“Analytic continuation” of $\mathcal{N} = 2$ minimal model

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In this paper we discuss what theory should be identified as the “analytic continuation” with $N \to -N$ of the $\mathcal{N} = 2$ minimal model with the central charge $\hat{c} = 1 - \frac{2}{N}$. We clarify how the elliptic genus of the expected model is written in terms of holomorphic linear combinations of the “modular completions” introduced in [T. Eguchi and Y. Sugawara, JHEP 1103, 107 (2011)] in the $SL(2)_{N+2}/U(1)$ supercoset theory. We further discuss how this model could be interpreted as a kind of model of the $SL(2)_{N+2}/U(1)$ supercoset in the $(\tilde{R}, \tilde{R})$ sector, in which only the discrete spectrum appears in the torus partition function and the potential IR divergence due to the non-compactness of the target space is removed. We also briefly discuss possible definitions of the sectors with other spin structures.

Subject Index A13, B24

1. Introduction

The $\mathcal{N} = 2$ minimal model is one of the most familiar rational superconformal field theories in two dimensions [1–6]. It is defined by the supercoset theory of $SU(2)_k/U(1)$ with level $k = N - 2$ [7] and also described as the IR fixed point of the $\mathcal{N} = 2$ Landau–Ginzburg (LG) model with the superpotential $W(X) = X^N + \text{[lower powers]}$ [8–10]. One of the good features of $\mathcal{N} = 2$ minimal models is a very simple formula for elliptic genus [11],

$$Z_{\text{min}}^{(N-2)}(\tau, z) = \frac{\theta_1(\tau, N \frac{1}{N} z)}{\theta_1(\tau, \frac{1}{N} z)}, \quad (1.1)$$

which behaves nicely under modular transformations as well as spectral flows. Namely, the function $Z_{\text{min}}^{(N-2)}(\tau, Nz)$ is a weak Jacobi form [12] of weight 0 and index $\frac{N(N-2)}{2}$.

An old question is what theory should be identified as the “analytic continuation” of the $\mathcal{N} = 2$ minimal model under $N \to -N$? A naive guess of the answer would be the $SL(2)_{N+2}/U(1)$ supercoset theory that has the expected central charge $\hat{c} (\equiv \frac{3}{2}) = 1 + \frac{2}{N}$. However, this is not completely correct, for the following reasons:

- The $SL(2)/U(1)$ supercoset theory contains both discrete and continuous spectra of primary fields, while the $\mathcal{N} = 2$ minimal model only has discrete spectra. It is not likely to be the case that two such theories are directly connected by an analytic continuation of the parameter of theory.

- It has been shown that the elliptic genera of the $SL(2)/U(1)$ supercoset theories show non-holomorphicity with respect to the modulus $\tau$ of the world-sheet torus [13,14], whereas the elliptic genera of the minimal model (1.1) are manifestly holomorphic.
In this paper we shall try to give a precise answer to this question. In other words, we will focus on the problems:

(1) What is the superconformal model with \( \hat{c} = 1 + \frac{2}{N} \) which has

\[
Z(\tau, z) \left( \alpha \left[ Z_{\min}^{(-N-2)}(\tau, z) \right] \right) = \frac{\theta_1(\tau, \frac{N+1}{N} z)}{\theta_1(\tau, \frac{1}{N} z)}
\]

as its elliptic genus?

(2) Then, does this model have any relationship with the \( SL(2)_{N+2}/U(1) \) supercoset?

This paper is organized as follows. In Sect. 2 we shall demonstrate our mathematical results. We make a detailed analysis of the holomorphic function (1.2), and prove the main theorem which addresses the precise relation between (1.2) and the “modular completions” introduced in [14] in the \( SL(2)_{N+2}/U(1) \) supercoset. We will further present a physical interpretation of this mathematical result and some discussions in Sect. 3. In Sect. 4 we present the summary and some comments.

2. Holomorphic linear combinations of modular completions

In this section we address some mathematical results. The main claims will be expressed in (2.24) and (2.31).

2.1. Preliminary

We first introduce the relevant notations. To begin with, we introduce the symbol of “IR part” just for convenience:

\[
[f(\tau, z)] := \lim_{\tau \to i\infty} f(\tau, z),
\]

(2.1)

where \( f(\tau, z) \) is assumed to be holomorphic around the cusp \( \tau = i\infty \).

2.1.1. A holomorphic Jacobi form. We consider a holomorphic function defined by

\[
\Phi^{(N)}(\tau, z) := \frac{\theta_1(\tau, \frac{N+1}{N} z)}{\theta_1(\tau, \frac{1}{N} z)}.
\]

(2.2)

This is obtained by the formal replacement \( N \to -N \) in the elliptic genus of the \( \mathcal{N} = 2 \) minimal model (1.1). The function (2.2) possesses the next modular and spectral flow properties with \( \hat{c} \equiv 1 + \frac{2}{N} \):

\[
\Phi^{(N)}(\tau + 1, z) = \Phi^{(N)}(\tau, z), \quad \Phi^{(N)}\left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = e^{i\pi \frac{\hat{c}^2}{\tau}} \Phi^{(N)}(\tau, z),
\]

(2.3)

\[
\Phi^{(N)}(\tau, z + m\tau + n) = (-1)^{m+n} q^{-\frac{\hat{c}^2}{2} m^2} y^{-i\hat{c} m} \Phi^{(N)}(\tau, z), \quad (m, n \in \mathbb{Z}).
\]

(2.4)

In other words, \( \Phi^{(N)}(\tau, Nz) \) is a weak Jacobi form of weight \( 0 \) and index \( \frac{N^2 \hat{c}^2}{2} \equiv \frac{N(N+2)}{2} \):

\[
\Phi^{(N)}(\tau, Nz) \in J\left[ 0, \frac{N(N+2)}{2} \right],
\]

where we set

\[
J[w, d] := \{ \text{weak Jacobi forms of weight } w \text{ and index } d \}.
\]

(2.5)

The IR part of (2.2) is evaluated as

\[
\left[ \Phi^{(N)}(\tau, z) \right] = y^{-\frac{1}{2}} \sum_{v=0}^{N} y^{\frac{v}{N}}.
\]

(2.6)
We next introduce the “spectral flow operator” $s_{(a,b)} (a, b \in \mathbb{Z})$ defined by
\[ s_{(a,b)} \cdot f(\tau, z) := (-1)^{a+b} q^{\frac{a}{2}y^{2}} e^{2\pi i \frac{ab}{y}} f(\tau, z + a\tau + b), \]
and set
\[ \Phi_{(a,b)}^{(N)}(\tau, z) := s_{(a,b)} \cdot \Phi_{(a,b)}^{(N)}(\tau, z) \quad (a, b \in \mathbb{Z}). \]
(2.8)

Since having a periodicity
\[ \Phi_{(a+Nm,b+Nn)}^{(N)}(\tau, z) = \Phi_{(a,b)}^{(N)}(\tau, z) \quad (\forall \ m, n \in \mathbb{Z}), \]
we may assume $a, b \in \mathbb{Z}_N$. Its modular property is written as
\[ \Phi_{(a,b)}^{(N)}(\tau + 1, z) = \Phi_{(a,a+b)}^{(N)}(\tau, z), \quad \Phi_{(a,b)}^{(N)} \left( \frac{-1}{\tau}, \frac{z}{\tau} \right) = e^{i\pi \frac{z^2}{\tau}} \Phi_{(b,-a)}^{(N)}(\tau, z). \]
(2.9)

The IR part of $\Phi_{(a,b)}^{(N)}(\tau, z)$ is computed as
\[ \left[ \Phi_{(a,b)}^{(N)}(\tau, z) \right] = \delta_{a,0} \sum_{v=1}^{N} e^{2\pi i \frac{b}{N} v y^{-\frac{1}{2} + \frac{v}{N}} + y^{-\frac{1}{2} + \frac{|a|}{N}}}, \]
(2.10)
where we introduced the notation $[a]$, defined by $[a] \equiv a (\text{mod } N), \ 0 \leq [a] \leq N - 1$, and $\delta_{a,s}$ denotes the “mod $N$ Kronecker delta”. In fact, we find
\[ \Phi_{(0,b)}^{(N)}(\tau, z) = \frac{y^{-\frac{N+1}{2N} e^{2\pi i \frac{b}{N}} \left( 1 - y^{\frac{N+1}{N} e^{2\pi i \frac{b}{N}}} \right)}}{y^{-\frac{1}{2} e^{2\pi i \frac{b}{N}} \left( 1 - y^{\frac{1}{N} e^{2\pi i \frac{b}{N}}} \right)}} + O(q) = \sum_{v=0}^{N} e^{2\pi i \frac{b}{N} v y^{-\frac{1}{2} + \frac{v}{N}}} + O(q), \]
and also (for $a \neq 0$),
\[ \Phi_{(a,b)}^{(N)}(\tau, z) = y^{-\frac{1}{2} + \frac{|a|}{N}} + O(q^{\frac{1}{N}}), \]
which proves (2.10).

2.1.2. Modular completions. Let us introduce the “modular completion” of the extended discrete characters (of the $\hat{\mathbb{R}}$ sector) [15–17] in the $SL(2)/U(1)$ supercosect according to [14]. For the case $\hat{c} = 1 + \frac{2}{\sqrt{N}}, (\forall \ N \in \mathbb{Z}_N)$, this function is defined as
\[ \chi_{\text{dis}}^{(N,1)}(v, a; \tau, z) := \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{n \in a+N\mathbb{Z}, \ r \in v+N\mathbb{Z}} \left\{ \int_{\mathbb{R}+i(N-0)} dp - \int_{\mathbb{R}-i0} dp \right\} \]
\[ \times \frac{e^{-\pi \tau \frac{z^2+2v}{N} (yq^n)^r}}{p - ir} \frac{y^{\frac{2a}{N} q^{\frac{2v}{N}}}}{1 - yq^n} \]
\[ = \chi_{\text{dis}}^{(N,1)}(v, a; \tau, z) + \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{n \in a+N\mathbb{Z}, \ r \in v+N\mathbb{Z}} \int_{\mathbb{R}-i0} dp \frac{e^{-\pi \tau \frac{z^2+2v}{N} (yq^n)^r}}{p - ir} \frac{y^{\frac{2a}{N} q^{\frac{2v}{N}}}}{1 - yq^n} \]
\[ (v, a \in \mathbb{Z}_N), \quad (2.11) \]
where $\chi_{\text{dis}}^{(N,1)}$ denotes the extended discrete character introduced in [15–17] (written in the convention of [14]):
\[ \chi_{\text{dis}}^{(N,1)}(v, a; \tau, z) = \sum_{n \in a+N\mathbb{Z}} \frac{(yq^n)^r}{1 - yq^n} \frac{y^{\frac{2a}{N} q^{\frac{2v}{N}}}}{1 - yq^n} \frac{\theta_1(\tau, z)}{\eta(\tau)^3}, \]
\[ (v = 0, 1, \ldots, N, \ a \in \mathbb{Z}_N). \]
(2.12)
The first line of (2.11) naturally appears through the analysis of the partition function of the $SL(2)/U(1)$ supercoset [14], and the second line just comes from the contour deformation.

