Introduction

It has been considered that the difficulties of the infinite self-energy and self-force of the point electron are reduced to the following two reasons: one, the classical point model of the electron, and the other, the zero-point fluctuation of the electromagnetic field which is inherent in quantum theory. Recently Dirac has tried to remove the latter difficulty by introducing the new concept of the negative energy photons. While on the former difficulty, many works had been already published. We can collect these works in two groups. One is as follows: we consider the point electron and it is assumed that the Maxwell-Lorentz equation is satisfied in all the space including the neighbouring points of the electron, and we try to give a special interpretation to the formulae, in other words, only to derive special solutions from the equations. Wentzel and Dirac's investigations belong to this type. While the other is that we try to establish new formulae which are nearly equivalent to Maxwell equations in all the space except in the neighbouring points of the electrons where the former deviate from the latter in such manners as to remove the above mentioned difficulties. This method had been considered at the first by Mie, and Born-Infeld, Bopp and recently Dirac had followed the same way. On the treatment of these two types it has been discussed by Prof. S. Sakata and his collaborators.

In the present paper it will be tried to follow the former method. The same problem was investigated quantum-mechanically by Wentzel. But we believe that it is more easy in the classical theory to understand the essential point of this problem. So we shall treat it classically throughout this paper.

§ 1. The equations of the electron and the electromagnetic field.

We express the space-time coordinates by \( x_\mu = (x, y, z, \mu t)(\mu = 1 \rightarrow 4) \),

\( 0,1 \) tlu Classical Theory of the Electron. And we shall assume that the Greek and Latin indices take on 1, 2, 3, 4 and 1, 2, 3 respectively, and further we shall adopt the convention that whenever a Greek (or Latin) index appears twice in a term that term is to be summed for values of the index 1, 2, 3, 4 (or 1, 2, 3).

The functions of the electromagnetic field are given by

\[
\varphi_{\mu}(x) = (\mathcal{A}, i\varphi) \quad \mu = 1, 2, 3, 4
\]

\[
f_{\mu \nu} = \partial \varphi_{\nu} / \partial x_{\mu} - \partial \varphi_{\mu} / \partial x_{\nu}
\]

and those of the electron field are

\[
\phi_{a}(x), \quad \phi_{a}^{*}(x), \quad (a = 1, 2, 3, 4)
\]

And we shall take the following as the Lagrangian of the total system:

\[
L = L_0 + L_1 + V
\]

\[
L_0 = -(1/4) \cdot f_{\mu \nu} f_{\mu \nu} - (1/2) \cdot (\varphi_{\mu \nu})^2
\]

\[
L_1 = \psi^* \left[ (\partial / \partial x_{\nu}) + \alpha \ \text{grad} + i \beta \gamma \right] \psi
\]

\[
V = -i e (\varphi_{\mu} \psi^* \psi - \varphi_{\mu} \psi^* \alpha^* \psi)
\]

\[
\gamma^a = i \gamma^a \quad k = 1, 2, 3 \quad \beta = \gamma^4
\]

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \delta_{\mu \nu}
\]

Here we adopt the natural unit (i.e. \( c = 1, \hbar = 1 \)), and assume that 1/\( x \) is the Compton wave length of the electron wave and \( e \) is the electronic charge.

The canonical conjugate momenta of \( \varphi_{\mu} \) and \( \phi_{a} \) are as usual given by as follows:

\[
E_{\mu} = i f_{\mu \nu} \quad E_{a} = i \psi_{\mu} \quad \text{and} \quad \phi_{a}^{*}
\]

And if we shall express the charge-current four-vector by

\[
S^{*} = i e \phi^* \alpha^* \phi
\]

\[
S_{a} = i e \psi^* \phi
\]

then the Hamiltonian of the total system is given by

\[
\bar{H} = \bar{H}_0 + \bar{H}_1 + \bar{V}
\]

\[
\bar{H}_0 = (1/2) \cdot E_{\mu} E_{\mu} + (1/4) \cdot f_{\mu \nu} f_{\mu \nu} + i (E_{k \varphi_{\mu \nu} - E_{\mu} \varphi_{k \nu})
\]

\[
\bar{H}_1 = -\psi^* \alpha \ \text{grad} \ \psi - i \psi^* \beta \psi
\]

\[
\bar{V} = -S_{\mu} \varphi_{\mu}
\]
where we adopt the convention that the space integral of any function $A(x, x_0)$ at any instant will be expressed by $\overline{A}(x_0) = \int A(x, x_0) dx dy dz$.

