]

where \( \eta(p \cdot u) \) contains the distribution functions of the particles in the medium, which are given by

\[
\eta(p \cdot u) = \frac{\theta(p \cdot u)}{e^{\beta(p \cdot u - \mu_\ell)} + 1} + \frac{\theta(-p \cdot u)}{e^{i\beta(p \cdot u - \mu_\ell)} + 1}. \tag{12}
\]

Here, \( \beta \) and \( \mu_\ell \) are the inverse of the medium temperature and the chemical potential of the charged lepton, respectively. By evaluating equation (6) explicitly, we obtain

\[
R e \Sigma(k) = R \{ a_{\perp} \xi + b \hat{\phi} + c \hat{\beta} \mathcal{L} \}, \tag{13}
\]

where the Lorentz scalars are given by \((Fraija 2014b)\)

\[
a_{\perp} = - \sqrt{2} G_p \left\{ E_{\eta} (n_e - \bar{n}_e) + \frac{E_\nu k_3}{m_\nu} (n_0^0 - \bar{n}_0^0) \right\} \nonumber \]

\[
\quad + \frac{eB}{2\pi^2} \int_0^\infty dp_3 \sum_{n=0}^\infty (2 - \delta_{n,0} (m_\nu^2/E_\nu - H/E_\nu) (f_{\bar{n},0} + \bar{f}_{\bar{n},0})). \nonumber \tag{14}
\]

\[
b = \sqrt{2} G_p \left\{ \left(1 + \frac{E_\nu}{m_\nu} \right) (n_e - \bar{n}_e) + \frac{E_\nu k_3}{m_\nu} (n_0^0 - \bar{n}_0^0) \right\} \nonumber \]

\[
\quad - \frac{eB}{\pi^2 m_\nu^2} \int_0^\infty dp_3 \sum_{n=0}^\infty (2 - \delta_{n,0}) E_\nu f_{\bar{n},0} \nonumber \times \left[ E_\nu \delta_{0,0} + \left( E_\nu - \frac{m_\nu^2}{2E_\nu} \right) (f_{\bar{n},0} + \bar{f}_{\bar{n},0}) \right] \tag{15}.
\]

Figure 5. Neutrino effective potentials at moderate (top) and weak (bottom) limits are plotted as a function of magnetic field for temperatures at keV energies (left) and different angles between the direction of neutrino propagation and magnetic field (right).
Figure 6. Contour lines of temperature and chemical potential as a function of neutrino energy for which the resonance condition is satisfied. We have used the neutrino effective potential at the moderate limit (equation 19) and the best-fitting values of the two-neutrino mixing (solar, top-left; atmospheric, top-right; accelerator, bottom-left) and three-neutrino mixing (bottom-right).

and

\[ c = \sqrt{2} G F \left( 1 - \frac{k_f^2}{m_W^2} \right) \left( n_0^2 - \bar{n}_e^2 \right) - \frac{E_{\nu e}^2}{m_W^2} (n_e - \bar{n}_e) \]

\[ - \frac{eB}{\pi m_W^2} \int_0^\infty dp_3 \sum_{n=0}^\infty (2 - \delta_{n,0}) \left[ \left( E_n - \frac{m_e^2}{E_n} \right) \delta_{n,0} + \left( E_n - \frac{3 m_e^2}{2 E_n} - \frac{H}{E_n} \right) \left( f_{e,n} + \bar{f}_{e,n} \right) \right] \]  

(16)

Here, the electron number density and electron distribution function are

\[ n_e(\mu, T, B) = \frac{eB}{2\pi^2} \sum_{n=0}^\infty (2 - \delta_{n,0}) \int_0^\infty \frac{dp_3}{e^{(E_n - \mu) / T} + 1} \]  

(17)

and

\[ f(E_{\nu e}, \mu) = \frac{1}{e^{(E_{\nu e} - \mu) / T} + 1} \]  

(18)

respectively, where \( f_{e,n}(\mu, T) = f_{e,n}(-\mu, T) \) and \( E_{\nu e} = \sqrt{p_3^2 + m_e^2 + H} \) with \( H = 2neB \). Solving the integral terms in equations (14), (15) and (16) and replacing them in equation (5), we calculate the neutrino effective potential for two cases: the moderate and the weak magnetic field limit.

