Half-Period Theorem of Binary Black Holes

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The merging of event horizons of binary black holes is investigated. Although the recent development of the numerical study of binary black hole coalescence has shown that their apparent horizons can orbit for many periods, we study the orbital motion of event horizons. We discuss how many periods the event horizons of binary black holes orbit before their coalescence. We find that they soon merge into one horizon and that black holes cannot orbit for half a period, while apparent horizons can orbit many times.

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§1. Introduction

Recently, the binary black hole system has been paid much attention as a candidate for gravitational wave source. Indeed, numerical studies of binary black hole coalescence have advanced so that it is now possible to predict the time profile of gravitational radiation.\textsuperscript{1} Usually, in numerical simulation, black hole formation is examined by determining an apparent horizon. This is because the existence of the apparent horizon strongly indicates that of the event horizon. The event horizon is meaningful for an asymptotic observer, since he can observe the outside of the event horizon but not the apparent horizon. Moreover, the event horizon is a gauge-invariant concept; as for the apparent horizon, even its existence depends on timeslicing. Hence, we discuss the event horizon for the binary black hole system.

Some authors have confirmed that two apparent horizons can orbit around each other in the case of binary black hole coalescence.\textsuperscript{2,3} We discuss whether this picture also holds for the event horizon. The reason why we suspect that the behavior of the event horizon differs from that of the apparent horizon is that, in some studies of black hole formation, it is asserted that the event horizon soon settles to a single sphere.\textsuperscript{4,5} In other words, we speculate that insufficient time remains for the two black holes to orbit around each other.

It is known that the event horizon is generated by null geodesics without a future endpoint.\textsuperscript{6} In particular, the event horizon is a null hypersurface that may not be smooth at the past endpoints of null geodesic generators. The set of the past endpoints of the null geodesic generators of the event horizon is called the crease set.\textsuperscript{5,7} Since the crease set is a subset of the null hypersurface, it is an achronal set\textsuperscript{5,8,9} (no two points are connected by a timelike curve), as illustrated in Fig. 1. The fact that the crease set is achronal roughly implies that two event horizons coalesce at superluminal speed.
As a rough argument, let two black holes be located with separation $2R$. They will coalesce within the time interval on the order of $\Delta t = R/c$; we cannot expect any circular motion with this time duration. Hence, we expect that there is an upper bound of the rotation angle of binary black holes before their coalescence.

In the present article, we assert the existence of the upper bound of the rotation angle and demonstrate a half-period theorem stating that binary black holes cannot orbit for half a period in terms of their event horizons, assuming reflection symmetry with respect to the orbital surface. This assumption is required for technical reasons to formulate the notion of the half period of the binary black holes without ambiguities. Clearly, it is meaningless to say that binary black holes make one full orbit or half an orbit in terms of a coordinate system. We should formulate the half period of binary black holes without referring to a specific coordinate system.

The outline of our argument is as follows. We first consider the foliation of spacetime by a family of timelike curves and choose a reference timelike curve as a center of binary motion. Next, we determine the opposite direction, which we call the light ray opposite, of each black hole with respect to the center. Then, whether or not a black hole orbits for half a period from the initial configuration can be determined in terms of the light ray opposite. Finally, we show that each black hole cannot orbit for half a period irrespective of the choice of the foliation and center. This gives a possible means of measuring the amount of binary motion in a coordinate-free way.

We first explain more details of our scheme in the next section. In §3 we give several definitions and set up the notion. The main theorem is stated in §4. In §5, we give some remarks and implications.

**§2. Schematic picture**

Here, we provide a schematic discussion based on the topological notion of the event horizon to illustrate that half a period of binary rotation is typically the upper limit for the duration of a binary rotating era. From a Newtonian picture, one may think that the coalescence of binary black hole event horizons occurs after two black hole event horizons orbit around each other for several periods. A simple picture of the orbiting event horizons, however, makes us suspect that they merge before several periods have elapsed.
Here we simply consider two black holes with identical masses fated to coalesce. They are on a binary orbit that shrinks with the energy loss caused by gravitational radiation. In a coordinate system \((t, x^i)\) in which each \(x^i = \text{const}\) line is timelike, every comoving observer inside a black hole at some moment will stay within the black hole region, while the black hole moves around its binary orbit. This implies that the Newtonian picture of binary motion in Fig. 2 is incorrect.

