The growth of chiral magnetic instability in a large-scale magnetic field

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The chiral magnetic effect emerges from a microscopic level, and its interesting consequences have been discussed in relation to the dynamics of the early universe, neutron stars, and quark–gluon plasma. An instability is caused by an anomalous electric current along the magnetic field. We investigate the effects of plasma motion on the instability in terms of linearized perturbation theory. A magnetic field can inhibit magnetohydrodynamic waves to a remarkable degree, and thereby affects the instability mode. We also found that the unstable mode consists of coupling between Alfvén and one of the magneto-acoustic waves. Therefore, the propagation of a mixed Alfvén wave driven by magnetic tension is very important. The direction of the unperturbed magnetic field favors the wave propagation of the instability mode, when Alfvén speed exceeds sound speed.

Subject Index  E14, E20, E32

1. Introduction

It is widely recognized that magnetohydrodynamics (MHD) is capable of describing a variety of astrophysical phenomena. The treatment is macroscopic, consisting of fluid motions coupled with electromagnetic forces. A set of MHD equations is scale independent, and may be applied to the laboratory and to astrophysical plasma. The electromagnetic fields are described by the classical Maxwell equations. They have a symmetry with respect to parity, that is, a transformation property under spatial inversion. Physical vectors can equate only vectors of the same kind. An example is Ohm’s law, \( \vec{j} = \sigma \vec{E} \) with a scalar \( \sigma \), where the electric field \( \vec{E} \) and electric current \( \vec{j} \) are polar vectors. The magnetic vector \( \vec{B} \) is an axial one, and is connected to the polar current vector \( \vec{j} \) as \( \nabla \times \vec{B} = 4\pi \vec{j} / c \) (see, e.g., Ref. [1]).

A peculiar form of electric current arises from a microscopic level:

\[
\vec{j} = \kappa \vec{B}. \tag{1}
\]

This form means that \( \kappa \) is not scalar but pseudo-scalar by the parity transformation. A possible origin of the form in Eq. (1) is a quantum anomaly known as the chiral magnetic effect (see, e.g., Refs. [2–6] and references therein). There is an imbalance between left-handed and right-handed particles in a quantum system, and current flow along the magnetic field emerges at a macroscopic level. It is known that the electric current along the magnetic field causes an instability, leading to a growth of the magnetic field (e.g., [7–9]). Magnetic helicity, which is an indicator of a global topology, is also changed as well as the field amplification. It has been discussed that the chiral magnetic effect leads to an inverse cascade, that is, energy transfer from small to large scales. The problem has been studied

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from various points of view [10–15]. This property is an important process in the self-organization of a turbulent structure.

In recent years, the chiral magnetic effect has been widely discussed in relation to quark–gluon plasma in heavy-ion collision experiments [16], astrophysical consequences in the early universe [7,10,17], core-collapse supernovae [18,19], and magnetars [20]. The electric current of Eq. (1) is also discussed in the context of a mean-field MHD dynamo (e.g., [21–23]). In the theory, the mean values of the variables can be distinguished from the fluctuating ones. Thus, the electric current in Eq. (1) on a large scale is caused as an ensemble of screw-like vortices in microscopic turbulence.

Direct numerical simulations of the MHD dynamo have been developed with the increase in computer power. In the approach, dynamics of all scales is simultaneously followed as far as small-scale waves are numerically resolved. Thus, a model of microscopic turbulence is no longer needed there. Indeed, some chiral MHD simulations have been performed in a non-relativistic framework [24,25] and in a relativistic framework [20]. Their remarkable results demonstrate the ability of the method. However, the computational cost may be high in high-resolution simulations. Another example of the form in Eq. (1) is force-free magnetic fields (e.g., [21,26]), in which the Lorentz force vanishes: \( \vec{j} \times \vec{B} = 0 \). In the stationary case, \( \kappa \) is constant along a magnetic field line. The magnetosphere around a star is modeled by a force-free approximation, in which the magnetic pressure is assumed to be much larger than the thermal.