3.1.1 Moderate magnetic field limit

In the moderate field approximation \( (B/B_c \leq 1) \), the Landau levels are discrete and can be described by sums \( \sum_n \) with \( n = 1, 2, 3, \ldots \).

In this regime, the neutrino effective potential is written as

\[ V_{\text{eff,mod}} = \sqrt{2} G F m_3^2 B \left[ \sum_{l=0}^\infty (-1)^l \sinh \alpha_l \left( F_m - G_m \cos \varphi \right) \right. \]

\[ - \left. 4 \frac{m_e^2}{m_W^2} E_{\nu e} \sum_{l=0}^\infty (-1)^l \cosh \alpha_l \left( J_m - H_m \cos \varphi \right) \right] \]  

(19)

where \( \alpha_l = \beta \mu (l + 1) \) and the functions \( F_m, G_m, J_m \) and \( H_m \) are given in Appendix . It is worth noting that as the magnetic field decreases, the effective potential will depend less on the Landau levels.

3.1.2 Weak magnetic field limit

In the weak field approximation \( (B/B_c \ll 1) \), all levels are full and overlap each other. In this regime, sums over the Landau levels
can be described and approximated by an integral $\sum_n \int \frac{d\mu}{n^2}$, and then the effective potential does not depend on the Landau levels. The potential in this regimen can be written as

$$V_{\text{eff,ss}} = \sqrt{2} G_F m_e \left[ \sum_{l=0}^{\infty} \frac{1}{(l+1)!} \sinh x_l (F_w - G_w \cos \varphi) - 4 \frac{m_e^2}{m_N^2} E_e \sum_{l=0}^{\infty} \frac{1}{(l+1)!} \cosh x_l (J_w - H_w \cos \varphi) \right], \quad (20)$$

where the functions $F_w, G_w, J_w$ and $H_w$ are shown in Appendix.

### 3.2 Density profiles of envelopes

Models of density distributions in CCSNe have been widely explored (Chevalier & Soker 1989; Bethe & Pizzochero 1990; Shigeyama & Nomoto 1990; Woosley, Langer & Weaver 1993). We use two models with density profiles $\rho \propto r^{-3}$ and $\rho \propto r^{-17/7}$. Explicitly, the first model corresponds to a polytropic hydrogen envelope

$$\rho_1(r) = 4.0 \times 10^{-6} \left( \frac{R_*}{r} - 1 \right)^3 \text{ g cm}^{-3}. \quad (21)$$

and the second model is a power-law fit with an effective polytropic index $n_{\text{eff}} = 17/7$ as done for SN 1987A (Chevalier & Soker 1989)

$$\rho_2(r) = 3.4 \times 10^{-5} \text{ g cm}^{-3} \times \begin{cases} \left( \frac{R_*}{r} \right)^{17/7} & 10^{10.8} \text{ cm} < r < r_b = 10^{12} \text{ cm}, \\ \left( \frac{r - R_b}{r_b} \right)^{17/7} & r > r_b. \end{cases} \quad (22)$$

In both cases, from the number density of electrons $N_e = N_a \rho(r) Y_e$, the neutrino effective potential can be written as

$$V_{\text{eff,ss}} = \sqrt{2} G_F N_e. \quad (23)$$
where \( N_a = 6.022 \times 10^{23} \text{ g}^{-1} \) is the Avogadro number, \( Y_e = 0.5 \) is the number of electrons associated per nucleon and \( \rho(r) \) is given by equations (21) and (22).

4 NEUTRINO RESONANT OSCILLATIONS

When neutrino oscillations take place in matter, a resonance could occur that would dramatically enhance the flavour mixing and could lead to a maximal conversion from one neutrino flavour to another. This resonance depends on the effective potential and neutrino oscillation parameters. The equation that determines the neutrino evolution in matter in the two- and three-flavour framework is (Fraija, Bernal & Hidalgo-Gaméz 2014)

\[
U \frac{1}{2E_\nu} M \cdot U^\dagger + \text{diag}(V_{\text{eff}} k, 0),
\]

where

\[
M = \begin{cases} 
(-\delta m^2_{12}, 0) & \text{for two flavours,} \\
(-\delta m^2_{21}, 0, \delta m^2_{31}) & \text{for three flavours,}
\end{cases}
\]

\( \delta m^2_{ij} \) is the mass difference (Giunti & Chung 2007), \( U \) is the two- and three-neutrino mixing matrix (see Appendix, equation B9), \( V_{\text{eff}} k \) is the neutrino effective potentials calculated in Section 3 (for \( k = \text{ss} \) and ss) and \( E_\nu \) is the neutrino energy. Hereafter, we use the first and second lines for two- and three-neutrino mixing, respectively.