We expect two possible scenarios for binary black hole coalescence, which we call the quasi-stationary scenario and rapid coalescence scenario. The quasi-stationary scenario is what we expect to occur in the quasi-stationary binary motion of black holes. In the quasi-stationary scenario, the event horizons form a toroidal event horizon along the binary orbit in half a period of binary motion (see Fig. 2). The rapid coalescence scenario is expected to occur when the binary black holes have insufficient orbital angular momentum. In this case, two black holes coalesce without forming a toroidal horizon (see Fig. 2). In both scenarios, binary black holes do not orbit for half a period before coalescence.

In the above argument, the term “half a period” is loosely used. It is in general difficult to determine whether the binary black holes make one full orbit without a specific coordinate system. The present work is our attempt to seek a rigorous statement corresponding to the above schematic discussion. For this purpose, we should study this problem in a generally covariant manner incorporating a technique based on causal studies of general relativity.

§3. Setup

First, we have to consider the amount of orbital rotation of binary black holes. This will be determined by introducing a global angular coordinate function. However, this is in general difficult owing to the lack of a standard coordinate system. For this reason, we give up considering one radian to be an upper bound of the orbital rotation angle and attempt to put a mark corresponding to half a period in order to examine the orbiting of black holes. We ought to at least be able to discuss half a period by introducing a ‘straight’ curve passing the antipodal point, even without the angular coordinate.

Nevertheless, each black hole does not always pass the opposite side of the straight curve in general cases. For simplicity in this work, we assume reflection symmetry such that the plane of symmetry, which we call the orbital surface, intersects binary black holes. With the reflection symmetry, it is enough to discuss the motion of the object only on the orbital surface.

We define a light ray opposite as the mark of half of the orbital period. Suppose that the spacetime \((M, g)\) is globally hyperbolic. Then, \((M, g)\) admits the timeslicing \(\{\Sigma(t)\}\) in terms of the global time function \(t : M \to \mathbb{R}\) and there is a timelike vector field \(T\) without zero points. Let the vector field \(T\) be normalized such that \(\langle T, dt \rangle = 1\) holds.

Let us consider a cylindrical region \(\mathcal{U}\) generated by the vector field \(T\) with a timelike side-boundary \(B_{\mathcal{U}}\), as illustrated in Fig. 3. Then the timelike vector field
Fig. 2. The figures on the left side illustrate a Newtonian picture of binary motion. The other figures are two possible scenarios of binary black hole coalescence. The center column is the quasi-stationary scenario and the right side is the rapid coalescence scenario.

$T$ determines the natural projection

$$\pi_t : \mathcal{U} \mapsto \mathcal{U} \cap \Sigma(t)$$

of the closed subset $\mathcal{U}$ of $M$ into each timeslice $\Sigma(t)$ along the integrated curve of $T$.

**Definition 1** (Comoving ball with the origin). Let $(M, g)$ be a globally hyperbolic spacetime. Let $t : M \to \mathbb{R}$ be a global time function in $M$. The sequence $\{\Sigma(t)\}$
of the $t = \text{const}$ hypersurface $\Sigma(t)$, $t \in [t_i, t_f]$ is called timeslicing. The spacetime $M$ admits a future-directed timelike vector field $T$ such that $\langle T, dt \rangle = 1$ holds. Let $U \subset \Sigma(t)$ be a topological 3-ball embedded in $\Sigma(t)$, $T \in (t_i, t_f)$. Let $\mathcal{U}$ be the closed subset of $M$ generated by $T$ such that $\mathcal{U} \cap \Sigma(t) = U$ holds. Let $B_{\mathcal{U}} \subset \mathcal{U}$ be the closed subset of $\mathcal{U}$, which is the product set $\mathcal{U} \times [t_i, t_f]$ generated by $T$, and let $o : [t_i, t_f] \to M; t \mapsto o(t)$ be the integral curve of $T$ which passes through an interior point $o(t)$ of $U$.