The modular completion of the Appell function (or the “Appell–Lerch” sum) $\mathcal{K}^{(2N)}(\tau, z)$ [18,19] is given as [20]:

$$\tilde{\mathcal{K}}^{(2N)}(\tau, z) := \mathcal{K}^{(2N)}(\tau, z) - \frac{1}{2} \sum_{m \in \mathbb{Z}_2} R_{m,N}(\tau) \Theta_{m,N}(\tau, 2z), \quad (2.13)$$

where we set

$$R_{m,N}(\tau) := \frac{1}{i\tau} \sum_{r \in \mathbb{Z}+2N\mathbb{Z}} \int_{\mathbb{R} - i0} dp \frac{e^{-\pi \tau_2 p^2 + 2\pi \tau p} q^{r - \frac{N}{2}}} {p - ir} q^{-\frac{r^2}{4N}}, \quad (2.14)$$

which is generically non-holomorphic due to explicit $\tau_2 (\equiv \text{Im } \tau)$ dependence.

The next “Fourier expansion relation” [14] will be useful for our analysis:

$$y^{\frac{1}{2N}} q^{\frac{1}{2} N \tilde{\mathcal{K}}^{(2N)}}(\tau, \frac{z + a \tau + b}{N}) \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} = \sum_{v \in \mathbb{Z}_N} e^{2\pi i \frac{a \theta}{N}} \tilde{\chi}_{\text{dis}}^{(N,1)}(v, a; \tau, z), \quad (2.15)$$

which is the “hatted” version of the similar relation between $\mathcal{K}^{(2k)}$ and $\chi_{\text{dis}}(v, a)$ given in [17].

The modular transformation formulas for the modular completions (2.11) and (2.13) are written as [14,20]

$$\tilde{\chi}_{\text{dis}}^{(N,1)}(v, a; -\frac{1}{\tau}, \frac{z}{\tau}) = e^{i\pi \frac{s^2}{2}} \sum_{v' = 0}^{N-1} \sum_{a' \in \mathbb{Z}_N} \frac{1}{N} e^{2\pi i v v' - (\tau + 2\pi)(v' + 2a') \frac{2\pi}{4N}} \tilde{\chi}_{\text{dis}}^{(N,1)}(v', a'; \tau, z), \quad (2.16)$$

$$\tilde{\chi}_{\text{dis}}^{(N,1)}(v, a; \tau + 1, z) = e^{2\pi i \frac{a \theta}{N}} \chi_{\text{dis}}^{(N,1)}(v, a; \tau, z), \quad (2.17)$$

$$\tilde{\mathcal{K}}^{(2N)}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \tau e^{i\pi \frac{2N}{\tau} \tilde{\chi}_{\text{dis}}^{(2N)}}(\tau, z), \quad (2.18)$$

$$\tilde{\mathcal{K}}^{(2N)}(\tau + 1, z) = \tilde{\mathcal{K}}^{(2N)}(\tau, z). \quad (2.19)$$

The spectral flow property is summarized as follows:

$$\tilde{\chi}_{\text{dis}}^{(N,1)}(v, a; \tau, z + r \tau + s) = (-1)^r s^s e^{2\pi i s \frac{2q}{N}} q^{-\frac{s^2}{2}} y^{-\frac{sr}{2}} \tilde{\chi}_{\text{dis}}^{(N,1)}(v, a + r; \tau, z), \quad (\forall r, s \in \mathbb{Z}), \quad (2.20)$$

$$\tilde{\mathcal{K}}^{(2N)}(\tau, z + r \tau + s) = q^{-N r^2} y^{-2N r} \tilde{\mathcal{K}}^{(2N)}(\tau, z), \quad (\forall r, s \in \mathbb{Z}). \quad (2.21)$$

More detailed formulas in general cases of $\tilde{\chi}_{\text{dis}}^{(N,K)}(v, a)$ with $\tilde{c} = 1 + \frac{2K}{N}$, $(N, K \in \mathbb{Z}_{>0})$ are summarized in Appendix B.

1 The relation to the notation given in Chapter 3 of [20] is as follows:

$$\mathcal{K}^{(2k)}(\tau, z) \equiv f^{(k)}(z; \tau), \quad \tilde{\mathcal{K}}^{(2k)}(\tau, z) \equiv \tilde{f}^{(k)}(z; \tau).$$
2.2. Fourier expansion of (2.2) and modular completions

Let us start our main analysis. We begin by introducing the holomorphic functions \( X^{(N)}(v, a; \tau, z) \) as the “Fourier transforms” of \( \Phi^{(N)}_{(a, b)}(\tau, z) \):

\[
X^{(N)}(v, a; \tau, z) := \frac{1}{N} \sum_{b \in \mathbb{Z}_N} e^{-2\pi i \frac{b}{N}(v + a)} \Phi^{(N)}_{(a, b)}(\tau, z); \tag{2.22}
\]

in other words,

\[
\Phi^{(N)}_{(a, b)}(\tau, z) = \sum_{v=0}^{N-1} e^{2\pi i \frac{b}{N}(v + a)} X^{(N)}(v, a; \tau, z). \tag{2.23}
\]

The main formula that we would like to prove in this section is

\[
X^{(N)}(v, a; \tau, z) = \hat{\chi}^{(N, 1)}_{\text{dis}}(v, a; \tau, z) + \hat{\chi}^{(N, 1)}_{\text{dis}}(N - v, a + v; \tau, z). \tag{2.24}
\]

In order to achieve this formula we first consider the elliptic genera of the \( SL(2)_{N+2}/U(1) \) supercoset with \( \hat{c} = 1 + \frac{2}{N} \) [13,14]. We set

\[
\mathcal{Z}(\tau, z) := \hat{K}^{(2N)}\left(\tau, \frac{z}{N}\right) \frac{\theta_1(\tau, z)}{\eta(\tau)^3}, \tag{2.25}
\]

and

\[
\tilde{\mathcal{Z}}(\tau, z) := \frac{1}{N} \sum_{a, b \in \mathbb{Z}_N} s_{(a, b)} \cdot \mathcal{Z}(\tau, z) = \frac{1}{N} \sum_{a, b \in \mathbb{Z}_N} q^{\frac{a^2}{N}} y^{\frac{2a}{N}} e^{2\pi i \frac{ab}{N}} \hat{K}^{(2N)}_{\tau}(\tau, \frac{z + a\tau + b}{N}) \frac{\theta_1(\tau, z)}{\eta(\tau)^3}. \tag{2.26}
\]

Here, \( \tilde{\mathcal{Z}}(\tau, z) \) is identified with the elliptic genus of the axial supercoset of \( SL(2)_{N+2}/U(1) \) (“cigar” [21–25]), while \( \mathcal{Z}(\tau, z) \) is associated with the vector supercoset:

\[
\text{[vector } SL(2)/U(1) \text{]} \cong \text{[\( \mathbb{Z}_N \)-orbifold of axial } SL(2)/U(1) \text{]} \cong \text{[\( N \)-fold cover of “trumpet”]},
\]

as shown in [26]. The following identities play crucial roles:

\[
\mathcal{Z}_{(a, b)}(\tau, z) := s_{(a, b)} \cdot \mathcal{Z}(\tau, z) = \sum_{a=0}^{N-1} e^{2\pi i \frac{b}{N}(v + a)} \hat{\chi}^{(N, 1)}_{\text{dis}}(v, a; \tau, z) \tag{2.27}
\]

\[
\tilde{\mathcal{Z}}_{(a, b)}(\tau, z) := s_{(a, b)} \cdot \tilde{\mathcal{Z}}(\tau, z) \left( \equiv \frac{1}{N} \sum_{\alpha, \beta \in \mathbb{Z}_N} e^{-2\pi i \frac{1}{N}(a\beta - b\alpha)} \mathcal{Z}_{(\alpha, \beta)}(\tau, z) \right)
\]

\[
= \sum_{v=0}^{N-1} e^{2\pi i \frac{b}{N}(v + a)} \hat{\chi}^{(N, 1)}_{\text{dis}}(N - v, a + v; \tau, z); \tag{2.28}
\]

these are proven in [14,26].
We also note that the IR parts of $Z_{(a,b)}$ and $\tilde{Z}_{(a,b)}$ are given as
\begin{equation}
Z_{(a,b)}(\tau, z) = \delta_{(a,0)}^{(N)} \left\{ \sum_{\nu=1}^{N-1} e^{2\pi i \frac{\nu}{N} y} y^{-\frac{1}{2} + \frac{\nu}{N}} + \frac{1}{2} \left( y^{-\frac{1}{2}} + y^{\frac{1}{2}} \right) \right\},
\end{equation}
(2.29)
\begin{equation}
\tilde{Z}_{(a,b)}(\tau, z) = y^{-\frac{1}{2} + \frac{\nu}{N}} + \delta_{(a,0)}^{(N)} \frac{1}{2} \left( y^{\frac{1}{2}} - y^{-\frac{1}{2}} \right).
\end{equation}
(2.30)

With these preparations we shall prove the next identity,
\begin{equation}
Z(\tau, z) + \tilde{Z}(\tau, z) = \Phi^{(N)}(\tau, z),
\end{equation}
(2.31)
from which the identity (2.24) is readily derived by using (2.27) and (2.28) as well as the definition $X^{(N)}(\nu, a)$ (2.22).

**Proof of (2.31).** We set $\Phi^{(N)'}(\tau, z) := Z(\tau, z) + \tilde{Z}(\tau, z)$, and will prove $\Phi^{(N)'}(\tau, z) = \Phi^{(N)}(\tau, z)$. We first enumerate relevant properties of $\Phi^{(N)'}(\tau, z)$.

(i) Modular and spectral flow properties: We first note that $\Phi^{(N)'}(\tau, z)$ possesses the expected modular and spectral flow properties:
\begin{equation}
\Phi^{(N)'}(\tau + 1, z) = \Phi^{(N)'}(\tau, z), \quad \Phi^{(N)'} \left( -\frac{1}{\tau}, -\frac{z}{\tau} \right) = e^{i\pi \frac{c}{\tau}} \Phi^{(N)'}(\tau, z),
\end{equation}
(2.32)
\begin{equation}
s_{(Na,Nb)} \cdot \Phi^{(N)'}(\tau, z) = \Phi^{(N)'}(\tau, z), \quad (\forall a, b \in \mathbb{Z}).
\end{equation}
(2.33)
These are shown from the same properties of $Z(\tau, z)$ as well as the fact that $s_{(a,b)}$ acts modular covariantly.