The canonical equations corresponding to (6) are

\[
\begin{align*}
\delta \varphi / \delta \varphi^* &= -i \text{grad} \varphi - i x \beta \varphi - i c \varphi_3 \alpha \beta + i c \varphi_3 \varphi^\star \\
\delta \varphi^* / \delta \varphi &= - \text{grad} \varphi^* \alpha + i x \beta \varphi^* \beta + i c \varphi_3 \varphi^\star \alpha - i e \varphi_3 \varphi^\star \\
\delta \varphi_\mu / \partial x_\mu &= \delta \overline{H} / \partial E_\mu = E_\mu - \varphi_\mu, \\
\delta \varphi / \partial x_\mu &= \delta \overline{H} / \partial E_\mu = E_\mu - i \varphi_\mu, \\
\delta E_\mu / \partial x_\mu &= - \delta \overline{H} / \partial \varphi_\mu = - \varphi_\mu / \varphi_\mu - \partial E_\mu / \partial x_\mu + S_\mu \\
\delta E_\mu / \partial x_\mu &= - \delta \overline{H} / \partial \varphi_\mu = i \partial E_\mu / \partial x_\mu + S_\mu
\end{align*}
\]

where $\delta \overline{H} / \partial \varphi_\mu(x, x_0)$ etc. represent the functional derivative of $\overline{H}$ with respect to $\varphi_\mu$ etc. at the world point $(x, x_0)$.

The Lorentz condition is

\[
\partial \varphi_\mu / \partial x_\mu = 0, \quad \partial (\partial \varphi_\mu / \partial x_\mu) / \partial x_0 = 0.
\]

and these are also represented as follows by using the field equations (5),

\[
-i E_\mu = \varphi_\mu, = 0 \quad -i \partial E_\mu / \partial x_\mu = \partial E_\mu / \partial x_\mu + S_\mu = 0. \quad (6)
\]

Considering the field equation

\[
\partial^2 E_\mu / \partial x_\mu = i \partial (\partial E_\mu / \partial x_\mu) / \partial x_\mu + i \partial S_\mu / \partial x_\mu = \Delta E_\mu + i (\partial S_\mu / \partial x_\mu + \partial S_\mu / \partial x_\mu)
\]

and the equation of continuity

\[
\partial S_\mu / \partial x_\mu + \partial S_\mu / \partial x_\mu = 0 \quad (7)
\]

which can be also derived from (5), we get

\[
\partial^2 E_\mu / \partial x_\mu = \Delta E_\mu.
\]

Therefore from (6) we obtain $\partial E_\mu / \partial x_\mu = 0$ and so

\[
-i E_\mu = \varphi_\mu, = 0
\]

is always satisfied.

Furthermore by eliminating $E_\mu$ from (5) we obtain

\[
\square \varphi_\mu = -S_\mu. \quad (8)
\]
§ 2. Canonical transformation.

Now we shall consider the following Hamilton-Jacobi's equation

$$\ddot{H}_a(E_p, \phi_p, \phi) + \ddot{H}_i(\phi, \phi) + \frac{\partial \bar{S}}{\partial x_o} = 0, \quad (9)$$

where

$$E_p = \frac{\partial \bar{S}}{\partial \phi_p}, \quad \phi_p^* = \frac{\partial \bar{S}}{\partial \phi_p}.$$

As the solution of this equation we shall adopt the so-called Hamilton's principal function

$$\bar{S} = \int_{x_0}^{x_1} (\varphi_p, \varphi) dt + \int_{x_0}^{x_1} \bar{E}_a(\phi, \partial \phi / \partial x_a) dt, \quad (10)$$

where we shall assume that $\varphi_p$ and $\phi_a$ in (10) are those which make the value of (10) to be extremal, in other words, we shall adopt the solution of the Euler equation for the variation of (10). Now we can regard $\bar{S}$ given by (10) as a functional of the functions $\varphi_p, \phi_a$ at the instant $x_0$, $\varphi_p'(x_c, x_{c}')$ at $x_{c}'$ and $\phi_a'(x, x_{a})$ at $x_{a}$, i.e.

$$\bar{S} = \bar{S}[\varphi_p(x_c, x_0), \phi_a(x_c, x_0), \varphi_p'(x_c, x_{c}'), \phi_a'(x, x_{a})].$$

where we shall use the convention that any functional $F$ of a function $f(x)$ is represented by $F = F[f(x)]$.