The macroscopic dynamics are governed by the same equations, although the transport coefficient \( \kappa \) determined by a microscopic process varies in magnitude. Bearing various astrophysical environments in mind, it is important to explore the instability in a wide range of parameters. In this paper, we consider normal mode analysis for a linearized system of chiral MHD equations. The background state is assumed to be homogeneous, with uniform magnetic field, and small perturbations propagate as MHD waves in the absence of the chiral magnetic effect. We study the modification of wave propagation and the instability caused by the chiral magnetic effect in Sect. 2. This problem has been partially studied [24], where the modification is found, but the general property is not clear. The reason will be discussed after our results. We extensively analyze it to explore the relevance of the instability in various astrophysical environments. We discuss our results in Sect. 3.

2. Waves in a linearized system

2.1. Equations

A set of chiral MHD equations have been discussed in the literature (e.g., [24,25]). The linear perturbation equations in non-relativistic dynamics are summarized here. We assume that the unperturbed state of the medium is static and homogeneous, i.e., with density \( \rho_0 \) and magnetic field \( \vec{B} = B_0 \hat{z} \), where \( \rho_0 \) and \( B_0 \) are constant. We write a small perturbation of a quantity \( Q \) as \( \delta Q \); the perturbation equations are then given by

\[
\frac{\partial}{\partial t} \delta \vec{B} = -\vec{\nabla} \times (c \delta \vec{E}),
\]

\[
c \delta \vec{E} = -\delta \vec{v} \times \vec{B}_0 + \eta \vec{\nabla} \times \delta \vec{B} - \kappa \delta \vec{B},
\]

\[
\rho_0 \frac{\partial}{\partial t} \delta \vec{v} = -\vec{\nabla} (c_s^2 \delta \rho) + \frac{1}{4\pi} (\vec{\nabla} \times \delta \vec{B}) \times \vec{B}_0,
\]

where \( c_s \) is the sound speed.
where an adiabatic relation $\delta p = c_s^2 \delta p$ and Ampère’s law $4\pi \delta j = c \vec{\nabla} \times \delta \vec{B}$ are used. An electric current parallel to the magnetic field is added by the chiral magnetic effect in Eq. (3). We denote the sound speed as $c_s$ and also the Alfvén speed as $c_a = (B_0^2/(4\pi \rho_0))^{1/2}$. There are two kinds of restoring forces, pressure and magnetic tension on the plasma motion. Their relative importance is inferred from the following ratio, which corresponds to the so-called plasma $\beta$: $\beta \approx (c_s/c_a)^2$. We assume that the electric resistivity $\eta$ and the coefficient $\kappa$ of the chiral magnetic effect are constant.

The latter is determined by the chiral chemical potential, i.e., the imbalance between left and right dispersion relation of Eq. (8) becomes

$$\text{Eigenvectors of perturbation functions satisfy}$$

$$\text{mode with } \lambda > \lambda_c \text{ decays. However, the mode with } \lambda = \lambda_c \equiv |\kappa|^{-1} \eta \text{ is unstable. Resistivity is inefficient for such a long-wavelength mode. Eigenvectors of perturbation functions satisfy}$$

$$\delta \vec{v} = (0, 0, 0), \quad \delta \vec{B} \propto (k_z, \pm ik, k_x).$$

2.2. **Chiral magnetic instability**

We first consider the case of $c_s = c_a = 0$. That is, there are no forces on plasma motions. The dispersion relation of Eq. (8) becomes

$$0 = [(\omega + i \kappa k^2 + \kappa^2 k^2) \omega^4]$$

Non-zero solutions are given by $\omega = -i \kappa k^2 \pm ik \kappa$. The mode with $\omega = -i (\kappa k + \eta k^2)$ always decays. However, the mode with $\omega = i (\kappa k - \eta k^2)$ grows for $k < \eta^{-1} \kappa$. That is, the long-wavelength mode with $\lambda > \lambda_c$ is unstable. Resistivity is inefficient for such a long-wavelength mode. Eigenvectors of perturbation functions satisfy

$$\delta \vec{v} = (0, 0, 0), \quad \delta \vec{B} \propto (k_z, \pm ik, k_x).$$
These functions mean that the disturbance is purely a magnetic one, and is transverse to the wave vector, i.e., \( \vec{k} \cdot \delta \vec{B} = 0 \) and also \( \vec{k} \cdot \delta \vec{j} = 0 \).