Figure 8. The flip probability is plotted as a function of neutrino energy for a strength of magnetic field in the moderate \( (B = 10^{-4} B_c) \) and weak \( (B = 1 \text{ G}) \) regime and at a distance of \( 10^{10.8} \) and \( 10^{12} \) cm. We have used the best-fitting values of the two-neutrino mixing (solar, top-left; atmospheric, top-right; accelerator, bottom-left) and three-neutrino mixing (bottom-right).

Figure 9. Contour lines of distance and neutrino energy as a function of neutrino oscillation parameters for which the resonance condition is satisfied. We have used the neutrino effective potential generated by the envelope of the star (equation 23) and the best parameters of neutrino oscillation for solar, atmospheric, accelerator and three flavours.
Figure 10. Neutrino flavour ratio expected on Earth as a function of neutrino energy when these are created on the surface of a WR (at $10^{11}$ and $10^{10.8}$ cm for the upper and second rows, respectively) and a BSG (at $10^{12.3}$ and $10^{12}$ cm for the third and bottom rows, respectively). We have used the neutrino effective potential at the moderate (second and bottom rows) and weak (upper and third rows) field limit for $\theta_{13} = 11^\circ$ (left) and $\theta_{13} = 2^\circ$ (right).
GeV–TeV neutrino oscillations

Figure 11. Neutrino flavour ratio expected on Earth as a function of neutrino energy when these are created on the surface of a BSG. We have considered that internal shocks take place at $r_j = 10^{12}$ cm with a physical width $\Delta r_j = 2 \times 10^{11}$ cm and the magnetic field is oriented to different angles $0^\circ \leq \varphi \leq 75^\circ$ with respect to neutrino direction. We have used the neutrino effective potential at the moderate-field limit (equation 19) and $\theta_{13} = 2^\circ$.

as written in equation (25). From the conversion probabilities, we find that the oscillation lengths are

$$L_{\text{res}} = 4 \pi E_\nu \times \left\{ \begin{array}{ll}
\frac{1}{\sqrt{(2E_\nu V_{\text{eff},k} - \delta m^2 \cos 2\theta)^2 + (\delta m^2 \sin 2\theta)^2}}, \\
\frac{1}{\sqrt{(2E_\nu V_{\text{eff},k} - \delta m_{12}^2 \cos 2\theta_{13})^2 + (\delta m_{12}^2 \sin 2\theta_{13})^2}},
\end{array} \right.$$ 

with the resonance conditions

$$2 \times 10^6 E_\nu V_{\text{eff},k} = \left\{ \begin{array}{ll}
\delta m^2 \cos 2\theta, \\
\delta m_{12}^2 \cos 2\theta_{13},
\end{array} \right.$$ 

with $\gamma \gg 1$ or the flip probability given by

$$P_f = e^{-\pi/2\gamma}.$$ 

In addition to the resonance condition, the dynamics of the transition from one flavour to another must be determined by adiabatic conversion through the adiabaticity parameter (Mohapatra & Pal 2004)

$$\gamma = \frac{1}{2E_\nu} \left| \frac{1}{(1/V_{\text{eff},k})(dV_{\text{eff},k}/dr)_r} \right| \left\{ \begin{array}{ll}
\delta m^2 \sin 2\theta \tan 2\theta, \\
\delta m_{12}^2 \sin 2\theta_{13} \tan 2\theta_{13},
\end{array} \right.$$ 