The 4-tuple $(\{\Sigma(t)\}, \mathcal{U}, T, o)$ is called the comoving ball with the origin in $(M, g)$.

It is difficult to give a precise notion of the orbital plane of binary black holes or the period of binary motion in general settings. A possible means of overcoming this difficulty is to impose a reflection symmetry on $(M, g)$. Although this restriction might be rather stringent, we could still consider a large class of spacetimes describing binary black holes. In the rest of this paper, we will assume that the spacetime $(M, g)$ admits a reflection symmetry with respect to a timelike hypersurface $O$ in $M$. The fixed point set $O$ under this isometry is called the orbital surface and the orbital surface at time $t$ is denoted as $O_t = O \cap \Sigma(t)$. All the settings including the global time function $t$ and the comoving ball with the origin $(\{\Sigma(t)\}, \mathcal{U}, T, o)$ are taken with respect to the reflection symmetry.

Here, we introduce the notion of the opposite side of a point $p$ beyond the origin $o$, which we call the light ray opposite (abbreviated to LRO) of $p$, in terms of a null geodesic generator of $\dot{J}^+(p)$ (see Fig. 4).

**Definition 2** (Light ray opposite). Let $(M, g)$ be a reflection-symmetric spacetime with respect to the timelike hypersurface $O$. Let $(\{\Sigma(t)\}, \mathcal{U}, T, o)$ be a comoving ball with its origin in $M$ with respect to the reflection symmetry. Let us call $\mathcal{O}_t = O \cap \Sigma(t)$ the orbital surface at the time $t$. For a point $p \in \mathcal{O}_t \cap \mathcal{U}$, $t_1 \in (t_i, t_f)$, $\gamma_p$ is defined to be the geodesic generator of $\dot{J}^+(p)$ from $p$, which passes through $o$ at time $t = t_c \in (t_1, t_f)$ if there is exactly one null geodesic generator of $\dot{J}^+(p)$ from $p$ through $o$. The null geodesic $\gamma_p$ will be within $O$ owing to the reflection symmetry. Then,
for a point \( p \) and time \( t_2 \in (t_1, t_f) \), the light ray opposite (LRO) \( \lambda(p, t_2) \) of \( p \) at time \( t = t_2 \) is defined by

\[
\lambda(p, t_2) = \pi_{t_2} \left[ \gamma_p \cap \bigcup_{t \in [t_c, t_f]} \Sigma(t) \right],
\]

where \( \pi_{t_2} \) denotes the projection \( U \to \Sigma(t_2) \) naturally defined by the timelike vector field \( T \), and the LRO of \( p \) is defined to be the empty set if \( \gamma_p \) is not defined.

\[\square\]

**Remark 1.** The LRO is the empty set when the origin is far from the reference point \( p \) such that \( o \cap J^+(p) = \emptyset \). Besides, there might be a situation where \( o \cap J^+(p) = \emptyset \) holds but \( o \cap \hat{J}^+(p) = \emptyset \) when the congruence of the light rays from \( p \) to \( o \) have caustics due to the local gravitational effects. We have precluded such a possibility for simplicity.

Next, we introduce the notion of coalescing binary black holes as follows (Fig. 5).

**Definition 3** (The binary black hole coalescence system). Let \( (M, g) \) be a black hole spacetime and let \( (\{\Sigma(t)\}, U, T, o) \) be a comoving ball with its origin in \( M \). A binary black hole coalescence system \( (H, \{\Sigma(t)\}) \) is defined to be a pair consisting of the event horizon \( H \) in \( M \) and the timeslicing \( \{\Sigma(t)\}, t \in [t_i, t_f] \), such that there is a coalescence time \( t' \in (t_i, t_f) \) decomposing \( H \cap U \) into a precoalescence part

\[
H_{pr} = H \cap U \cap \left[ \bigcup_{t \in [t_i, t']} \Sigma(t) \right],
\]

which has a pair of connected components, and a postcoalescence part

\[
H_{po} = H \cap U \cap \left[ \bigcup_{t \in (t', t_f]} \Sigma(t) \right],
\]

which is connected by the spatial hypersurface \( \Sigma(t') \).

\[\square\]
Note that the concept of black hole coalescence depends on the choice of the timeslicing. For a different timeslicing, the black hole coalescence system can always be regarded as the formation of a single black hole. Here, we consider the typical case \( S^2 \sqcup S^2 \rightarrow S^2 \) for the transition of the horizon topology.