Our concern is the chiral instability mode due to \( \kappa \neq 0 \), so that from now on we neglect the resistivity \( \eta = 0 \). The approximation is valid in the long-wavelength mode \( \lambda \gg \lambda_c \). This approximation simplifies the dispersion relation of Eq. (8), which is reduced to a cubic equation in \( \omega^2 \).

### 2.3. Effect of thermal pressure

By setting \( \eta = 0 \) and \( c_a = 0 \) in Eq. (8), we have the relation

\[
0 = (\omega^2 + \kappa^2 k^2) (\omega^2 - c_s^2 k^2) \omega^2. \tag{11}
\]

There are two non-trivial solutions. One is a chiral magnetic mode \( (\omega^2 = -\kappa^2 k^2) \), and the other a sound wave mode \( (\omega^2 = c_s^2 k^2) \). The sound wave is produced by compressional motion of matter and hence is a longitudinal mode, i.e., \( \vec{k} \cdot \delta \vec{v} \neq 0 \). We explicitly check this fact by the eigenfunctions:

\[
\delta \vec{v} \propto \vec{k} = (k_x, 0, k_z), \quad \delta \vec{B} = 0. \tag{12}
\]

As discussed in the previous subsection, the chiral mode is a transverse mode, \( \vec{k} \cdot \delta \vec{B} = 0 \). These two modes are completely decoupled. The growth rate of the chiral magnetic mode is not affected by the pressure.

### 2.4. Effect of a uniform magnetic field

In the case of \( c_s = 0 \) and \( \eta = 0 \), the dispersion relation of Eq. (8) is reduced to

\[
0 = [(\omega^2 - c_a^2 k_z^2)(\omega^2 - c_a^2 k_z^2) + \kappa^2 k^2 \omega^2] \omega^2. \tag{13}
\]

It is clear that two waves, the Alfvén mode and the fast MHD mode (or fast magneto-acoustic mode) are coupled by a \( \kappa \) term. The frequency of the slow one is zero \( (\omega^2 = 0) \) in the limit of \( c_s = 0 \). Two non-zero solutions are given by

\[
\frac{\omega^2}{k^2} = \frac{1}{2} [(1 + \cos^2 \theta) c_a^2 - \kappa^2 \pm Q^{1/2}], \tag{14}
\]

where

\[
Q = [(1 - \cos \theta)^2 c_a^2 - \kappa^2][(1 + \cos \theta)^2 c_a^2 - \kappa^2]. \tag{15}
\]

The function \( Q \) becomes negative in the range

\[
\frac{1}{(1 + |\cos \theta|)^2} < \frac{c_a^2}{\kappa^2} < \frac{1}{(1 - |\cos \theta|)^2}, \tag{16}
\]