By considering that the flux ratio of $\dot{N}_{\nu_\mu} \simeq \dot{N}_{\bar{\nu}_\mu} \simeq 2 \dot{N}_{\nu_e} \simeq 2 \dot{N}_{\bar{\nu}_e}$ is created in the internal shocks, neutrinos first oscillate in matter due to the magnetized and thermal plasma and, secondly, oscillate to the star envelope. In vacuum, after neutrinos have left the star, they start oscillating to the Earth. Hence, from these three effects (i.e.
internal shocks, the envelope of the star and vacuum), the flavour ratio expected on Earth will be
\[
\begin{pmatrix}
    \nu_e \\
    \nu_{\mu} \\
    \nu_{\tau}
\end{pmatrix}_{\text{Earth}} =
\begin{pmatrix}
P_{11}^* & P_{12}^* & P_{13}^* \\
P_{21}^* & P_{22}^* & P_{23}^* \\
P_{31}^* & P_{32}^* & P_{33}^*
\end{pmatrix}
\begin{pmatrix}
    1 \\
    2 \\
    0
\end{pmatrix}_c,
\]

where the probabilities $P_{ij}^*$ are derived in Appendix B.

The best-fitting values of the two neutrino mixing are: solar neutrinos, $\delta m^2 = (5.6^{+1.9}_{-1.0}) \times 10^{-5}$ eV$^2$ and $\tan^2 \theta = 0.427^{+0.033}_{-0.032}$ (Aharmim et al. 2013); atmospheric neutrinos, $\delta m^2 = (2.1^{+0.5}_{-0.2}) \times 10^{-3}$ eV$^2$ and $\sin^2 2\theta = 1.0^{+0.00}_{-0.07}$ (Abe et al. 2011); accelerator neutrinos 0.5 eV$^2$ and $\sin^2 \theta = 0.0049$ (Araki et al. 2005). Combining solar, atmospheric, reactor and accelerator parameters, the best-fitting values of the three neutrino mixings are: for $\sin^2 \theta_{13} < 0.053$, $\Delta m^2_{21} = (7.41^{+0.21}_{-0.19}) \times 10^{-5}$ eV$^2$ and $\tan^2 \theta_{12} = 0.446^{+0.030}_{-0.029}$ for $\sin^2 \theta_{13} < 0.04$, $\Delta m^2_{23} = (2.1^{+0.5}_{-0.2}) \times 10^{-3}$ eV$^2$ and $\sin^2 \theta_{23} = 0.50^{+0.063}_{-0.063}$ (Wendell et al. 2010; Aharmim et al. 2013).

5 RESULTS AND CONCLUSIONS

In this analysis, we have considered HE neutrinos created in the energy range of 100 GeV $\leq E_\nu \leq$ 100 TeV (Razzaque & Smirnov 2010; Murase & Ioka 2013; Fraija 2014a) and we have also assumed (in the CCSNe–GRB connection) progenitors such as Wolf–Rayet (WR) and blue supergiant (BSG) stars with radii $R = 10^{11}$ cm and $R = 3 \times 10^{12}$ cm, respectively, with formation of jets leading to internal shocks inside them.
In internal shocks, energy is equipartitioned to generate and/or amplify the magnetic field and to accelerate particles. Electrons and protons are expected to be accelerated in these shocks, and later to be cooled down by synchrotron radiation, inverse Compton and hadronic processes (pγ and p–hadron interactions). Photons produced by electron synchrotron radiation are thermalized at keV energies and serve as targets for production of HE neutrinos through K±, π± and μ± decay products in the proton–γ and proton–hadrons interactions. Therefore, this plasma is endowed with a magnetic field and made of protons, mesons, electrons, positrons, photons and neutrinos.

First, we consider those internal shocks that take place inside progenitors (rj < R⋆), as plotted in Fig. 2. In this figure, we show the contour lines of bulk Lorentz factors and variability timescales for different internal shock radii. In these plots, we observe that for a typical value of variability in the range of 10−3 s ≤ τv ≤ 1 s, the values of bulk Lorentz factors are Γ ≤ 34 for a WR (left-hand panel) and Γ ≤ 163 for a BSG (right-hand panel). Taking into account internal shocks at rj = 1010.8 cm (left-hand panel) and rj = 1015.2 cm (right-hand panel), we can see that the physical width of the internal shocks is restricted to ∆rJ ≤ 1.8 × 1010 cm and ∆rJ ≤ 2.5 × 1011 cm, respectively. Once we have obtained the values of Γ and τv for internal shocks inside the progenitor, we compute the range of values associated with the magnetic field and temperature of the plasma, as shown in Figs 3 and 4, respectively. We plot the contour lines of the magnetic fields (Fig. 3) and thermalized photons (Fig. 4) for values of luminosity in the range 1040 ≤ Lγ ≤ 1052 erg s−1. Colours in light- and dark-grey backgrounds represent the regions of a WR and a BSG, respectively.