(ii) Holomorphicity: Recall the fact that $Z(\tau, z)$ is written as
\begin{equation}
Z(\tau, z) = \frac{\theta_1(\tau, z)}{\eta(\tau)^3} \left[ K^{(2N)}(\tau, \frac{z}{N}) - \frac{1}{2} \sum_{m \in \mathbb{Z}_N} R_{m,N}(\tau) \Theta_{m,N} \left( \tau, \frac{2z}{N} \right) \right],
\end{equation}
(2.34)
and the second term is non-holomorphic. Let us consider the “$\mathbb{Z}_N$-orbifold action,” that is
\begin{equation}
\frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} s_{(a,b)} \cdot [\ast],
\end{equation}
to this non-holomorphic correction term. Because of the simple identity
\begin{equation}
\frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} q^{\frac{a}{N} \pi \frac{2b}{N}} e^{2\pi i \frac{ab}{N}} \Theta_{m,N} \left( \tau, \frac{2z}{N} \right) = \Theta_{-m,N} \left( \tau, \frac{2z}{N} \right),
\end{equation}
(2.35)
we obtain
\begin{equation}
[\text{non-hol. corr. term}] \mathbb{Z}_N\text{-orbifolding} \quad \frac{1}{N} \sum_{m \in \mathbb{Z}_N} R_{m,N}(\tau) \Theta_{-m,N} \left( \tau, \frac{2z}{N} \right) \frac{\theta_1(\tau, z)}{\eta(\tau)^3} = -\frac{1}{2} \sum_{m \in \mathbb{Z}_N} R_{-m,N}(\tau) \Theta_{m,N} \left( \tau, \frac{2z}{N} \right) \frac{\theta_1(\tau, z)}{\eta(\tau)^3} = \frac{1}{2} \sum_{m \in \mathbb{Z}_N} \left[ R_{m,N}(\tau) - 2\delta_{m,0}^{(N)} \right] \Theta_{m,N} \left( \tau, \frac{2z}{N} \right) \frac{\theta_1(\tau, z)}{\eta(\tau)^3}.
\end{equation}
Therefore, potential non-holomorphic terms in \( \Phi^{(N)'}(\tau, z) \) are strictly canceled out, and we can rewrite it as

\[
\Phi^{(N)'}(\tau, z) = \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \left[ K^{(2N)}(\tau, z) + \frac{1}{N} \sum_{a, b \in \mathbb{Z}_N} q^{a^2 + \frac{2a}{N} + \frac{2b}{N}} e^{2\pi i \frac{ab}{N}} \right] + \kappa^{(2N)}(\tau, z) + \sum_{a, b \in \mathbb{Z}_N} q^{a^2 + \frac{2a}{N} + \frac{2b}{N}} e^{2\pi i \frac{ab}{N}} \left( \frac{z}{N} - \frac{2z}{N} \right) + \theta_{0,N}(\tau, z).
\]

This is manifestly holomorphic.

(iii) IR part: Recall

\[
[Z(\tau, z)] = \frac{1}{2} \left( y^\frac{1}{2} + y^{-\frac{1}{2}} \right) + \sum_{v = 1}^{N-1} y^{-\frac{1}{2} + \frac{v}{N}}, \quad [\bar{Z}(\tau, z)] = \frac{1}{2} \left( y^\frac{1}{2} + y^{-\frac{1}{2}} \right),
\]

and thus

\[
\left[ \Phi^{(N)'}(\tau, z) \right] = \sum_{v = 0}^{N} y^{-\frac{1}{2} + \frac{v}{N}} \left( \equiv \left[ \Phi^{(N)}(\tau, z) \right] \right).
\]

Moreover, we can show

\[
\left[ \Phi^{(N)'}_{(a,b)}(\tau, z) \right] = \delta_{a,0} \sum_{v = 1}^{N} e^{2\pi i \frac{b}{N}} y^{-\frac{1}{2} + \frac{v}{N}} + y^{-\frac{1}{2} + \frac{[a]}{N}} \left( \equiv \left[ \Phi^{(N)}_{(a,b)}(\tau, z) \right] \right),
\]

due to (2.29), (2.30), and (2.10).

In this way, we can conclude that both of \( \Phi^{(N)}(\tau, Nz) \) and \( \Phi^{(N)'}(\tau, Nz) \) are holomorphic Jacobi forms of weight 0 and index \( \frac{N^2}{2} \equiv \frac{N(N+2)}{2} \) which share the IR part, namely,

\[
\Phi^{(N)}(\tau, Nz), \quad \Phi^{(N)'}(\tau, Nz) \in \mathcal{J}\left[ 0, \frac{N(N+2)}{2} \right],
\]

\[
\left[ \Phi^{(N)}(\tau, Nz) \right] = \left[ \Phi^{(N)'}(\tau, Nz) \right] = \sum_{v = 0}^{N} y^{-\frac{v}{N}}.
\]

Consequently, if setting

\[
F(\tau, z) := \frac{\Phi^{(N)'}(\tau, Nz)}{\Phi^{(N)}(\tau, Nz)},
\]

then \( F(\tau, z) \) is an elliptic, modular invariant function with the IR part \( [F(\tau, z)] = 1 \). Thus, if we would succeed in proving the holomorphicity of \( F(\tau, z) \) \( \forall \tau \in \mathbb{H} \cup \{i\infty\}, \forall z \in \mathbb{C} \), we can conclude \( F(\tau, z) \equiv 1 \) and the proof will be completed.

\( \Phi^{(N)'}(\tau, Nz) \) is obviously holomorphic \( \forall \tau \in \mathbb{H} \cup \{i\infty\}, \forall z \in \mathbb{C} \), and \( \Phi^{(N)}(\tau, Nz) \) has simple zeros only at \( N(N+2) \) points

\[
z_{a,b} := \frac{a\tau + b}{N+1}, \quad a, b = 0, 1, \ldots, N, \quad (a, b) \neq (0, 0),
\]

in the fundamental region of double periodicity of \( F \). Hence the function \( F(\tau, z) \) at most possesses simple poles at \( z = z_{a,b} \) in the fundamental region, and the following lemma is enough for completing the proof:

[**Lemma**] All the residues of \( F(\tau, z) \) at \( z = z_{a,b} \) vanish.
Let us denote
\[ R_{a,b}(\tau) := \text{Res}_{z=z_{a,b}} \{ F(\tau, z) \} = \frac{1}{2\pi i} \oint_{C_{a,b}(\tau)} dz \, F(\tau, z), \]  
(2.42)
where \( C_{a,b}(\tau) \) is a small contour encircling \( z_{a,b} \). Because of the modular invariance of \( F(\tau, z) \), we find the modular properties of \( R_{a,b} \) as
\[ R_{a,b}(\tau + 1) = R_{a,a+b}(\tau), \quad R_{a,b} \left( -\frac{1}{\tau} \right) = \frac{1}{\tau} R_{b,-a}(\tau). \]  
(2.43)
In fact, the first formula is trivial, and the second formula is proven as follows:
\[
R_{a,b} \left( -\frac{1}{\tau} \right) = \frac{1}{2\pi i} \oint_{C_{a,b}(\tau)} dz \, F \left( -\frac{1}{\tau}, z \right) = \frac{1}{2\pi i} \oint_{C_{b,-a}(\tau)} dz' \frac{dz'}{z'} F \left( -\frac{1}{\tau}, \frac{z'}{z} \right) = \frac{1}{\tau} \oint_{C_{b,-a}(\tau)} dz' \frac{dz'}{z'} F \left( \tau, z' \right) \quad \text{\( : \text{ modular invariance of } F \)}
\]
It is obvious that \( R_{a,b}(\tau) \) is holomorphic over \( \mathbb{H} \), since \( z = z_{a,b} \) is at most a simple pole and \( \Phi^{(N)}(\tau, N z) \) is holomorphic \( \forall \, \tau \in \mathbb{H}, \forall \, z \in \mathbb{C} \). Moreover, one can show
\[
\lim_{\tau \to \infty} \left| R_{a,b}(\tau) \right| < \infty \quad (\forall \, a, b),
\]  
(2.44)
with the help of (2.38) and (2.39). The proof of (2.44) is straightforward, and we shall present it in Appendix C.

Now, we define the “modular orbit” \( O_r \subset \mathbb{Z}_{N+1} \times \mathbb{Z}_{N+1} \, \forall \, r \in \mathcal{D}(N + 1) \equiv \{ r = 1, 2, \ldots, N; \, r \, | \, (N + 1) \} \) as
\[
O_r := \left\{ (a, b) \in \mathbb{Z}_{N+1} \times \mathbb{Z}_{N+1} : (a, b) \equiv (r, 0)A \, (\text{mod } (N + 1) \times (N + 1)), \, \exists \, A \in SL(2, \mathbb{Z}) \right\}.
\]
Then,
\[
\mathbb{Z}_{N+1} \times \mathbb{Z}_{N+1} - \{(0, 0)\} = \bigsqcup_{r \in \mathcal{D}(N+1)} O_r
\]
holds, and
\[
R^{(r,k)}(\tau) := \sum_{(a,b) \in O_r} \left( R_{a,b}(\tau) \right)^{2k}
\]  
(2.45)
should be a modular form of weight \(-2k\). Therefore, \( R^{(r,k)}(\tau) \) has to vanish everywhere over \( \mathbb{H} \cup \{ \infty \} \) for arbitrary \( r \in \mathcal{D}(N + 1) \) and \( k \in \mathbb{Z}_{\geq 0} \). This is sufficient to conclude that \( R_{a,b}(\tau) \equiv 0 \) for every pole \( z_{a,b} \).

In this way, the identity (2.31) has been proven, leading to the main formula (2.24). \qed

3. Physical interpretation

In this section we try to make a physical interpretation of the main result given above, that is, the identities (2.24) or (2.31). In other words, we discuss what superconformal system leads to \( \Phi^{(N)}(\tau, z) \) (2.2) as its elliptic genus.
3.1. Some properties of the function $X^{(N)}(v, a)$

We first note several helpful facts about the expected “building block” $X^{(N)}(v, a)$ (2.22).

1. Modular and spectral flow properties: Once we achieve the identity (2.24), we can readily derive the formulas of modular and spectral flow transformations of $X^{(N)}(v, a)$ by using (2.16), (2.17), and (2.20), which are written as

$$
X^{(N)}(v, a; \tau + 1, z) = e^{2\pi i \frac{v}{N}} X^{(N)}(v, a; \tau, z)
$$

(3.2)

$$
X^{(N)}(v, a; \tau, z + r\tau + s) = (-1)^r e^{2\pi i \frac{v r^2}{N}} q^{- \frac{r^2}{2}} y^{-\frac{r}{2}} X^{(N)}(v, a + r; \tau, z), \quad (\forall r, s \in \mathbb{Z}),
$$

(3.3)

In other words,

$$
s_{(r,s)} \cdot X^{(N)}(v, a) = e^{2\pi i \frac{v + 2s + r}{N}} X^{(N)}(v, a + r) \quad (\forall r, s \in \mathbb{Z}).
$$

(3.4)

As a consistency check, one may also derive these formulas directly from the “Fourier expansion relation” (2.23).

2. Manifestly holomorphic expression: In place of (2.24), $X^{(N)}(v, a)$ can be rewritten in a manifestly holomorphic expression:

$$
X^{(N)}(v, a; \tau, z) = \chi^{(N,1)}_{\text{dis}}(v, a; \tau, z) + \chi^{(N,1)}_{\text{dis}}(N - v, a + v; \tau, z).
$$

(3.5)

In fact, the non-holomorphic correction terms in the right-hand side of (2.24) are found to be canceled out precisely, as in (2.36). One can achieve this identity by making the Fourier expansion of (2.36) and recalling the identity $\Phi^{(N)}(\tau, z) = \Phi^{(N)}(\tau, z)$ proven in the previous section.