and hence \( \omega \) becomes a complex number. Outside the range of Eq. (16), \( \omega \) is either real or pure imaginary. The nature of the solution \( \omega \) is thus classified into three regions in the \( c_a \kappa^{-1} - \theta \) plane, as shown in Fig. 1. In region I (the left part of the figure), where \( c_a \kappa^{-1} < (1 + |\cos \theta|)^{-1} \), the solution is pure imaginary, i.e., \( \omega_R = 0 \). On the other hand, \( \omega \) is real \( (\omega = 0) \) in region III (the top-right part of the figure). That is a stable wave region in which \( c_a \kappa^{-1} > (1 - |\cos \theta|)^{-1} \). In the intermediate region II, the frequency is a complex number, \( \omega = \omega_R + i \omega_I \). The mode becomes an oscillatory instability.
As wave parallel to the magnetic field is never stabilized for any value of \( \omega/\kappa \). Other curve plotted by a horizontal dotted line represents the Alfvén wave, which is expressed as a curve with constant velocity growing modes, and their characteristic growth rates are where the two functional forms correspond to the upper and lower signs in Eq. (14). There are two branches, which are approximated by Eq. (17). In the intermediate region II, the growth rate strongly depends on the propagation angle \( \theta \). Therefore, a curve with constant velocity \( \omega_R/(kk) \) becomes almost vertical in the left panel of Fig. 1. The other curve plotted by a horizontal dotted line represents the Alfvén wave, which is expressed as \( (\omega/k)^2 \approx c_a^2 \cos^2 \theta - k^2 \cot^2 \theta \) for \( c_a k^{-1} \ll 1 \).

Next we discuss the imaginary part, \( \omega_I/(kk) \). For small \( c_a k^{-1} \), the two solutions in Eq. (14) are approximated as

\[
\frac{\omega^2}{k^2} = \begin{cases} 
-k^{-2}c_a^4 \cos^4 \theta + \cdots, \\
-k^{-2} + (1 + \cos^2 \theta)c_a^2 + \cdots,
\end{cases} \tag{17}
\]

where the two functional forms correspond to the upper and lower signs in Eq. (14). There are two growing modes, and their characteristic growth rates are \( \omega_I \approx c_a^2 k^{-1} \) (slowly growing mode) and \( \omega_I \approx \kappa k \) (rapidly growing mode). The frequency of the former vanishes in the limit of \( c_a = 0 \). As \( c_a k^{-1} \) increases, both growth rates approach each other, and match on the critical line \( c_a k^{-1} = (1 + |\cos \theta|)^{-1} \). In the right panel of Fig. 1, some lines with constant \( \omega_I/(kk) \) are plotted. In region I there are two branches, which are approximated by Eq. (17). In the intermediate region II, the growth rate strongly depends on the propagation angle \( \theta \). For example, the wave perpendicular to the unperturbed magnetic field, i.e. \( \theta = \pi/2 \), is stabilized for \( c_a k^{-1} \geq 1 \). On the other hand, the wave parallel to the magnetic field is never stabilized for any value of \( c_a k^{-1} \).

We show the coupling of the Alfvén and fast MHD modes, which causes an unstable mode for \( c_a k^{-1} < 1 \). Figure 2 displays how the phase velocity \( \omega/k \) normalized by \( \kappa \) changes with the Alfvén velocity \( c_a \), for fixed propagation angle \( \theta = \pi/4 \). In the large limit of \( c_a k^{-1} \) there are two different modes, which are described by positive velocities. There are also negative-velocity modes, but they are physically the same as the positive ones. These different modes represent stable Alfvén and fast MHD waves. As \( c_a k^{-1} \) decreases, two velocities agree at a certain point \( (c_a k^{-1} \approx 3) \), where two modes convert to one oscillatory growing mode and one oscillatory decaying mode in a region of \( c_a k^{-1} < 3 \). The velocity \( \omega_R/(kk) \) further goes to 0, and \( \omega_R = 0 \) at \( c_a k^{-1} \approx 0.6 \). Two propagating