In Figs 3 and 4, we can see that the values of magnetic field and thermalized photons lie in the ranges 1010 ≤ B ≤ 1014 G and 0.1 ≤ T ≤ 30 keV, respectively. It is important to clarify that the magnetic field amplified in the internal shocks falls out of them to the magnetic field endowed by the progenitor (black hole or magnetar) (Razzaque & Smirnov 2010).

Following Fraija (2014b) and taking into account the fact that the range of neutrino energy considered is larger than the W-boson mass (Eν ≤ mW), we have obtained the neutrino effective potential up to an order mW4 in the moderate regime below B/Bc ∼ 10−3 and the weak B/Bc ∼ 10−15 regime as a function of the observable quantities in the internal shocks: thermalized photons, magnetic field, neutrino energy and the angle between the direction of neutrino propagation and the magnetic field. We plot the neutrino effective potential in both limits (moderate and weak field limits) as shown in Fig. 5. The neutrino effective potential at moderate limit (upper figures) and weak limit (lower figures) are plotted for a magnetic field in the range of 10−6 Bc ≤ B < 10−4 Bc and 10−15 Bc ≤ B < 10−12 Bc, respectively. In both cases, we use the values of temperature T = (20, 24, 27, 30) keV, angle ϕ = (0°, 30°, 60°, 90°) and the neutrino energy Eν = 10 TeV. The neutrino effective potential at the weak limit is smaller than at the moderate limit. It is worth mentioning that in the range of the magnetic field considered, the contribution of Landau levels to the effective potential at the moderate-field limit is not significant due to ∑∞ l=−∞ δl Kν l (m/λν) ∼ 0 for λν = 0 l+1 B/Bc and δl = β_m (l+1). From these plots, we can observe that the neutrino effective potential is positive. Therefore, due to its positivity (Veff (r, ν) > 0) for k = m and w, neutrinos can oscillate resonantly. From the resonance condition (equation 27) and the neutrino effective potential at moderate (equation 19) and weak (equation 20) limit, we plot the contour lines of temperature and chemical potential as a function of neutrino energy for which the resonance condition is satisfied, as shown in Figs 6 and 7, respectively. From these figures, we can see that temperature is a decreasing function of chemical potential and neutrino energy. As neutrino energy increases, temperature decreases steadily. Considering the values of neutrino energy (Eν = 100 GeV, 500 GeV, 10 TeV and 100 TeV) and ϕ = 90°, we see that the temperature and chemical potential are in the range 10 to ∼100 keV and 60 eV to 50 keV, respectively. For instance, taking into account a neutrino energy of 10 TeV, from Fig. 6 we can see that the temperature lies in the range 22.2–15.3 keV for solar, 23.4–16.2 keV for atmospheric, 40–26.4 keV for accelerator and 28.3–15.4 keV for three-neutrino parameters. As shown in Fig. 7, temperature lies in the range 18.3–14.1 keV for solar, 19.8–15.1 keV for atmospheric, 32.1–21.9 keV for accelerator and 24.5–18.3 keV for three-neutrino parameters. In addition, we have obtained the resonance lengths, which are shown in Table 1. As shown in this table, the resonance lengths lie in the range lres ∼ (1010–1015) cm. Hence, depending on the progenitor associated and the oscillation parameters, neutrinos would leave the internal shock region in different flavours of 1: 2: 0. For instance, taking into account the parameters of three-neutrino mixing, neutrinos with energy less than 0.5 (10) TeV will oscillate resonantly with a resonance length equal or less than the radius of the progenitor, either a WR or BSG. Considering parameters of accelerator experiments, neutrino energy around 100 TeV will oscillate in a BSG star before leaving it.