Then we get the following relations:

$$\frac{\partial \bar{S}}{\partial \varphi_p(x_c, x_0)} = E_p(x_c, x_0), \quad \frac{\partial \bar{S}}{\partial \phi_a(x_c, x_0)} = \phi_a^*(x, x), \quad (11)$$

and if we put

$$\frac{\partial \bar{S}}{\partial \varphi_p'(x, x_{c}')} = -E_p'(x, x_{c}'), \quad \frac{\partial \bar{S}}{\partial \phi_a'(x, x_{a}'')} = -\phi_a'^*(x, x_{a}'').$$

then we obtain two canonical transformations, one of which transforms a set of canonical variables $(\varphi_p(x, x_0), E_p(x, x_0))$ at the instant $x_0$ into another set $(\varphi_p'(x, x_{c}'), E_p'(x, x_{c}'))$ at the other instant $x_{c}'$, and the other one transforms $(\phi(x, x_0), \phi^*(x, x_0))$ at the instant $x_0$ into $(\phi'(x, x_{a}''), \phi'^*(x, x_{a}''))$ at the other instant $x_{a}''$.

And further $\bar{S}$ satisfies the following two Hamilton-Jacobi's equations.
\[
\begin{align*}
\frac{\partial (-\hat{S})}{\partial x'_o} + \bar{H}_0 \left[ \frac{\delta (-\hat{S})}{\delta \varphi'_\mu}, \varphi'_\nu, \varphi'_\nu \right] &= 0 \\
\frac{\partial (-\hat{S})}{\partial x''_o} + \bar{H}_1 \left[ \frac{\delta (-\hat{S})}{\delta \psi}, \psi, \psi' \right] &= 0.
\end{align*}
\]

The canonical equations corresponding to (12) are given by,

\[
\begin{align*}
\frac{\partial \varphi'_\mu}{\partial x'_o} &= \frac{\delta \bar{H}_0}{\delta \varphi'_\mu}, \\
\frac{\partial \psi'}{\partial x'_o} &= \frac{\delta \bar{H}_1}{\delta \psi'}. \end{align*}
\]

where (11) is used. As the initial conditions of (13) we must put as follows at the instant \(x'_o = x_o\)

\[
\varphi'_\mu(x, x_o') = \varphi'_\mu(x, x_o) \quad \text{and at } x''_o = x_o
\]

\[
\psi(x, x_o') = \psi(x, x_o) \quad \text{and at } x''_o = x_o.
\]

Solving (13) under the condition (14), we get the following solutions;

\[
\begin{align*}
\varphi'_\mu(x, x_o') &= \varphi'_\mu(x, x_o', x_o - x_o, \varphi(x, x_o), \varphi(x, x_o)), \\
E'_\mu(x, x_o') &= E'_\mu(x, x_o', x_o - x_o, \varphi(x, x_o), \varphi(x, x_o)) \\
\psi(x, x_o'') &= \psi(x, x_o', x_o - x_o, \psi(x, x_o), \psi(x, x_o)) \\
\psi^*(x, x_o'') &= \psi^*(x, x_o', x_o - x_o, \psi(x, x_o), \psi^*(x, x_o)).
\end{align*}
\]

Now the new Hamiltonian \(\bar{K}\) transformed by (11) is given by as follows provided that (9) is satisfied,

\[
\bar{K} = \bar{H}_0 + \bar{H}_1 + \bar{V} + \frac{\partial \hat{S}}{\partial x_o} = -S'[\psi', \psi'^*]p'_\mu, \psi'_\mu, E'_\mu, (15)
\]

where \(S'_\mu[\psi', \psi'^*]\) is

\[
-\epsilon \psi^*[x, x_o - x_o', \psi(x, x_o'), \psi^*(x, x_o')] \times
\times \gamma r_p \psi[x, x_o - x_o', \psi(x, x_o')] \psi^*(x, x_o').
\]