3. Interpretation as “analytic continuation” of the character of $\mathcal{N} = 2$ minimal model: It would be worthwhile to recall the basic facts of the $\mathcal{N} = 2$ minimal model, namely, the $SU(2)_N \times \bar{U}(1)$ supercoset which has $\tilde{c} = 1 - \frac{2}{N}$. As we mentioned at the beginning of this paper, the elliptic genus of the minimal model is given as [11]

$$
Z^{(N-2)}_{\text{min}}(\tau, z) = \frac{\theta_1(\tau, \frac{N-1}{N} z)}{\theta_1(\tau, \frac{z}{N})} = \sum_{\ell=0}^{N-2} \text{ch}_{\ell, \ell+1}(\tilde{R}, \tau, z),
$$

(3.6)

where $\text{ch}_{\ell, m}(\tau, z)$ denotes the $\tilde{R}$-character of the $\mathcal{N} = 2$ minimal model which has the Witten index

$$
\text{ch}_{\ell, m}(\tau, 0) = \delta^{(2N)}_{m, \ell} - \delta^{(2N)}_{m, -(\ell+1)}.
$$

(3.7)
Moreover, the spectrally flowed elliptic genus is Fourier expanded in terms of the $\tilde{R}$-characters as follows ($a, b \in \mathbb{Z}_N$):

$$Z^{(N-2)}_{\text{min}(a,b)}(\tau, z) \equiv (-1)^{a+b} q^{\frac{N-2}{2} \tau^2} y^{a} e^{-2\pi i \frac{ab}{N}} Z^{(N-2)}_{\text{min}}(\tau, z + a\tau + b)$$

$$= \sum_{\ell=0}^{N-2} e^{2\pi i \frac{\ell}{N}(\ell+1-a)} \text{ch}_{\ell,\ell+1-2a}^{(\tilde{R})}(\tau, z). \quad (3.8)$$

In this way, by comparing (2.23) with (3.8) one would find a similarity between the function $X^{(N)}(v, a; \tau, z)$ and the minimal character $\text{ch}_{\ell,m}^{(\tilde{R})}(\tau, z)$ with the correspondence

$$N \to -N, \quad \ell \to -v - 1, \quad m \to -v - 2a. \quad (3.9)$$

Furthermore, let us recall the modular transformation formula of $\text{ch}_{\ell,m}^{(\tilde{R})}(\tau, z)$:

$$\text{ch}_{\ell,m}^{(\tilde{R})}(\tau + 1, z) = e^{2\pi i \tau z} \sum_{\ell'=0}^{N-2} \sum_{\ell' \in \mathbb{Z}_N} \text{ch}_{\ell',m}^{(\tilde{R})}(\tau, z)$$

$$= e^{2\pi i \frac{(\ell+1)^2 - 2m^2}{4N}} \text{ch}_{\ell+1, m}^{(\tilde{R})}(\tau, z) \quad (\text{with } m \equiv \ell + 2a), \quad (3.10)$$

$$\text{ch}_{\ell,m}^{(\tilde{R})}(\tau, z) = \left(-i\right)e^{i\frac{\pi}{N} \ell^2} \sum_{\ell'=0}^{N-2} \sum_{\ell' \in \mathbb{Z}_N} \frac{1}{N} \sqrt{2} \sin \left(\frac{\pi (\ell + 1)(\ell' + 1)}{N}\right) \text{ch}_{\ell',m}^{(\tilde{R})}(\tau, z)$$

$$= e^{i\frac{\pi}{N} \ell^2} \sum_{\ell'=0}^{N-2} \sum_{\ell' \in \mathbb{Z}_N} \frac{1}{N} e^{-2\pi i \frac{(\ell + 1)(\ell' + 1) - mm'}{2N}} \text{ch}_{\ell',m}^{(\tilde{R})}(\tau, z). \quad (3.11)$$

The first line of (3.11) is familiar formula and we have made use of the property of minimal character,

$$\text{ch}_{N-2-\ell,m+N}^{(\tilde{R})}(\tau, z) = -\text{ch}_{\ell,m}^{(\tilde{R})}(\tau, z), \quad (3.12)$$

to derive the second line.\footnote{We have an obvious identity for $X^{(N)}(v, a; \tau, z)$:

$$X^{(N)}(N - v, v + a; \tau, z) = X^{(N)}(v, a; \tau, z),$$

which again corresponds to (3.12) under (3.9) up to the overall sign.}

### 3.2. “Compactified $SL(2)/U(1)$ supercoset” in the $(\tilde{R}, \tilde{R})$ sector

Now, let us discuss what the superconformal model is whose modular invariant is built up from the functions $X^{(N)}(v, a) \ (2.24)$.

Let $\mathcal{H}_{\text{a}}^{(R)}$ be the Hilbert space of the axial supercoset of $SL(2)_{N+2}/U(1)$ in the (R, R) sector, and $\mathcal{H}_{\text{v}}^{(R)}$ be that of the vector supercoset. In other words, $\mathcal{H}_{\text{a}}^{(R)}$ corresponds to the cigar-type
superconformal model with the asymptotic radius $\sqrt{N\alpha'}$, whereas $\mathcal{H}_{V}^{(R)}$ should be associated with $[\text{cigar}]/\mathbb{Z}_N \cong [N\text{-fold cover of trumpet}]$.

as already mentioned in Sect. 2. This means that the torus partition functions in the $\tilde{\mathbb{R}}, \tilde{\mathbb{R}}$ sector of these theories are schematically written as (3.14)

\[
Z_{A}^{(R)}(\tau, \tilde{\tau}; z, \tilde{z}) = e^{2\pi \frac{c}{12}} \sum_{v=0}^{N-1} \sum_{a \in \mathbb{Z}_N} \hat{\chi}_{\text{dis}}^{(N,1)}(v, a; \tau, z) \left[ \hat{\chi}_{\text{dis}}^{(N,1)}(v, a; \tau, z) \right]^* + [\text{cont. terms}],
\]

(3.14)

The discrete part of (3.13) obviously looks like the diagonal modular invariant, whereas that of (3.14) is anti-diagonal only in the case of integer levels (i.e. $K = 1$), while the vector type (3.13) is always diagonal.

\[
\hat{\chi}_{\text{dis}}^{(N,1)}(v, a; \tau, z) = \hat{\chi}_{\text{dis}}^{(N,1)}(N - v, -a; \tau, z)
\]

The quantum number $m \equiv v + 2a$ adopted in [26].

The “anomaly factors” $e^{-2\pi \frac{c}{12}}$, $e^{2\pi \frac{c}{12}}$, which ensure the modular invariance, originate precisely from the path integrations, and they differ due to the gauged WZW actions of vector and axial types [14,26]. Note that these factors just get the common form $e^{2\pi \frac{c}{12}}$, if we replace $z$ with $-\tilde{z}$ in the axial model. Thus it would be useful to rewrite (3.14) as

\[
Z_{A}^{(R)}(\tau, \tilde{\tau}; -z, \tilde{z})
\]

in the second line.

\[\text{3 The quantum number } m \text{ labels the } U(1)_R \text{ charges appearing in the spectral flow orbit defining } \hat{\chi}_{\text{dis}}^{(N,1)}(v, a).\]

As mentioned in [26], the axial type (3.14) is anti-diagonal only in the case of integer levels (i.e. $K = 1$), while the vector type (3.13) is always diagonal.
Moreover, it is found that the continuous terms appearing in (3.13) and (3.15) are written in precisely the same functional form with inverse sign. We will prove this fact in the next subsection, and here address our main result in this section:

\[ Z_{cSL(2)/U(1)}(\tau, \bar{\tau}; z, \bar{z}) := Z^{(R)}_V(\tau, \bar{\tau}; z, \bar{z}) + Z^{(R)}_A(\tau, \bar{\tau}; -z, \bar{z}) = \frac{1}{2} e^{-2\pi \tau\bar{\tau}^2} \sum_{v=0}^{N-1} \sum_{a \in \mathbb{Z}_N} |X^{(N)}(v, a; \tau, z)|^2. \]  

(3.17)

Note that, even though both \(Z^{(R)}_V\) and \(Z^{(R)}_A\) include contributions of continuous characters with non-trivial coefficients showing IR divergence, the combined partition function (3.17) is written in terms only of a finite number of holomorphic building blocks \(X^{(N)}(v, a; \tau, z)\) given in (2.24) or (3.5). This fact strongly suggests that the modular invariant partition function (3.17) would define an \(\mathcal{N} = 2\) superconformal model with \(\hat{c} = 1 + \frac{2}{N}\) associated with some compact background. Therefore, we tentatively call this model the “compactified \(SL(2)/U(1)\) supercoset model” here. It is obvious that the elliptic genus of this model is given by the holomorphic function \(\Phi^{(N)}(\tau, z)\) (2.2):

\[ Z_{cSL(2)/U(1)}(\tau, \bar{\tau}) = \Phi^{(N)}(\tau, z) = \frac{\theta_1(\tau, \frac{N+1}{N}z)}{\theta_1(\tau, \frac{1}{N}z)}. \]  

(3.18)

### 3.3. Cancellation of continuous terms

In this subsection, we prove the precise cancellation of the continuous terms potentially appearing in the first line of (3.17).

We start with the explicit form of the partition function \(Z^{(R)}_V\), which is evaluated in [26] by means of the path integration \((u \equiv s_1 \tau + s_2)\):

\[ Z^{(R)}_V(\tau, \bar{\tau}; z, \bar{z}; \epsilon) = N e^{\frac{2\pi i}{\tau_2 \bar{\tau}_2} \frac{N+4}{N} z^2} \sum_{n_1, n_2 \in \mathbb{Z}} \int_{\Sigma(z, \epsilon)} |\theta_1(\tau, u + \frac{N+2}{N}z)|^2 \int_{\Sigma(z, \epsilon)} |\theta_1(\tau, u + \frac{2}{N}z)|^2 e^{2\pi i n_1 s_1} e^{2\pi i n_2 s_2}. \]  

(3.19)

Here we introduced the IR regularization adopted in [14,26], which removes the singularity of the integrand originating from the non-compactness of target space. Namely, we set

\[ \Sigma(z, \epsilon) \equiv \Sigma \setminus \left\{ u = s_1 \tau + s_2 : -\epsilon - \frac{2}{k} \xi_1 < s_1 < \frac{\epsilon}{2} - \frac{2}{k} \xi_1, \quad 0 < s_2 < 1 \right\}, \]  

(3.20)

where we set \(z \equiv \xi_1 \tau + \xi_2, \xi_1, \xi_2 \in \mathbb{R}\), and \(\epsilon(>0)\) denotes the regularization parameter.