- **Fig. 1.** Contours of \( \omega_R/(kk) \) (left panel) and \( \omega_I/(kk) \) (right panel) in the \( c_a k^{-1} - \theta \) plane. In region I of the left panel, \( \omega_R/(kk) \) is 0, but \( \omega_R/(kk) \) increases with \( c_a k^{-1} \) in region II. Constant lines with \( \omega_R/(kk) = 1 \), 2 are plotted. In region III there are two stable waves, fast MHD and Alfvén waves for a fixed velocity \( \omega_R/(kk) \). In region I of the right panel, there are two kinds of growing modes. In region III of the right panel, \( \omega_I/(kk) \) is zero. Some constant lines are plotted for values of \( \omega_i/(kk) \) as labeled in the figure.
waves merge and change as standing waves for \( c_a \kappa \sim 0.6 \). Toward \( c_a \kappa \rightarrow 0 \) after that point, the frequencies change as \( \omega_I \rightarrow \pm \omega \) or \( \omega_I \rightarrow \pm \omega R \) with \( \omega R = 0 \). The perturbation amplitudes satisfy

\[
\delta v_x = -\frac{c_a^2 k}{\omega \cos \theta} \frac{\delta B_x}{B_0}, \quad \delta v_y = -\frac{c_a^2 k}{\omega} \frac{\delta B_y}{B_0}, \quad \delta v_z = 0,
\]

\[
(\omega^2 - c_a^2 k^2) \delta B_x = k \omega k \cos \theta \delta B_y, \quad \delta B_z = -\tan \theta \delta B_x.
\]  

The perturbation of the magnetic field is always perpendicular to the wave, since \( \vec{k} \cdot \delta \vec{B} \propto \vec{\nabla} \cdot \delta \vec{B} = 0 \). In order to study the direction of the plasma motion, we consider two limiting cases. For the wave parallel to the magnetic field \( (\theta = 0) \), plasma motion is also transverse to wave propagation, since \( \delta v_z = 0 \). There is no compression of matter, \( \vec{\nabla} \cdot \delta \vec{v} = 0 \) in this case. As \( \theta \rightarrow \pi/2 \), we have \( \delta v_y \rightarrow 0 \) as well as \( \delta v_z = 0 \). This means that the plasma motion is longitudinal, \( \delta \vec{v} \propto \vec{k} \), and is compressional \( \vec{\nabla} \cdot \delta \vec{v} \neq 0 \) in this case. At intermediate angles of wave propagation, the instability mode is a mixture of the properties of the two limiting cases.

### 2.5. Magnetohydrodynamical effects

In previous subsections, we have separately considered the effects of plasma motion driven by thermal pressure or magnetic tension on the chiral instability. We consider here the combined effect of \( c_a \neq 0 \) and \( c_s \neq 0 \). The dispersion relation of Eq. (8) is a cubic equation in \( \omega^2 \), so that a pair \((\pm \omega)\) is always a solution of it. It is also easy to understand that there is at least one solution of \( \omega^2 > 0 \), i.e., a stable wave.

Figure 3 shows the maximum growth rate \( \omega_I/(k \kappa) \) among four solutions in the \( c_s \kappa^{-1} - c_a \kappa^{-1} \) plane for the propagation angle \( \theta = \pi/4 \). A stable wave-propagation region is expressed by the upper part of the “\( \nu \)” shape. It is natural that there is a different nature that depends on the dominant force. In the high-\( \beta \) region (the lower right part of Fig. 3) there is an unstable mode. The growth rate is \( \omega_I/(k \kappa) \approx 1 \) in the limit of \( c_a \kappa^{-1} = 0 \), irrespective of \( c_a \kappa^{-1} \). Plasma motion driven by the dominant pressure force does not affect the instability, as discussed in Sect. 2.3. As \( c_a \kappa^{-1} \) increases with a fixed value of \( c_s \kappa^{-1} (> 1.5) \), \( \omega_I/(k \kappa) \) decreases and becomes 0 at \( c_a \kappa^{-1} \approx c_s \kappa^{-1} \). However, the
Fig. 3. Color contour of the maximum growth rate $\text{Im}(\omega/(k\kappa))$ in the $c_s\kappa^{-1}-c_a\kappa^{-1}$ plane. The propagation angle of the perturbation is $\theta = \pi/4$. All modes become stable propagation waves in the upper "n"-shaped region (the blue region in the figure).