Since the reflection symmetry of \( M \) is imposed, it is enough to consider the causal structure of the orbital surface \( \mathcal{O} \). In other words, we will concentrate on the section of the event horizon by \( \mathcal{O} \). In the following, we assume that \( H_{pr} \cap \mathcal{O}_t \) consists of a pair of circles \( S^1 \sqcup S^1 \) and that \( H_{po} \cap \mathcal{O}_t \) consists of a single circle \( S^1 \) in the binary black hole coalescence system.

We want to define the half cycle of the binary black hole system by saying that a black hole makes half an orbit if all the infinitesimal-area elements of the black hole also make half an orbit. For this purpose, we need to specify the trajectory of each point on the black hole. At first sight, the null geodesic generators of the event horizon seem to naturally determine each orbit. However, this is not appropriate, since new null geodesic generators emerge incessantly. Instead, we consider an arbitrarily chosen one-parameter family of homeomorphisms, \( \phi_t : S^1 \rightarrow S^1 \) between \( H_{pr} \cap \mathcal{O}_{t_i} \) and \( H_{pr} \cap \mathcal{O}_{t_f} \), \( t \in [t_i, t_f] \), which are continuous with respect to \( t \). Such a one-parameter family of homeomorphisms \( \phi_t \) determines the motion of each infinitesimal-area element of the black hole. Note that each orbit determined by \( \phi_t \) necessarily exceeds or equals the speed of light, because it lies on the null hypersurface.

The following is the definition of half a period of binary black holes in an orbital surface, where each black hole event horizon moves as shown in Fig. 6.

**Definition 4** (Half a period of binary coalescence system). Let \( (H, \{ \Sigma(t) \}) \) be a binary black hole coalescence system with the reflection symmetry with respect to the orbital surface \( \mathcal{O} \) with the coalescence time \( t' \in (t_i, t_f) \). Let the pre-coalescence part \( H_{pr} \cup \mathcal{O} \) in \( \mathcal{O} \) consist of a disconnected sum \( H_{pr} \cup \mathcal{O} = H_I \sqcup H_{II} \) such that \( H_I \cup \mathcal{O}_t \) and \( H_{II} \cup \mathcal{O}_t \), \( t \in [t_i, t_f] \), are both circles. Let \( \phi_t^A : H_A \cap \mathcal{O}_{t_1} \rightarrow H_A \cap \mathcal{O}_t \) \( (A = I, II) \) be a one parameter family of homeomorphisms such that \( \phi_{t_1}^I \) is the identity map of \( H_A \) and \( \phi_t^A \) is continuous with respect to \( t \). For \( t_i < t_1 < t_2 < t_f \), we say that half...
a period of the binary coalescence system has elapsed during \((t_1, t_2)\) if the following statement holds for both \(A = I\) and \(II\). There is a point \(p\) on \(H_A \cap \mathcal{O}_{t_1}\) such that every orbit of a point on \(H_A \cup \mathcal{O}_{t_1}\) intersects the LRO \(\lambda(p, t)\) of \(p\) during \(t_1 < t < t_2\).

Note that whether or not the half period has elapsed does not depend on the choice of correspondence \(\{\phi^I, \phi^{II}\}\).

§4. Half-period theorem

Now we show that half a period of the binary coalescence system does not elapse before the coalescence. Firstly, it is easily seen that two black holes before the coalescence are causally separated from each other, as shown in Fig. 7.