As the dynamics of resonant transitions is not only determined by the resonance condition, but also by adiabatic conversion, we analyse the flip probability (equation 29) to find the regions for which neutrinos can oscillate resonantly. First, we derive the neutrino effective potential as the function of magnetic field dVeff/dr = ∂Veff/∂B × ∂B/∂r, and assume that at internal shocks (1010 cm for WR and 1012 cm for BSGs), magnetic fields change a 10 per cent of any variation around the radius shock scale. For instance, for a WR star, ∂B/∂r = 0.1 × 10−4 Bc/1010 cm = 6.99 × 10−3 G cm−1. We plot the flip probability as a function of neutron energy for two and three flavours (Fig. 8). We divide each plot of flip probability into three regions in order to analyse the whole range of probabilities: less than 0.2 (PΓ ≤ 0.2, a pure adiabatic conversion), between 0.2 and 0.8 (0.2 < PΓ < 0.8 represents the transition region) and greater than 0.8 (PΓ ≥ 0.8 is a strong violation of adiabaticity) (Dighe & Smirnov 2000). In Fig. 8, we use two flavours: solar (top-left panel), atmospheric (top-right panel), accelerator (bottom-left panel) and three flavour (bottom-right panel). When we use solar parameters, a pure adiabatic conversion occurs in a WR (BSG) star for neutrino energies of less than 1010.5 (1011.7) eV and 1011.5 (1012.7) eV, which are endowed with B = 10−8 Bc and B = 10−13 Bc, respectively. Considering atmospheric parameters, only a pure adiabatic conversion takes place in a WR (BSG) star for neutrino energies less than 1013.6 (1014.8) eV and 1014.3 (> 1015) eV, which are endowed with B = 10−4 Bc and B = 10−13 Bc, respectively. Taking into account accelerator parameters, a pure adiabatic conversion occurs in a WR (BSG) star for neutrino energies of less than 1011.8 (1013.1) eV and 1012.7 > 1015 eV with B = 10−13 Bc and B = 10−13 Bc, respectively. Once again, considering three neutrino mixing, a pure adiabatic conversion occurs in a WR (BSG) star for neutrino energy of less than 1011.1 (1012.5) eV and 1012.1 (1013.5) eV with B = 10−8 Bc and B = 10−13 Bc, respectively. Higher energies to those considered are found in regions of transition and/or those prohibited.

In addition, we have studied the HE neutrino oscillations from the neutrino effective potential generated in the star envelope (equation 23), as shown in Fig. 9. From the resonance condition (equation 27), we obtain the contour plots of radius as a
function of neutrino energy. We can see that for neutrino energy in the range 100 GeV < Eν < 100 TeV, the radius lies in the range 10^{0.8} < r < 10^{1.25} cm. The flip probability for neutrino oscillations in the envelope of a star was studied by Fraija (2014a), who plotted this probability as a function of neutrino energy for density profiles (equation 21) and (equation 22) and neutrino oscillation parameters. From this analysis, Fraija (2014a) showed that neutrinos can oscillate depending on their energy and the parameters of neutrino experiments, finding that neutrinos with energies above dozens of TeV can hardly oscillate.

Finally, considering a flux ratio Nν/ν ∼ 2Nν/ν ∼ 2Nν, we estimate the neutrino flavour ratio coming from the surface of a WR and BSG to Earth, as shown in Fig. 10. In this estimation, we take into account the contribution of thermal and magnetized plasma at moderate and weak B limit generated by internal shocks; weak limit) and neutrino energy in the range 10^{10.8} cm (second panel), 10^{11} cm (upper panel), 10^{12.5} cm (bottom panel) and 10^{13.5} cm (third panel), the effective potential due to the envelope of star and oscillation neutrinos in vacuum, due to the path up to Earth. In this figure, we take into account two values of θ13 mixing angle: 2° (left column) and 11° (right column). As shown, we can observe that a non-significant deviation of the standard ratio (φν/φν, 1:1:1) is expected, less than 10 per cent for θ13 = 11° and only 2 per cent for θ13 = 2°. In addition, we plot the neutrino flavour ratio expected on Earth as a function of neutrino energy when the magnetic field is oriented to different angles 0° ≤ θ ≤ 75° concerning neutrino direction, as shown in Figs 11 and 12. In Fig. 11, we consider the neutrino effective potential at the moderate-field limit and internal shocks at r = 10^{12} cm with a physical width Δr = 2 × 10^{11} cm. In Fig. 12, we consider the neutrino effective potential at the weak-field limit and internal shocks at r = 10^{10.8} cm with a physical width Δr = 1.5 × 10^{10} cm. From both figures, we can see that although the neutrino flavour ratio changes at different angles, distances of internal shocks, strength of magnetic field (moderate and weak limit) and neutrino energy in the range 10^{11} ≤ Eν ≤ 10^{14} eV, this flavour ratio expected on Earth lies between 0.98 and 1.02, and we can hence conclude that the directionality of magnetic fields does not affect our results. Although, currently, neutrino oscillations can hardly be detected, new techniques in the near future will allow us to perceive these oscillations and to put limits on the neutrino mixing angles. Finally, it is worth noting that the estimated values of the bulk Lorentz factor, in particular those relying on variability time measurements, are only raw approximations, and variations by a factor of a few cannot be ruled out by existing data.