In the same way, the axial partition function \(Z^{(R)}_A\), which describes the cigar background, is written as [14]:

\[ Z^{(R)}_A(\tau, \bar{\tau}; z, \bar{z}; \epsilon) = N e^{\frac{2\pi i}{\tau_2 \bar{\tau}_2} \frac{N+1}{N} z^2} \sum_{m_1, m_2 \in \mathbb{Z}} \int_{\Sigma(z, \epsilon)} |\theta_1(\tau, u + \frac{N+2}{N}z)|^2 \int_{\Sigma(z, \epsilon)} |\theta_1(\tau, u + \frac{2}{N}z)|^2 e^{2\pi i m_1 s_1} e^{2\pi i m_2 s_2}. \]  

(3.21)
As demonstrated in [14], one can make “character decompositions” of the partition functions (3.19) and (3.21). They are schematically expressed as

\[ Z_V^R(\tau, \tilde{\tau}; z, \bar{z}; \epsilon) = e^{-2\pi i \frac{\epsilon}{2}} \left[ \sum_{v=0}^{N-1} \sum_{a \in \mathbb{Z}_N^2} \hat{\chi}^{(N,1)}_{\text{dis}}(v, a; \tau, z) \left[ \hat{\chi}^{(N,1)}_{\text{dis}}(v, a; \tau, z) \right]^{*} \right. \]

\[ + \sum_{m \in \mathbb{Z}_N^2} \int dp_L \int dp_R \left\{ \rho_1(p_L, p_R, m; \epsilon) \chi_{\text{con}}^{(N,1)}(p_L, p_R, m; \tau, z) \left[ \chi_{\text{con}}^{(N,1)}(p_L, p_R, m; \tau, z) \right]^{*} \right\}, \]

\[ Z_A^R(\tau, \tilde{\tau}; z, \bar{z}; \epsilon) = e^{2\pi i \frac{\epsilon}{12}} \left[ \sum_{v=0}^{N-1} \sum_{a \in \mathbb{Z}_N^2} \hat{\chi}^{(N,1)}_{\text{dis}}(v, -v - a; \tau, z) \left[ \hat{\chi}^{(N,1)}_{\text{dis}}(v, a; \tau, z) \right]^{*} \right. \]

\[ + \sum_{m \in \mathbb{Z}_N^2} \int dp_L \int dp_R \left\{ \rho_1'(p_L, p_R, m; \epsilon) \chi_{\text{con}}^{(N,1)}(p_L, -p_R, m; \tau, z) \left[ \chi_{\text{con}}^{(N,1)}(p_L, -p_R, m; \tau, z) \right]^{*} \right\}, \]

\[ + \rho_2(p_L, p_R, m; \epsilon) \chi_{\text{con}}^{(N,1)}(p_L, -p_R, m; \tau, z) \left[ \chi_{\text{con}}^{(N,1)}(p_L, -p_R, m; \tau, z) \right]^{*} \right\}, \]

where \( \chi_{\text{con}}^{(N,1)}(p, m) \) denotes the extended continuous character (B.1) explicitly written as

\[ \chi_{\text{con}}^{(N,1)}(p, m; \tau, z) = q^{\frac{\tau}{2\pi}} \Theta_{m, N} \left( \frac{2z}{N} \right) \theta_1(\tau, z) \frac{1}{\eta(\tau)^2}. \]

In the continuous part, the “density functions” \( \rho_1 (\rho_1') \) and \( \rho_2 (\rho_2') \) have rather complicated forms. As expected, \( \rho_1 (\rho_1') \) includes the logarithmically divergent term as the leading contribution,

\[ \rho_1(p_L, p_R, m; \epsilon) = C |\ln \epsilon| \delta(p_L - p_R) + \cdots, \]

with some constant \( C \), which corresponds to the strings freely propagating in the asymptotic region, and \( C |\ln \epsilon| \) is roughly identified as an infinite volume factor. However, both \( \rho_1 (\rho_1') \) and \( \rho_2 (\rho_2') \) include subleading, non-diagonal terms with \( p_L \neq p_R \) and considerably non-trivial dependence on \( m \), as mentioned in [14].

Now, we would like to prove the equalities

\[ \rho_1(p_L, p_R, m; \epsilon) = \rho_1'(p_L, p_R, m; \epsilon), \quad \rho_2(p_L, p_R, m; \epsilon) = \rho_2'(p_L, p_R, m; \epsilon). \]  

(3.24)

If this is the case, by using the “charge conjugation relation”

\[ \chi_{\text{con}}^{(N,1)}(p, m; \tau, -z) = -\chi_{\text{con}}^{(N,1)}(p, -m; \tau, z), \]

as well as (3.16), we obtain

\[ Z_A^R(\tau, \tilde{\tau}; -z, \bar{z}; \epsilon) = e^{-2\pi i \frac{\epsilon}{12}} \left[ \sum_{v=0}^{N-1} \sum_{a \in \mathbb{Z}_N^2} \hat{\chi}^{(N,1)}_{\text{dis}}(N - v, v + a; \tau, z) \left[ \hat{\chi}^{(N,1)}_{\text{dis}}(v, a; \tau, z) \right]^{*} \right. \]

\[ - \sum_{m \in \mathbb{Z}_N^2} \int dp_L \int dp_R \left\{ \rho_1(p_L, p_R, m; \epsilon) \chi_{\text{con}}^{(N,1)}(p_L, p_R, m; \tau, z) \left[ \chi_{\text{con}}^{(N,1)}(p_L, p_R, m; \tau, z) \right]^{*} \right\}, \]

\[ + \rho_2(p_L, p_R, m; \epsilon) \chi_{\text{con}}^{(N,1)}(p_L, p_R, m; \tau, z) \left[ \chi_{\text{con}}^{(N,1)}(p_L, p_R, m; \tau, z) \right]^{*} \right\}, \]

which leads to the desired formula (3.17).

---

4 The minus sign just originates from the \( \theta_1 \) factor.
Proof of (3.24). We start by recalling the “orbifold relation” between $Z^\mathcal{R}(A)$ and $Z^\mathcal{R}_V$, which is shown in [14,26]:

$$Z^\mathcal{R}_A(\tau, \bar{\tau}, z, \bar{z}; \epsilon) = \frac{\epsilon^{2\pi i |z|^2}}{N} \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_N} Z_{V,(\alpha_1, \alpha_2)}(\tau, \bar{\tau}, z, \bar{z}; \epsilon), \quad (3.27)$$

where we introduced the “twisted partition function”⁵ in the right-hand side of (3.27):

$$Z^\mathcal{R}_V(\tau, \bar{\tau}; z, \bar{z}; \epsilon) = N \epsilon^{\frac{2\pi N^2 + 2}{N^2}} \sum_{n_1, n_2 \in \mathbb{Z}} \int_{\Sigma(z, \epsilon)} d^2u \frac{e^{-4\pi u^2/\tau_2}}{|\theta_1(\tau, u + \frac{N^2 + 2}{N})|^2} \times e^{-\pi i (n_1 + \alpha_1)\tau + (n_2 + \alpha_2)^2} e^{2\pi i (Nn_2 + \alpha_2)x_1 - (Nn_1 + \alpha_1)x_2}. \quad (3.28)$$

We can rewrite (3.28) by means of the Poisson resummation:

$$Z^\mathcal{R}_V(\tau, \bar{\tau}; z, \bar{z}; \epsilon) = e^{-\frac{2\pi i N^2}{N^2} \sum_{\ell \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} e^{2\pi i i N^2 (\ell + \ell')}} \cdot \frac{1}{2\pi i} \left[ \int_{\mathbb{R} - i0} dp e^{-\epsilon(v + i\ell)} yq^{\ell} \left[ \chi(\tau, z; \epsilon) \right]^* - \int_{\mathbb{R} + i(N - 0)} dp e^{\epsilon(v + i\ell)} \right] \times e^{-\frac{2\pi i N^2}{N^2} \sum_{p - iv} (yq^{\ell})^* \left[ \chi(\tau, z; \epsilon) \right]^* - \sum_{p - iv} \frac{yq^{\ell} yq^{\ell'}}{1 - yq^{\ell}} \left[ \chi(\tau, z; \epsilon) \right]^*}, \quad (3.29)$$

where we set $\epsilon := 2\pi \frac{N}{N} \epsilon$. A closely related analysis is given in [14]. We obviously find

$$Z^\mathcal{R}_V((0, 0); \tau, \bar{\tau}; z, \bar{z}; \epsilon) = Z^\mathcal{R}_V(\tau, \bar{\tau}; z, \bar{z}; \epsilon),$$

and (3.29) immediately implies the relation

$$Z^\mathcal{R}_V((0, 0); \tau, \bar{\tau}; z, \bar{z}; \epsilon) = Z^\mathcal{R}_V(\tau, \bar{\tau}; z, \bar{z}; \epsilon), \quad (3.30)$$

where $s_{(\alpha_1, \alpha_2)} (\alpha_i \in \mathbb{Z})$ denotes the spectral flow operator (2.7) acting only on the left-mover:

$$s_{(\alpha_1, \alpha_2)} \cdot f(\tau, \bar{\tau}, z, \bar{z}) := (-1)^{\alpha_1 + \alpha_2} q^{2\bar{a}_1} y^{\bar{a}_1} e^{2\pi i \frac{1}{N} \frac{\alpha_2}{N} \frac{\alpha_2}{N}} f(\tau, \bar{\tau}, z + \alpha_1 \tau + \alpha_2, \bar{z}). \quad (3.31)$$

Finally, by using the identities

$$\frac{1}{N} \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_N} s_{(\alpha_1, \alpha_2)} \cdot \chi^{(N, 1)}_\text{dis}(v, a; \tau, z) = \chi^{(N, 1)}_\text{dis}(v, -v - a; \tau, z), \quad (3.32)$$

$$\frac{1}{N} \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_N} s_{(\alpha_1, \alpha_2)} \cdot \chi^{(N, 1)}_\text{con}(p, m; \tau, z) = \chi^{(N, 1)}_\text{con}(p, -m; \tau, z), \quad (3.33)$$

which are easily proven by direct calculation,⁶ we can achieve the required identities (3.24). □

---

⁵ The relation of the notations here and in [26] is as follows:

$$Z^\mathcal{R}_A(\tau, \bar{\tau}; z, \bar{z}; \epsilon) = Z^\text{reg}_\text{reg}(\tau, z; \epsilon), \quad Z^\mathcal{R}_V(\tau, \bar{\tau}; z, \bar{z}; \epsilon) = Z^\text{reg}_\text{reg}(\tau, z; \epsilon),$$

$$Z^\mathcal{R}_V(\beta_1, \beta_2; z, \bar{z}; \epsilon) = Z^\text{reg}_\text{reg}(\tau, z; \beta_1, \beta_2; \epsilon).$$

⁶ Equation (3.33) is essentially the same identity as (2.35).
3.4. Other spin structures

We finally briefly discuss other spin structures. We shall assume the diagonal spin structures and the non-chiral GSO projection.