**Proposition 1.** Let \(\mathcal{B}\) be the black hole region in the binary black hole system with reflection symmetry, and let \(\mathcal{O}_{pr}\) be the part of \(\mathcal{O} \cap \mathcal{U}\) before the coalescence time defined by \(\mathcal{O}_{pr} = \bigcup_{t \in [t_i, t')} \mathcal{O}(t) \cap \mathcal{U}\), so that the black hole region \(B = \mathcal{B} \cap \mathcal{O}_{pr}\) in the orbital surface is composed of two black hole regions, namely, \(B_I \simeq D^2 \times [t_i, t')\) and \(B_{II} \simeq D^2 \times [t_i, t')\) without intersection, where \(D^2\) denotes the closed 2-disk. Then,

\[
\forall p_I \in B_I, \forall p_{II} \in B_{II}, \quad \mathcal{O}_{pr} \cap J^+(p_I) \cap J^+(p_{II}) = \emptyset
\]

holds.

**Proof.** Since \(p_A (A = I, II)\) is a point in \(B_A\), and \(B_A\) is a future set in \(\mathcal{O}_{pr}\), \(J^+(p_A) \cap \mathcal{O}_{pr} \subset B_A\) holds. Then, \(B_I \cap B_{II} = \emptyset\) implies that \((J^+(p_I) \cap \mathcal{O}_{pr}) \cap (J^+(p_{II}) \cap \mathcal{O}_{pr}) = \emptyset\).

This proposition simply reflects the fact that a black hole region is a future set. The following corollary immediately follows.

**Corollary 1.** A timelike curve \(o\) in \(\mathcal{O}\) does not intersect with \(J^+(p_I) \cap \mathcal{O}_{pr}\) or \(J^+(p_{II}) \cap \mathcal{O}_{pr}\) for any \(p_I \in B_I\) and \(p_{II} \in B_{II}\).

Next, we prove the following lemma.

**Lemma 1.** For every pair of points \(p_I \in B_I \cap \mathcal{O}_{t_1}\) and \(p_{II} \in B_{II} \cap \mathcal{O}_{t_1}\), either of the following statements holds.

1. The LRO \(\lambda(p_I, t_2)\) of \(p_I\) does not intersect with \(J^+(p_I)\) for any \(t_2 \in (t_1, t')\).
2. The LRO \(\lambda(p_{II}, t_2)\) of \(p_{II}\) does not intersect with \(J^+(p_{II})\) for any \(t_2 \in (t_1, t')\).
**Half-Period Theorem**

**Proof.** It follows from Cor. 1 that, for $p_I \in B_I \cap \partial t_1$ and $p_{II} \in B_{II} \cap \partial t_1$, $o$ does not intersect with $J^+(p_I) \cap \partial t_1$ or $J^+(p_{II}) \cap \partial t_1$. Assume that $J^+(p_I) \cap \partial t_1$ does not intersect $o$. Let LRO $\lambda(p_I,t_2)$ have an intersection with $J^+(p_I) \cap \partial_{pr}$, then $\lambda(p_I,t_2)$ starts from $o(t_2)$ and extends to the point $q$ on $J^+(p_I) \cap \partial_{pr}$. It follows from the definition of the LRO that $q$ is on the past directed timelike curve generated by $T$ starting from point $p$ on $J^+(p_I)$. By slightly deforming the causal curve from $p_I$ to $p$ obtained by joining the null geodesic generator from $p_I$ to $q$ and the timelike curve from $q$ to $p$, one can construct a timelike curve from $p_I$ to $p$. It follows that there is an open neighborhood $U$ of $p$ such that $U$ is contained in the chronological future $I^+(p_I)$ of $p_I$. This contradicts the fact that the neighborhood $U$ of the boundary point $p$ of $J^+(p_I)$ necessarily contains an exterior point of $J^+(p_I)$.

We are now at a position to state the main theorem.

**Theorem 1** (The half-period theorem). If the half period of the binary coalescence system elapses during $(t_1,t_2)$, two black holes merge into a single black hole at a time $t' \in (t_1,t_2)$.

**Proof.** First note that the restriction $H \cap \partial$ of $H$ on $\partial$ is a null hypersurface in $\partial$. 

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Fig. 7. Two black holes are causally separated.