ACKNOWLEDGEMENTS

We thank A. M. Sodelberg, J. Nieves, B. Zhang, K. Murase, W. H. Lee, F. de Colle, E. Moreno and A. Marinelli for useful discussions. NF gratefully acknowledges a Luc Binette-Fundaci UNAM postdoctoral fellowship. This work was supported by the projects IG100414 and Conacyt 101958.

REFERENCES

Abe K. et al., 2011, Phys. Rev. Lett., 107, 241801
Aharmim B. et al., 2013, Phys. Rev. C, 88, 025501

MNRA 450, 2784–2798 (2015)
APPENDIX A: EFFECTIVE POTENTIAL

The functions of the neutrino effective potential at a moderate magnetic field limit are

\[
F_m = \left( 1 + 2 \frac{E^2}{m_W} \right) K_1(\sigma_l) + 2 \sum_{n=1}^\infty \lambda_n \left( 1 + \frac{E^2}{m_W^2} \right) K_1(\sigma_l \lambda_n) \times K_1(\sigma_l) \lambda_n,
\]

\[
G_m = \left( 1 - 2 \frac{E^2}{m_W} \right) K_1(\sigma_l) - 2 \sum_{n=1}^\infty \lambda_n \frac{E^2}{m_W^2} K_1(\sigma_l \lambda_n)
\]

\[J_m = \frac{3}{4} K_0(\sigma_l) + K_1(\sigma_l) \sigma_l + \sum_{n=1}^\infty \lambda_n \left[ K_1(\sigma_l \lambda_n) \sigma_l - K_0(\sigma_l \lambda_n) \frac{2 \lambda_n^2}{\sigma_l^2} \right] \times \frac{K_0(\sigma_l \lambda_n) \sigma_l - K_0(\sigma_l \lambda_n) \frac{2 \lambda_n^2}{\sigma_l^2}}{\sigma_l^2},
\]

\[H_m = K_1(\sigma_l) \sigma_l + \sum_{n=1}^\infty \lambda_n^2 \left[ K_1(\sigma_l \lambda_n) \sigma_l - K_0(\sigma_l \lambda_n) \frac{2 \lambda_n^2}{\sigma_l^2} \right]
\]

and at weak magnetic field limit

\[
F_w = \left( 2 + 2 \frac{E^2}{m_W^2} \right) \left( \frac{K_0(\sigma_l)}{\sigma_l} + 2 \frac{K_1(\sigma_l)}{\sigma_l^2} \right) \frac{B_c}{B} - K_1(\sigma_l)
\]

\[
G_w = K_1(\sigma_l) - 2 \frac{B_c}{B} \frac{E^2}{m_W^2} \left( \frac{K_0(\sigma_l)}{\sigma_l} + 2 \frac{K_1(\sigma_l)}{\sigma_l^2} \right)
\]

\[J_w = \left( \frac{1}{2} + \frac{3 B_c}{B \sigma_l^2} \right) K_0(\sigma_l) + \frac{B_c}{B} \left( 1 + \frac{6}{\sigma_l^2} \right) K_1(\sigma_l)
\]

\[H_w = \left( \frac{1}{2} + \frac{B_c}{B \sigma_l^2} \right) K_0(\sigma_l) + \frac{B}{B_c} \left( \frac{2}{\sigma_l^2} - \frac{1}{2} \right) K_1(\sigma_l)
\]

where \(\lambda_n^2 = 1 + 2n B / B_c\), \(K_i\) is the modified Bessel function of integral order \(i\), \(\alpha_i = \beta \mu(i+1)\) and \(\sigma_i = \beta m_i(l+1)\).