Fig. 4. Three-dimensional display of mode coupling as a function of $c_a\kappa^{-1}$, which is chosen as the y axis ($0 \leq c_a\kappa^{-1} \leq 4$). The real and imaginary parts of a mode are shown on the x ($-4 \leq \text{Re}(\omega/k\kappa) \leq 4$) and z ($0 \leq \text{Im}(\omega/k\kappa) \leq 1$) axes. The left panel is for $c_a\kappa^{-1} = 0.75$, while the right one is for $c_a\kappa^{-1} = 2$. For large $c_a\kappa^{-1}$, all modes are stable waves characterized by a real frequency. They are identified by their propagation velocity as F (fast MHD), S (slow MHD), and A (Alfvén) waves, as labeled in the figure.

The unstable bound of $\omega_I > 0$ shifts to a larger value of $c_a\kappa^{-1}$ for a small $c_s\kappa^{-1}$, that is, the fat part of Fig. 3, where $c_a\kappa^{-1} > c_s\kappa^{-1}$.

In order to study the unstable mode, we show in Fig. 4 how the frequency of a mode changes with $c_a\kappa^{-1}$ for a fixed $c_s\kappa^{-1}$. In the case of $c_s\kappa^{-1} = 0.75$ (left panel of Fig. 4), all modes become stable waves for $c_a\kappa^{-1} \geq 3$. Their phase velocities $\omega/k$ characterize the waves, so that we identify the fast MHD, Alfvén, and slow MHD waves according to the absolute value of the velocity. It is also found that the unstable mode is caused by a coupling of the fast and Alfvén waves as $c_a\kappa^{-1}$ decreases. The slow one is always decoupled, and is a stable wave. This situation is the same as that considered in the previous subsection ($c_s = 0$), where the slow mode is decoupled as $\omega = 0$.

In the right panel of Fig. 4, we show the case of $c_s\kappa^{-1} = 2$. Like the previous case, three stable waves are identified for $c_a\kappa^{-1} \geq 3$. As $c_a\kappa^{-1}$ decreases, the coupling occurs between the slow and
Fig. 5. Stable wave region in the parameter space of $c_s/\kappa$ and $c_a/\kappa$. The region between the two “n”-shaped curves denotes stable wave propagation for a fixed angle. Three curves are plotted, for $\theta = \pi/8$, $\pi/4$, and $3\pi/8$. The stable wave region increases with an increase of the propagation angle $\theta$.

Alfvén modes. The fast mode is always decoupled. This point differs from that for $c_s\kappa^{-1} = 0.75$. The unstable region in Fig. 3 changes by an MHD mode, which the Alfvén mode couples with.

The Alfvén mode couples with the slow one in the high-$\beta$ region ($c_s > c_a$), whereas it couples with the fast one in the low-$\beta$ region ($c_a > c_s$). It is interesting to observe the behavior in the small-$c_a\kappa^{-1}$ region in the left panel of Fig. 4 (the case of $c_s\kappa^{-1} = 0.75$). The phase velocity $\omega_R/(k\kappa)$ of the instability mode sharply decreases around $c_a\kappa^{-1} = 0.75$. At that point, the velocity of the slow mode sharply increases. That is, the wave nature is exchanged. The unstable mode originates from a coupling of fast and Alfvén waves in a low-$\beta$ region, but the nature changes to be like the slow one in the high-$\beta$ region. At the same time, the stable mode behaves like the slow one in the large-$c_a$ region, but behaves like the fast one in the small-$c_a$ region.

We discuss how the stable wave region changes with the propagation angle $\theta$. Figure 5 shows the region for $\theta = \pi/8$, $\pi/4$, and $3\pi/8$. The instability is almost unchanged in the high-$\beta$ region ($c_s\kappa^{-1} > c_a\kappa^{-1}$), since $c_a$ is unimportant. However, the growth rate significantly depends on the angle $\theta$ in the low-$\beta$ region ($c_a\kappa^{-1} > c_s\kappa^{-1}$), where the Alfvén wave propagation affects the instability. As discussed in Sect. 2.4, the unstable region diminishes with an increase of $\theta$ for $c_a\kappa^{-1} > 1$. The Alfvén mode velocity goes to 0 in the direction orthogonal to the unperturbed magnetic field, $\theta \to \pi/2$. Accordingly, the growth is suppressed in the low-$\beta$ region with $c_a\kappa^{-1} > 1$.