And the new canonical equations are

\[
\frac{\partial \psi_0'}{\partial x_0} = \frac{\partial V}{\partial \psi_0'} \quad \text{and} \quad \frac{\partial \psi_0'}{\partial x_0} = -\frac{\partial V}{\partial \psi_0'}
\]

(16)

In a word, the dependence of \( \psi_0' \) on \( x_0 \) and \( x_0' \) are given by (16) and (13) respectively, and it is the same as in the case of \( \psi' \) (Fig. 1). Now the canonical equation (13) of \( \psi_0' \) is reduced to the following by eliminating \( E_\mu' \) from it;

\[
\Box' \psi_0' = \frac{\partial^2}{\partial x_\mu' \partial x_\nu'} \psi_0' = 0. \quad (17)
\]

And this is not identical with the Maxwell equation in free space. To identify (17) with the latter it is necessary to introduce the Lorentz condition. Now expanding \( \psi_0'(x, x') \) in power series of \( (x_0' - x_0) \), and using (13), we obtain

\[
\psi_0'(x', x_0) = \sum_{n=0}^{\infty} \frac{(x_0' - x_0)^n}{n!} \frac{\partial^n \psi_0'(x', x_0)}{\partial x_0^n}
\]

\[
= \sum_{n=0}^{\infty} \left[ (x_0' - x_0)^n \frac{\partial^n \psi_0(x', x_0) + (x_0' - x_0)^{n+1}}{(2n+1)!} \times \right.
\]

\[
\left. \cdot \cdot \cdot \cdot \cdot \cdot \cdot \right] 
\]

It is also the same as in the case of \( \psi_0' \). Using these results, the Lorentz condition in this case can be written in the following form;

\[
X = \frac{\partial \psi_0'(x', x_0)}{\partial x_\mu'} = \sum_{n=0}^{\infty} \left[ \frac{(x_0' - x_0)^n}{(2n+1)!} \frac{\partial E_\mu'(x', x_0)}{\partial x_\mu'} 
\right.
\]

\[
- i \frac{(x_0' - x_0)^n}{(2n)!} \frac{\partial^2 E_\mu'(x', x_0)}{\partial x_\mu'} \left] 
\]

Taking notice of that \( E_\mu'(x', x_0) = 0, \frac{\partial E_\mu'(x', x_0)}{\partial x_\mu'} = -S_\mu' \) it becomes

\[
X = - \sum_{n=0}^{\infty} \frac{(x_0' - x_0)^n}{(2n+1)!} \frac{\partial^n S_0(x', x_0)}{\partial x_\mu'} 
\]
\[ \frac{\phi_{\mu}'}{\mu} \delta(x' - x, x' - x') dx' = 0, \quad \delta(x - x, x - x) dx = 0, \]

where

\[ D(x, r) = \frac{1}{(2\pi)^3} \int \frac{e^{ikx}}{k^2} \sin k_0 \, dk_0, \quad k_0 = \sqrt{k^2} \] (19)

Because of (18) being derived by using the Lorentz condition at \( x_0' = x_0 \) as its initial condition, this is not the necessary consequence of the field equations but is a supplementary condition. Therefore it is necessary to give a proof for the compatibility of it with the field equations (13). And this proof is easily given by noticing the following formulae:

\[ \lim_{\mu' \to \mu} \left[ \frac{\partial \phi_{\mu}'}{\partial x_0'} \frac{\partial \phi_{\mu}'}{\partial x_0'} \right] + \int S_0(x; x_0) \frac{\partial}{\partial x_0} D(x' - x, x_0 - x_0') dx = 0, \]

the latter is given by the condition (6).

§ 3. Poisson Bracket.