We start with the ansatz of the total partition function:

\[ Z(\tau, \bar{\tau}; z, \bar{z}) = \frac{1}{2} \sum_{\sigma = \text{NS, NS, R, \bar{R}}} \left[ Z^{(\sigma)}_V(\tau, \bar{\tau}; z, \bar{z}) + \varepsilon(\sigma) Z^{(\sigma)}_A(\tau, \bar{\tau}; -z, -\bar{z}) \right], \tag{3.34} \]

where the partition functions with left–right symmetric spin structures \( \sigma = \text{NS, NS, R, \bar{R}} \) are evaluated by means of path integration as in (3.19) and (3.21). Relevant calculations for general spin structures as well as helpful formulas are presented in [31].  

We set the sign factor \( \varepsilon(\bar{R}) = 1 \) to reproduce (3.17) when setting \( \sigma = \bar{R} \), and we have the following two possibilities of \( \varepsilon(\sigma) \) for other spin structures that are compatible with the modular invariance:

(i) \( \varepsilon(\sigma) = 1, \forall \sigma : \) “non-geometric deformation of \( SL(2)/U(1) \) supercoset”: With this naive choice, the continuous parts of partition functions of the vector and axial types are common and appear with the same sign, contrary to the \( \bar{R} \) case. This feature just originates from the simple fact that \( \theta_3(z), \theta_4(z), \theta_2(z) \) are even functions of \( z \) although \( \theta_1(z) \) is an odd function. Thus, there are continuous excitations in the sectors \( \sigma = \text{NS, NS, \bar{R}} \) as in the standard \( SL(2)/U(1) \) supercoset.

On the other hand, the discrete part of each spin structure is described by the following building blocks \( (v \in \mathbb{Z}_N, a \in \frac{1}{2} + \mathbb{Z}_N \text{ for } \sigma = \text{NS, NS, } a \in \mathbb{Z}_N \text{ for } \sigma = \text{R, \bar{R}}) \):

\[
\begin{align*}
X_{+}^{(\text{NS})}(v, a; \tau, z) &:= \hat{\chi}_{\text{dis}}^{(N,1)[\text{NS}]}(v, a; \tau, z) + \hat{\chi}_{\text{dis}}^{(N,1)[\text{NS}]}(N - v, a + v; \tau, z), \\
X_{-}^{(\text{NS})}(v, a; \tau, z) &:= \hat{\chi}_{\text{dis}}^{(N,1)[\text{NS}]}(v, a; \tau, z) - \hat{\chi}_{\text{dis}}^{(N,1)[\text{NS}]}(N - v, a + v; \tau, z), \\
X_{-}^{(\text{R})}(v, a; \tau, z) &:= \hat{\chi}_{\text{dis}}^{(N,1)[\text{R}])(v, a; \tau, z) + \hat{\chi}_{\text{dis}}^{(N,1)[\text{R}])(N - v, a + v; \tau, z), \\
X_{+}^{(\text{R})}(v, a; \tau, z) &:= \hat{\chi}_{\text{dis}}^{(N,1)[\text{R}])(v, a; \tau, z) + \hat{\chi}_{\text{dis}}^{(N,1)[\text{R}])(N - v, a + v; \tau, z) \equiv (2.24),
\end{align*}
\tag{3.35}
\]

where we have explicitly indicated the spin structure. The explicit definitions of modular completions with general spin structures are presented in [31]. We remark that the functions \( X_{+}^{(\sigma)}(v, a) \) appearing in (3.35) are generically non-holomorphic except for the \( \bar{R} \)-sector.

This model shares the asymptotic cylindrical region with the radius \( 1 \sqrt{N \alpha'} \equiv \sqrt{1 \sqrt{N \alpha'}} \) with the standard \( SL(2)/U(1) \) (of the vector type) and the aspect of the propagating strings is almost the same. However, we have non-trivial modifications in the discrete spectrum, which leads us to the holomorphic elliptic genus (3.18) and would be non-geometric since they are never realized only within the cigar theory (or the trumpet theory).

(ii) \( \varepsilon(\sigma) = -1 \text{ for } \sigma = \text{NS, NS, R, and } \varepsilon(\bar{R}) = 1 : \) “Compactified \( SL(2)/U(1) \) supercoset”: This second possibility is more curious. In this case, the continuous sectors are canceled out for all

\[ \hat{c} = 1 + \frac{2K}{N}, K \in 2\mathbb{Z}_{>0} \text{ (} N \text{ and } K \text{ are not necessarily co-prime) as in [31]. This assumption is necessary when considering the chiral GSO projection.} \]
the spin structures, and the discrete parts are described respectively by

\[ X^{(N)\text{NS}}_{-}(v, a; \tau, z) := \tilde{X}^{(N,1)\text{NS}}_{\text{dis}}(v, a; \tau, z) - \tilde{X}^{(N,1)\text{NS}}_{\text{dis}}(N - v, a + v; \tau, z), \]

\[ X^{(N)\tilde{\text{NS}}}_{+}(v, a; \tau, z) := \tilde{X}^{(N,1)\tilde{\text{NS}}}_{\text{dis}}(v, a; \tau, z) + \tilde{X}^{(N,1)\tilde{\text{NS}}}_{\text{dis}}(N - v, a + v; \tau, z), \]

\[ X^{(N)\text{R}}_{-}(v, a; \tau, z) := \tilde{X}^{(N,1)\text{R}}_{\text{dis}}(v, a; \tau, z) - \tilde{X}^{(N,1)\text{R}}_{\text{dis}}(N - v, a + v; \tau, z), \]

\[ X^{(N)\tilde{\text{R}}}_{+}(v, a; \tau, z) := \tilde{X}^{(N,1)\tilde{\text{R}}}_{\text{dis}}(v, a; \tau, z) + \tilde{X}^{(N,1)\tilde{\text{R}}}_{\text{dis}}(N - v, a + v; \tau, z). \]

(3.36)

Namely, we achieve the total partition function in a very simple form:\(^8\)

\[
Z_{cSL(2)/U(1)}(\tau, \bar{\tau}; z, \bar{z}) = \frac{1}{4} e^{-\frac{\pi}{2} \bar{\tau} \bar{z}^2} \sum_{v \in \mathbb{Z}_N} \sum_{a \in \mathbb{Z}_N} \left[ X^{(N)\text{NS}}_{-}(v, a + \frac{1}{2}; \tau, z) \right]^2 + \left[ X^{(N)\text{R}}_{+}(v, a; \tau, z) \right]^2 + \left[ X^{(N)\tilde{\text{R}}}_{+}(v, a; \tau, z) \right]^2.
\]

(3.37)

This is a natural extension of (3.17) including all the spin structures, and should be directly compared with the \( N = 2 \) minimal model (with level \( N - 2 \)):

\[
Z_{\text{min}}(\tau, \bar{\tau}; z, \bar{z}) = \frac{1}{4} e^{-\frac{\pi}{2} \bar{\tau} \bar{z}^2} \left( 1 - \frac{3}{N} \right) \sum_{\sigma = \text{NS, NS}, \text{NS, R, R}} \sum_{m = 0}^{N-2} \left| \text{ch}^{(\sigma)}_{\ell, m}(\tau, z) \right|^2.
\]

(3.38)

All of the building blocks \( X^{(N)\sigma}_{\pm}(v, a) \) in (3.37) are holomorphic and can be rewritten in terms only of the extended characters (that are not modular completed) as in (3.5). In fact, the functions (3.36) for \( \sigma = \text{NS, NS}, \tilde{\text{NS}}, \text{R} \) are reproduced by the “half spectral flows” \( z \mapsto z + \frac{\ell + 1}{2}, z \mapsto z + \frac{\ell}{2}, z \mapsto z + \frac{1}{2} \). The absence of non-holomorphic corrections means that they are directly associated with some infinitely reducible representations of \( \mathcal{N} = 2 \) SCA with \( \hat{c} = 1 + \frac{2}{N} \). These representations are, however, non-unitary due to the relative minus sign appearing in the NS and R sectors. This aspect is in sharp contrast to the \( \mathcal{N} = 2 \) minimal model, in which the partition function (3.38) only includes the characters of unitary irreducible representations.

4. Summary and comments

In this paper, we have studied a possible “analytic continuation” with \( N \rightarrow -N \) of the \( \mathcal{N} = 2 \) minimal model with the central charge \( \hat{c} = 1 - \frac{2}{N} \). Namely, we have examined the problem of what is the superconformal system with \( \hat{c} = 1 + \frac{2}{N} \) that has (1.2) as its elliptic genus.

Our main results are summarized as follows:

1. The “Fourier expansion” of the function (1.2) is rewritten by a holomorphic linear combination of the modular completions of the extended discrete characters of the \( SL(2)/U(1) \) model [14].

---

\(^8\) The overall factor 1/4 is naturally interpreted as follows: 1/2 originates from the non-chiral GSO projection, while the remaining 1/2 can avoid the over-counting due to the obvious \( \mathbb{Z}_2 \) symmetry.

\[ X^{(N)\sigma}_{-}(v, a) = \pm X^{(N)\sigma}_{-}(N - v, a + v). \]
This result is exhibited in terms of the formulas (2.24) or (2.31), equivalently. This is similar to the fact that the elliptic genus of $\mathcal{N} = 2$ minimal model $Z_{\text{min}}^{(N-2)}(\tau, z)$ (1.1) is Fourier expanded by the characters associated with the Ramond ground states [11].

(2) The superconformal system corresponding to (1.2) is identified with a “compactified” model of the $SL(2)/U(1)$ supercoset, as is given by (3.17).

(3) Two possibilities of extension to general spin structures have been presented. One is a non-compact model regarded as a “non-geometric deformation” of the $SL(2)/U(1)$ supercoset, and the other is the natural extension of the compactified model (3.17). The latter is quite similar to the $\mathcal{N} = 2$ minimal model, although it is not a unitary theory.

The partition function (3.17) looks very like those of RCFTs. We only possess finite conformal blocks that are holomorphically factorized in the usual sense. However, there is a crucial difference from generic RCFTs defined axiomatically. The partition function (3.17) or the elliptic genus (3.18) does not include the contributions from the Ramond vacua saturating the unitarity bound $Q = \pm \frac{c}{2}$. This implies that the Hilbert space of normalizable states does not contain the NS vacuum ($h = Q = 0$) which should correspond to the identity operator. Of course, this feature is common with the spectrum of the original $SL(2)/U(1)$ supercoset read off from the torus partition function evaluated in [16] (see also [27–30]). It may be an interesting question whether or not the finiteness of conformal blocks without the identity representation, which is observed in our “compactified $SL(2)/U(1)$ model”, unavoidably leads to a non-unitarity of the spectrum in general conformal field theories.

A natural extension of this work would be the study of the cases of “fractional levels” $\hat{c} = 1 + \frac{2K}{N}$ ($K \geq 2$, $\text{GCD}(N, K) = 1$). In other words, one may seek a theory of which the elliptic genus would be

$$Z(\tau, z) = K \Phi^{(N/K)}(\tau, z) \equiv \frac{\theta_1(\tau, \frac{N+K}{N}z)}{\theta_1(\tau, \frac{K}{N}z)},$$

which has the Witten index $Z(\tau, z = 0) = N + K$. However, the function $\Phi^{(N/K)}(\tau, z)$ is only meromorphic with respect to the angle variable $z$, and such a function is not likely to be realized as the elliptic genus of any superconformal field theory.