Fig. 8. The LRO of the point $p$ does not belong to the causal future of $p$ itself.
generated by null geodesics, each without a future end point. From the definition of the half period of the binary coalescence system, there is a point $p$ on $H \cap \mathcal{O}$ such that the trajectory of its LRO $\bigcup_{t \in (t_1, t_2)} \lambda(p, t)$ intersects with the orbit $\bigcup_{t \in (t_1, t_2)} \phi^I_t(p')$ of every point $p'$ on $H$. This implies that a null geodesic generator of $J^+(p)$, which is also a null geodesic generator of $H \cap \mathcal{O}$ through $p$, intersects with its own LRO $\lambda(p, t_p)$ at a time $t_p \in (t_1, t_2)$. In the same way, there is a point $q$ on $H_\Pi \cap \mathcal{O}$ such that a null geodesic generator of $J^+(q)$ intersects with the LRO $\lambda(q, t_q)$ of $q$ at a time $t_q \in (t_1, t_2)$. The fact that this is impossible is an immediate consequence of Lemma 1.

\[\square\]

§5. Discussion

First, we comment on the general covariance of our result. The setup of the problem here seems to depend on a specific coordinate system. In fact, we have prepared a time function and a timelike vector field for defining the LRO. This obviously corresponds to a specific choice of the time coordinate and $x^i = \text{const.} (i = 1, 2, 3)$ lines. Then, the LRO of a point $p$ is regarded as the line $x^i = x^i(s)$ on a $t = \text{const}$ surface obtained by projecting points on the future-directed null geodesic $x^\mu = x^\mu(s) (\mu = 0, 1, 2, 3)$ starting from $p$ into the $t = \text{const}$ surface along the $x^i = \text{const}$ lines. In this sense, the definition of the LRO, and hence that of the half-period, depends on the coordinate system chosen. Nevertheless, the half-period theorem is formulated in a covariant manner in the sense that it holds for arbitrary choices of such an ordinary coordinate system, where by an ordinary coordinate we mean that a $t = \text{const}$ surface is a spacelike hypersurface and an $x^i = \text{const}$ line is a timelike curve.

There are at least a couple of shortcomings in the definition of the LRO. One is that the LRO of a point $p$ can be the empty set when the congruence of light rays from $p$ to the orbit $o$ of the origin has caustics before reaching $o$. Another is that the LRO might not be a sufficiently long curve, so that it cannot be used as a line indicating half an orbit.

The statement of the half-period theorem might sound extraordinary, since it seems to contradict the existence of the quasi-stationary phase of the binary black hole system. However, we would like to emphasize that the theorem states an ordinary thing that a comoving observer of a black hole does not exceed the speed of light, and that it does not contradict the quasi-stationary motion of binary black holes. Recently, there have appeared numerical results\cite{2, 3} showing that binary black holes orbit many times, which apparently contradicts our theorem. In these numerical computations, apparent horizons, not event horizons, are searched for in the numerical spacetime. There are two possibilities explaining this apparent contradiction. The first is that we expect that each apparent horizon is surrounded by event horizons. Then, it is possible that a pair of apparent horizons orbiting around each other is already enclosed by a single event horizon. In other words, while apparent horizons orbit many times, their event horizons quickly merge into one. The
second possibility is that the coordinate system used in the numerical simulation is superluminal, where $x^i = \text{const}$ lines become spacelike.

We expect that at least the second possibility is correct. First, the apparent horizon, when regarded as a dynamical surface, becomes a spacelike hypersurface in spacetime. This means that the apparent horizon has a confinement property. Therefore, each comoving observer, once it has entered the trapped region, never exits unless the apparent horizon disappears. Hence, if the apparent horizons seem to orbit many times, the coordinate system describing this will be a superluminal one.

Finally, we speculate on the implications of our theorem to astrophysical observation. We cannot directly observe the event horizon. What we can observe are light rays that miss the event horizon. Let us consider the situation where we can directly observe the shadows of a binary system owing to the existence of a bright background light. This is possible, at least in principle. If the binary system consists of ordinary dark stars other than black holes and we observe it in the orbital surface (or with the maximal inclination angle), we will observe two shadows that intersect each other many times. However, if it consist of black holes, we will only observe two shadows merging into one exactly once.

References

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