APPENDIX B: PROBABILITIES

The flavour ratio at the internal shocks and the envelope of the star are

\[
\begin{pmatrix}
  v_e \\
v_\mu \\
v_\tau
\end{pmatrix}_{ss} =
\begin{pmatrix}
P_{e\mu ss} & P_{e\tau ss} & P_{e\nu ss} \\
P_{\mu e ss} & P_{\mu\tau ss} & P_{\mu\nu ss} \\
P_{\tau e ss} & P_{\tau\mu ss} & P_{\tau\nu ss}
\end{pmatrix}
\begin{pmatrix}
v_e \\
v_\mu \\
v_\tau
\end{pmatrix}_{is}, 
\]
where the neutrino mixing matrix $U_{ij}$ is given by Gonzalez-Garcia & Nir (2003); Akhmedov et al. (2004); Gonzalez-Garcia (2011)

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}s_{13} & c_{12}s_{13} & 0 \\ c_{23}s_{12} & s_{23}s_{12} & 0 \end{pmatrix}. \quad (B9)$$

Here, $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$ and we have taken the Dirac phase $\delta = 0$. Taking into account the effect of internal shocks, the envelope of the star and the vacuum, the probabilities are given by

$$P_{11} = 0.82 P_{11} + 0.55 P_{21} + 0.19 P_{31},$$

$$P_{12} = 0.82 P_{12} + 0.55 P_{22} + 0.19 P_{32},$$

$$P_{13} = 0.82 P_{13} + 0.55 P_{23} + 0.19 P_{33},$$

$$P_{21} = -0.51 P_{11} + 0.51 P_{21} + 0.69 P_{31},$$

$$P_{22} = -0.51 P_{12} + 0.51 P_{22} + 0.69 P_{32},$$

$$P_{23} = -0.51 P_{13} + 0.51 P_{23} + 0.69 P_{33},$$

$$P_{31} = 0.28 P_{11} - 0.66 P_{31} + 0.69 P_{31},$$

$$P_{32} = 0.28 P_{12} - 0.66 P_{22} + 0.69 P_{32},$$

$$P_{33} = 0.28 P_{13} - 0.66 P_{23} + 0.69 P_{33}. \quad (B10)$$

This paper has been typeset from a TeX/LaTeX file prepared by the author.

where

$$P_{11} = P_{ee,ss} P_{ee,ss} + P_{ei,ss} P_{ei,ss} + P_{et,ss} P_{te,ss},$$

$$P_{12} = P_{ee,ss} P_{ep,ss} + P_{ei,ss} P_{mp,ss} + P_{et,ss} P_{te,mp},$$

$$P_{13} = P_{ee,ss} P_{ett,ss} + P_{ei,ss} P_{ett,ss} + P_{et,ss} P_{ett,ss},$$

$$P_{21} = P_{ep,ss} P_{ee,ss} + P_{mp,ss} P_{mp,ss} + P_{mr,ss} P_{mr,ss},$$

$$P_{22} = P_{ep,ss} P_{ep,ss} + P_{mp,ss} P_{mp,ss} + P_{mr,ss} P_{mr,ss},$$

$$P_{23} = P_{ep,ss} P_{et,ss} + P_{mp,ss} P_{mr,ss} + P_{mr,ss} P_{tr,ss},$$

$$P_{31} = P_{et,ss} P_{ee,ss} + P_{et,ss} P_{ep,ss} + P_{te,ss} P_{te,ss},$$

$$P_{32} = P_{et,ss} P_{et,ss} + P_{mr,ss} P_{mr,ss} + P_{tr,ss} P_{tr,ss},$$

$$P_{33} = P_{et,ss} P_{et,ss} + P_{mr,ss} P_{mr,ss} + P_{tr,ss} P_{tr,ss}. \quad (B11)$$