We also point out that the cancellation of continuous parts such as (3.17) does not seem to happen in that case. This fact suggests that the “compactification” of the $SL(2)/U(1)$ supercoset works only for integer levels, that is, $\hat{c} = 1 + \frac{2}{N}.

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Appendix A. Conventions for theta functions

We assume $\tau \equiv \tau_1 + i\tau_2, \tau_2 > 0$ and set $q := e^{2\pi i \tau}, y := e^{2\pi i z}$.

$$\theta_1(\tau, z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1-q^m)(1-yq^m)(1-y^{-1}q^m),$$

$$\theta_2(\tau, z) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1-q^m)(1+yq^m)(1+y^{-1}q^m),$$

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\[ \theta_3(\tau, z) = \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^m - 1/2)(1 + y^{-1}q^m - 1/2), \]

\[ \theta_4(\tau, z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^m - 1/2)(1 - y^{-1}q^m - 1/2). \tag{A.1} \]

\[ \Theta_{m,k}(\tau, z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2})^2} y^{k(n+\frac{m}{2})}. \tag{A.2} \]

We also set

\[ \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{A.3} \]

The spectral flow properties of theta functions are summarized as follows \((m, n, a \in \mathbb{Z}, k \in \mathbb{Z}_{>0})\):

\[ \begin{align*}
\theta_1(\tau, z + m\tau + n) &= (-1)^{m+n} q^{-\frac{m^2}{2}} y^{-m} \theta_1(\tau, z), \\
\theta_2(\tau, z + m\tau + n) &= (-1)^n q^{-\frac{m^2}{2}} y^{-m} \theta_2(\tau, z), \\
\theta_3(\tau, z + m\tau + n) &= q^{-\frac{m^2}{2}} y^{-m} \theta_3(\tau, z), \\
\theta_4(\tau, z + m\tau + n) &= (-1)^n q^{-\frac{m^2}{2}} y^{-m} \theta_4(\tau, z), \\
\Theta_{a,k}(\tau, 2(z + m\tau + n)) &= q^{-km^2} y^{-2km} \Theta_{a+2km,k}(\tau, 2z). \tag{A.4} 
\end{align*} \]

**Appendix B. Summary of modular completions**

In this appendix we summarize the definitions as well as useful formulas for the “extended discrete characters” and their modular completions of the \(N = 2\) superconformal algebra with \(\hat{c}(\equiv \frac{c}{k}) = 1 + \frac{2}{k}\). We focus only on the \(\hat{R}\) sector,\(^9\) and when treating the extended characters, we assume \(k = N/K, (N, K \in \mathbb{Z}_{>0})\) (but not assuming that \(N\) and \(K\) are co-prime).

**Extended continuous (non-BPS) characters \([15,16]\)**

\[ \chi_{\text{con}}^{(N,K)}(p, m; \tau, z) := q^{\frac{p^2}{2N^2}} \Theta_{m,NK} \left( \frac{2z}{N} \right) \frac{\theta_1(\tau, z)}{i\eta(\tau)^3}. \tag{B.1} \]

This corresponds to the spectral flow sum of the non-degenerate representation with \(h = \frac{p^2 + m^2}{4NK} + \frac{\hat{c}}{8}, \)
\(Q = \frac{m}{N} \pm \frac{1}{2} (p \geq 0, m \in \mathbb{Z}_{2NK})\), whose flow momenta are taken to be \(n \in N\mathbb{Z}\). The modular and spectral flow properties are simply written as

\[ \begin{align*}
\chi_{\text{con}}^{(N,K)} \left( p, m; -\frac{1}{\tau}, \frac{z}{\tau} \right) &= (-i)e^{i\pi \frac{z}{2}} \int_{-\infty}^{\infty} dp' \\
&\times \sum_{m' \in \mathbb{Z}_{2NK}} e^{2\pi i \frac{p' - mm'}{2NK}} \chi_{\text{con}}^{(N,K)}(p', m'; \tau, z). \tag{B.2} 
\end{align*} \]

\(^9\)In this paper we shall use the convention of \(\hat{R}\)-characters with the same sign as \([26]\), and the inverse sign compared to those of \([14,16,17]\), so that the Witten indices appear with a positive sign. (See (B.30) below.)
\[ \chi_{\text{con}}^{(N,K)}(p, m; \tau + 1, z) = e^{2\pi i \frac{p^2 + m^2}{4NK}} \chi_{\text{con}}^{(N,K)}(p, m; \tau, z), \]  
\[ \chi_{\text{con}}^{(N,K)}(p, m; \tau, z + r\tau + s) = (-1)^{r+s} e^{2\pi i \frac{p\tau}{N}} q^{-\frac{r^2}{2}} y^{-r} \times \chi_{\text{con}}^{(N,K)}(p, m + 2Kr; \tau, z) \quad (\forall r, s \in \mathbb{Z}). \]  

\textit{Extended discrete characters} \[15-17\]

\[ \chi_{\text{dis}}^{(N,K)}(v, a; \tau, z) := \sum_{n \in a + N\mathbb{Z}} (y^q)^{\frac{y^n}{\eta}} q^{2kn} \theta_1(\tau, z) \big/ \eta(\tau)^3, \]  

This again corresponds to the sum of the Ramond vacuum representation with \( h = \frac{c}{8}, \ Q = \frac{v}{N} - \frac{1}{2} \) \( (v = 0, 1, \ldots, N) \) over spectral flow with flow momentum \( m \) taken to be mod \( N \), as \( m = a + N\mathbb{Z} \) \( (a \in \mathbb{Z}_N) \). If one introduces the notation of the Appell function or Lerch sum with level 2k \[18-20\],

\[ K^{(2k)}(\tau, z) := \sum_{n \in \mathbb{Z}} q^\frac{y^{2kn}n^{k}}{1 - yq^n}. \]  

\( \chi_{\text{dis}}(v, a) \) is identified as its Fourier expansion:

\[ y^{\frac{2K}{N}a} q^{\frac{k}{N}a^2} K^{(2N\mathbb{K})} \left( \frac{z + a\tau + \beta}{N} \right) \theta_1(\tau, z) = \sum_{n=0}^{N-1} e^{2\pi i \frac{\beta n}{N}} \chi_{\text{dis}}^{(N,K)}(v, a; \tau, z). \]  

We also note

\[ \chi_{\text{dis}}^{(N,K)}(N, a; \tau, z) = \chi_{\text{dis}}^{(N,K)}(0, a; \tau, z) - \Theta_{2K,a,NK} \left( \frac{2z}{N} \right) \theta_1(\tau, z) \big/ \eta(\tau)^3. \]  

The modular transformation formulas of \( \chi_{\text{dis}}^{(N,K)}(v, a) \) and \( K^{(2k)} \) can be expressed as \[15-17,19,20\]:

\[ \chi_{\text{dis}}^{(N,K)}(v, a; -\frac{1}{\tau}, \frac{z}{\tau}) = e^{i\pi \frac{c}{2} \frac{z^2}{\tau}} \left[ \sum_{v=0}^{N-1} \sum_{a \in \mathbb{Z}_N} \frac{1}{N} e^{2\pi i \frac{v^{+} - v + 2K\alpha(v + 2K\alpha)}} K^{(2N\mathbb{K})} \chi_{\text{dis}}^{(N,K)}(v, a; \tau, z) \right. \]

\[ - \frac{i}{2NK} \sum_{m' \in \mathbb{Z}_{2N\mathbb{K}}} e^{-2\pi i \frac{v^{+} + 2K\alpha}{2N\mathbb{K}}} \int_{\mathbb{R} + i0} dp' \frac{e^{-2\pi i \frac{v^{+} + 2K\alpha}{2N\mathbb{K}}} \chi_{\text{con}}^{(N,K)}(p', m'; \tau, z)}{1 - e^{-2\pi i \frac{v^{+} + 2K\alpha}{2N\mathbb{K}}}} \right], \]  

\[ K^{(2k)}(v, a; \tau + 1, z) = e^{2\pi i \frac{a}{K} (v + K\alpha)} K^{(2k)}(v, a; \tau, z), \]  

\[ K^{(2k)}(\tau, z + r\tau + s) \big/ \eta(\tau)^3 \]  

\[ \chi_{\text{dis}}^{(N,K)} \bigg( \frac{1}{\tau}, \frac{z}{\tau} \bigg) = \tau e^{i\pi \frac{c}{2} \frac{z^2}{\tau}} \left[ K^{(2k)}(\tau, z) - \frac{i}{\sqrt{2k}} \sum_{m \in \mathbb{Z}_{2k}} \int_{\mathbb{R} + i0} dp' \frac{q^{\frac{1}{2} p'^2}}{1 - e^{-2\pi i \frac{p' + i\pi}{2}} - \Theta_{m,k}(\tau, 2z)} \right], \]  

\[ K^{(2k)}(\tau + 1, z) = K^{(2k)}(\tau, z). \]
The spectral flow property is also expressed as
\[
\chi_{\text{dis}}^{(N,K)}(v, a; \tau, z + r\tau + s) = (-1)^{r+s} e^{2\pi i \frac{v+r+2Kas}{\pi}} q^{-\frac{s^2}{2}} y^{-ir} \times \chi_{\text{dis}}^{(N,K)}(v, a + r; \tau, z) \quad (\forall r, s \in \mathbb{Z}),
\]
\[
\mathcal{K}^{(2k)}(\tau, z + r\tau + s) = q^{-k\tau^2} y^{-2kr} \mathcal{K}^{(2k)}(\tau, z) \quad (\forall r, s \in \mathbb{Z}).
\]

**Modular completion of the extended discrete characters**

The modular completion of the discrete character $\chi_{\text{dis}}$ is defined as follows:

\[
\hat{\chi}_{\text{dis}}^{(N,K)}(v, a; \tau, z) := \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{a+N \in \mathbb{Z}} \left\{ \int_{\mathbb{R}+i(N-0)} dp - \int_{\mathbb{R}-i0} dp (yq^n) \right\}
\]
\[
\times e^{-\pi \frac{v^2 + r^2 + 2Kas}{\eta}} q^{\frac{K}{\pi} n^2} \frac{K}{p - i\tau} (yq^n) \frac{2K}{\eta} q^{\frac{K}{\pi} n^2}
\]
\[
= \chi_{\text{dis}}^{(N,K)}(v, a; \tau, z) + \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{a+N \in \mathbb{Z}} \int_{\mathbb{R}-i0} dp e^{-\pi \frac{v^2 + r^2}{\eta}} (yq^n) \frac{2K}{\eta} q^{\frac{K}{\pi} n^2},
\]
\[
(B.15)
\]

This expression (B.15) is obviously periodic with respect to both of $v$ and $a$:

\[
\hat{\chi}_{\text{dis}}^{(N,K)}(v + Nm, a + Nn; \tau, z) = \hat{\chi}_{\text{dis}}^{(N,K)}(v, a; \tau, z) \quad (\forall m, n \in \mathbb{Z}).
\]
\[
(B.16)
\]

In particular, we have

\[
\hat{\chi}_{\text{dis}}^{(N,K)}(N, a; \tau, z) = \hat{\chi}_{\text{dis}}^{(N,K)}(0, a; \tau, z),
\]
\[
(B.17)
\]

in spite of (B.8).