A peculiar thing should be noted in the limit of $\theta = \pi/2$ ($\cos \theta = 0$). The behavior of $\cos \theta = 0$ differs from that of $\cos \theta \approx 0$. The dispersion relation for $\cos \theta = 0$ is analytically expressed, and shows that there is always one growing mode, for any values of $c_a \neq 0$ and $c_s \neq 0$. In an exactly perpendicular direction, the Alfvén wave propagation is prohibited, and the instability grows irrespective of magnetic field strength.

3. Summary and discussion

The chiral magnetic instability is inherent in an electromagnetic field with the electric current parallel to the magnetic field. We have taken into account the plasma motion in order to study its relevance in various environments. Assuming that disturbances are small, linearized equations of chiral MHD are examined. All modes are described by solving the resultant dispersion relation, no matter whether they are stable or unstable. The analysis is not new, as mentioned in the introduction. We should therefore discuss previous results [24], and especially insufficient points of the previous analysis.
The dispersion relation depends on three parameters: a set of independent ones is chosen as $c_a/\kappa$, $c_s/\kappa$, and $\theta$ in this paper. The authors of Ref. [24] fixed the ratio of $c_a/c_s$, and plotted the phase velocity of MHD waves as a function of propagation angle $\theta$ for $(c_s/\kappa)^2 = 0.1, 1, 10$. We found that this choice is not good to grasp the whole structure of the normal frequencies, as inferred from Fig. 3. Their parameters were also limited to the stable wave region, and growth rates were never discussed. The relation to the instability is unclear, although modified MHD waves are shown by weak chiral magnetic effect. Our choice is fixing the propagation angle first, and all normal frequencies are calculated in a wider parameter space. The growth rates are illustrated in a three-dimensional representation. Thus, we have successfully explored the mode coupling in the unstable region.

We next summarize our findings. When the chiral magnetic effect is small enough, small disturbances are described by three waves: Alfvén, and fast and slow MHD waves. An unstable mode originates from a coupling of the Alfvén and one of the magneto-acoustic waves. The coupling condition is determined by matching the phase velocities, and therefore the counterpart is the slow one for high-$\beta$ plasma ($c_s > c_a$), and the fast one for low-$\beta$ plasma ($c_s < c_a$). Astrophysically, the high-$\beta$ plasma is relevant to the early universe, and core-collapse supernovae and neutron stars, while low-$\beta$ plasma is relevant to a force-free magnetosphere. The Alfvén wave plays a very important role, since the magnetic perturbations are always transverse waves, both in the pure Alfvén mode and in the pure chiral mode.

The unstable mode grows regardless of the propagation direction in a high-$\beta$ plasma, where pressure is a dominant force and does not hinder the growth. As the value of $\beta \approx (c_s/c_a)^2$ decreases, magnetic tension becomes an important force on the plasma motion. Accordingly, the wave propagation velocity depends on the direction. In a low-$\beta$ regime with $c_a > \kappa$, three stable waves like in ordinary MHD appear by mismatching their phase velocities. The propagation of an unstable mode perpendicular to the background magnetic field is strongly constrained, i.e., disturbances propagate as stable waves in the direction.

In this way, we found that the wave propagation and growth of the unstable mode are significantly affected in the presence of a magnetic field on a large scale. In particular, in a low-$\beta$-regime, the direction parallel to the field is favored for unstable wave propagation, that is, the instability grows anisotropically. The situation may be related to structure formation with a larger coherent length of magnetic field, but the issue is a non-linear process and is beyond the scope of this paper.

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