The modular completion of Appell function $\mathcal{K}^{(2k)}(\tau, z)$ is given as [20]:

\[
\hat{\mathcal{K}}^{(2k)}(\tau, z) := \mathcal{K}^{(2k)}(\tau, z) - \frac{1}{2} \sum_{m \in \mathbb{Z}_{2k}} R_{m,k}(\tau) \Theta_{m,k}(\tau, 2z),
\]
\[
(B.18)
\]

where we set

\[
R_{m,k}(\tau) := \frac{1}{i\tau} \sum_{r \in m+2k \mathbb{Z}} \int_{\mathbb{R}-i0} dp e^{-\pi \frac{r^2 + 2\tau r}{\eta}} q^{-r^2/4\tau},
\]
\[
(B.19)
\]

which is generically non-holomorphic due to the $\tau_2$ dependence.

One can easily show

\[
R_{m+2ks,k}(\tau) = R_{m,k}(\tau) \quad (\forall s \in \mathbb{Z}), \quad R_{m,k}(\tau) = 2\theta_{m,0}^{(2k)}(\tau) - R_{-m,k}(\tau),
\]
\[
(B.20)
\]

and thus

\[
R_{0,k}(\tau) \equiv 1, \quad R_{k,k}(\tau) \equiv 0,
\]
\[
(B.21)
\]

holds in particular.
The “Fourier expansion relation” (B.7) is inherited to the modular completions:

\[
y^{2K}a qK a^2 \hat{\mathcal{K}}(2NK) \left( \tau, \frac{z + a\tau + b}{N} \right) \frac{\theta_1(\tau, z)}{\eta(\tau)^3} = \sum_{v \in \mathbb{Z}_N} e^{2\pi i \frac{v}{N}} \hat{\mathcal{X}}_{\text{dis}}^{(N,K)}(v, a; \tau, z).
\]  

(B.22)

The modular transformation formulas for the modular completions (B.15) and (B.18) are written as

\[
\hat{\mathcal{X}}_{\text{dis}}^{(N,K)}(v, a; -\frac{1}{\tau}, \frac{z}{\tau}) = e^{i\pi \frac{v}{\tau}} \sum_{v' = 0}^{N-1} \sum_{a' \in \mathbb{Z}_N} \frac{1}{N} e^{2\pi i \frac{v' - (v + ka)(v' + ka')}{2NK}} \hat{\mathcal{X}}_{\text{dis}}^{(N,K)}(v', a'; \tau, z).
\]

(B.23)

\[
\hat{\mathcal{X}}_{\text{dis}}^{(N,K)}(v, a; \tau + 1, z) = e^{2\pi i \frac{v}{N} (v + Ka)} \hat{\mathcal{X}}_{\text{dis}}^{(N,K)}(v, a; \tau, z),
\]

(B.24)

\[
\hat{\mathcal{K}}^{(2k)} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = \tau e^{i\pi \frac{v}{\tau} z^2} \hat{\mathcal{K}}^{(2k)}(\tau, z),
\]

(B.25)

\[
\hat{\mathcal{K}}^{(2k)}(\tau + 1, z) = \hat{\mathcal{K}}^{(2k)}(\tau, z).
\]

(B.26)

When compared with (B.9) and (B.11), the S-transformation formulas have been simplified because of the absence of continuous terms.

Also, the spectral flow property is preserved by taking the completion

\[
\hat{\mathcal{X}}_{\text{dis}}^{(N,K)}(v, a; \tau, z + r \tau + s) = (-1)^{r+s} e^{2\pi i \frac{v+2ka}{N} \tau} q^{-\frac{kr^2}{2}} y^{-kr} \times \hat{\mathcal{X}}_{\text{dis}}^{(N,K)}(v, a + r; \tau, z) \quad (\forall r, s \in \mathbb{Z}),
\]

(B.27)

\[
\hat{\mathcal{K}}^{(2k)}(\tau, z + r \tau + s) = q^{-kr^2} y^{-2kr} \hat{\mathcal{K}}^{(2k)}(\tau, z) \quad (\forall r, s \in \mathbb{Z}).
\]

(B.28)

Note that the IR part of modular completions are evaluated as

\[
\left[ \hat{\mathcal{X}}_{\text{dis}}^{(N,K)}(v, a; \tau, z) \right] \left( \equiv \lim_{\tau_2 \to \infty} \hat{\mathcal{X}}_{\text{dis}}^{(N,K)}(v, a; \tau, z) \right) = \begin{cases} 
\delta_{a,0}^{(N)} \frac{1}{2} (y^{-\frac{1}{2}} + y^{\frac{1}{2}}) & (v = 0, N) \\
\delta_{a,0}^{(N)} y^{\frac{v}{N} - \frac{1}{2}} & (v = 1, \ldots, N - 1),
\end{cases}
\]

(B.29)

while we have

\[
\left[ \hat{\mathcal{X}}_{\text{dis}}^{(N,K)}(v, a; \tau, z) \right] = \delta_{a,0}^{(N)} y^{\frac{v}{N} - \frac{1}{2}}, \quad (v = 0, \ldots, N).
\]

(B.30)

**Appendix C. Finiteness of $R_{a,b}(\tau)$ at the cusp $\tau = i\infty$**

In this appendix we confirm the finiteness of the residue function $R_{a,b}(\tau)$ (2.42) at the cusp $\tau = i\infty$. Namely, we will prove

\[
\lim_{\tau \to i\infty} \left| R_{a,b}(\tau) \right| < \infty,
\]

(C.1)

$\forall (a, b) \in \mathbb{Z}_{N+1} \times \mathbb{Z}_{N+1} - \{(0, 0)\}$.

We classify $z_{a,b}$ (2.41) into three groups and examine the behavior of $F(\tau, z)$ (2.40) around them separately:
(i) \( a = 0, b = 1, \ldots, N \): In this case, we have \( z_{0,b} \equiv \frac{b}{N+1}, \ (b = 1, \ldots, N) \). All of them are simple zeros of the function

\[
\left[ \Phi^{(N)}(\tau, Nz) \right] = \left[ \Phi^{(N)'}(\tau, Nz) \right] = y^{-\frac{N}{2}} \sum_{j=0}^{N} y^{j} = y^{-\frac{N}{2}} \frac{1 - y^{N+1}}{1 - y}.
\]

This means

\[
\lim_{\tau \to i\infty} F(z_{0,b}) = 1 \quad (\forall \ b = 1, \ldots, N),
\]

and thus we obtain

\[
\lim_{\tau \to i\infty} \left| R_{0,b}(\tau) \right| = 0 \quad (\forall \ b = 1, \ldots, N).
\]

(ii) \( a = 1, \ldots, N-1, b = 0, \ldots, N \): In this case, we first note

\[
\zeta_{a,b} = \xi_{a,b} + \tilde{\zeta}_{a,b}, \quad \xi_{a,b} := \frac{a\tau + b}{N}, \quad \tilde{\zeta}_{a,b} := -\frac{a\tau + b}{N(N+1)}, \quad (C.2)
\]

and the term \( \xi_{a,b} \) can be interpreted as the spectral flow caused by \( s_{(a,b)} \). Therefore, it is enough to compare the behaviors of \( \Phi^{(N)}(a,b)(\tau, Nz) \) and \( \Phi^{(N)'}(a,b)(\tau, Nz) \) around \( z = \zeta_{a,b} \) in place of examining \( F(\tau, z) \) around \( z = \zeta_{a,b} \).

Recalling

\[
\Phi^{(N)}(a,b)(\tau, Nz) = y^{a} \theta_{1}(\tau, (N+1)z + \xi_{a,b}), \quad (a \neq 0)
\]

we obtain

\[
\Phi^{(N)}(a,b)(\tau, Nz) \sim (z - \zeta_{a,b}) q^{-\frac{a^{2}}{N(N+1)} + \frac{a}{N(N+1)}}, \quad (z \sim \zeta_{a,b}, \quad \tau \sim i\infty). \quad (C.3)
\]

On the other hand, since \( \left[ \Phi^{(N)'}(a,b)(\tau, Nz) \right] = y^{-\frac{N}{2}} + a, \ (a \neq 0) \) holds, we also obtain

\[
\Phi^{(N)'}(a,b)(\tau, Nz) \sim \left( e^{2\pi i \bar{\zeta}_{a,b}} \right)^{-\frac{N}{2} + a} q^{-\frac{a^{2}}{N(N+1)} + \frac{a}{N(N+1)}}, \quad (z \sim \bar{\zeta}_{a,b}, \quad \tau \sim i\infty). \quad (C.4)
\]

We thus conclude

\[
\lim_{\tau \to i\infty} \left| R_{a,b}(\tau) \right| < \infty \quad (a = 1, \ldots, N-1, \ b = 0, \ldots, N).
\]

(iii) \( a = N, b = 0, \ldots, N \): Around \( z = z_{N,b} \), we find \( \theta_{1}(\tau, (N+1)z) \sim (z - z_{N,b}) q^{-\frac{N^{2}}{2} + \frac{1}{2}} \) and \( \theta_{1}(\tau, z) \sim q^{-\frac{N^{2}}{2} + \frac{1}{2}} \), and hence

\[
\Phi^{(N)}(\tau, Nz) \sim (z - z_{N,b}) q^{-\frac{N^{2}}{2} + \frac{N}{N(N+1)}}, \quad (z \sim z_{N,b}, \quad \tau \sim i\infty). \quad (C.5)
\]

On the other hand, by using the decomposition (C.2), we can evaluate \( \Phi^{(N)'}(\tau, Nz) \) around \( z = z_{N,b} \) as follows:

\[
\Phi^{(N)'}(\tau, Nz) = \Phi^{(N)'}(\tau, N(\xi_{N,b} + \tilde{\zeta}_{N,b})) \\
\sim q^{-\frac{1}{2}N(N+2)} e^{-2\pi iN(\xi_{N,b} + \tilde{\zeta}_{N,b})} \times \left[ \Phi^{(N)'}(\tau, N\tilde{\zeta}_{N,b}) \right] \\
\sim q^{-\frac{1}{2}N(N+2)} \left( 2\pi i \bar{\xi}_{N,b} \right)^{-N(N+2) + \frac{N}{2}} \\
\sim q^{-\frac{N^{2}}{2} + \frac{N}{N(N+1)}}, \quad (z \sim z_{N,b}, \quad \tau \sim i\infty). \quad (C.6)
\]

Consequently, we again obtain

\[
\lim_{\tau \to i\infty} \left| R_{N,b}(\tau) \right| < \infty \quad (b = 0, \ldots, N).
\]
In this way, we have shown that the residue function $R_{a,b}(\tau)$ is finite at the cusp $\tau = i\infty$ for every simple pole $z = z_{a,b}$.

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**References**