Now we shall define the “Poisson Bracket” in the classical theory of fields as follows; being \( A \) and \( B \) any functionals of \( \phi_{\mu}', E_{\mu}' \), and if the space integrals of them are given by \( \overrightarrow{A}(x_0') \) and \( \overrightarrow{B}(x_0'') \) respectively, then we define

\[ \overrightarrow{[A(x_0'), B(x_0'')]} = \sum_{\mu=1}^{4} \int \frac{\delta \overrightarrow{A}}{\delta \phi_{\mu}'(x, x_0)} \cdot \frac{\delta \overrightarrow{B}}{\delta \phi_{\mu}'(x, x_0)} \, dx \]

\[ \quad - \sum_{\mu=1}^{4} \int \frac{\delta \overrightarrow{A}}{\delta \phi_{\mu}'(x, x_0)} \cdot \frac{\delta \overrightarrow{B}}{\delta E_{\mu}'(x, x_0)} \, dx \] (20)

Now it is completely arbitrary what value to adopt for \( x_0 \) if we consider the following fact that the variation of the state of the considering system
with the lapse of time is given by the integration of the infinitesimal canonical transformation, the substitution function of which is the Hamiltonian of the system, and furthermore the Poisson Blacket is invariant for the canonical transformation. Therefore we can put \( x_0 \) equal to \( x'_o \). For example, if we put \( A \) and \( B \) as follows

\[
A(x_0') = \delta(\omega' - \xi)E_x'(\xi, x_0') dx = E_x'(\omega', x_0'),
\]

\[
B(x_0'') = \int \delta(\omega'' - \xi)\varphi_y'(\xi, x_0'') d\xi = \varphi_y'(\omega'', x_0''),
\]

then putting \( x_o = x_o' = x''_o \), we obtain for the electromagnetic field

\[
[A_x'(\omega'x'_o), \varphi_y'(\omega''x''_o)] = \delta_{\omega''}(\omega' - \omega'').
\]

And by the same way we obtain for the electron field,

\[
[B_x'(\omega'x'_o), \varphi_y'(\omega''x''_o)] = \delta_{\omega''}(\omega' - \omega'').
\]

All the other couples vanish. In the next step, we consider the Poisson Blacket of any two quantities at different instances. By the completely same way as in the case of obtaining the Lorentz condition, we expand \( \varphi \varphi ', E \varphi ' \) in power series of time, and by using the field equation (13) we substitute all the time derivatives of \( \varphi \varphi ' \) with the spatial derivatives of \( \varphi \varphi ' \) and \( E \varphi ' \). Thus we can reduce the Poisson Blacket referring to different instances to those at same instance. In this way we can finally obtain the following results:

\[
[A_x'(\omega'x'_o), \varphi_y'(\omega''x''_o)] = \delta_{\omega''}(\omega' - \omega''), \quad (21)
\]

By the analogous way we obtain for the electron field

\[
[B_x'(\omega'x'_o), \varphi_y'(\omega''x''_o)] = \begin{array}{c}
- \partial \\
3x''_o
\end{array} D_1(\omega' - \omega'', x'_o - x''_o) \quad \text{ for } x = x''_o + i \theta D_1(\omega' - \omega'', x'_o - x''_o) \quad (23)
\]

where

\[
D_1(x, \tau) = \frac{1}{(2\pi)^2} \int \frac{e^{ikr}}{k_0} \sin k_0 \tau, \quad d\theta 
\]

\[
k_0 = \sqrt{x^2 + |k|^2}
\]

Now using the "Poisson Blacket" we can rewrite (16) and (13) as follows:
And in general we obtain the following for any functional $A$ of field quantities $\varphi_\mu'(x,x_0)E_\mu'(x,x_0)$, $\psi'(x,x_\nu')\psi'^*(x,x_\nu')$,

$$
\frac{\partial A}{\partial x_\mu} = [\bar{V}, A] \quad \frac{\partial A}{\partial x_\nu'} = [\bar{H}_\psi', A] \quad \frac{\partial A}{\partial x_\eta'} = [\bar{H}_E', A]
$$

where

$$
\bar{V} = - \int S_\mu'(x,x_0)\varphi_\mu'(x,x_0)dx.
$$

If we put $A$ equal to $\varphi_\mu'(x',x_0)$, then

$$
\frac{\partial \varphi_\mu'(x',x_0)}{\partial x_\nu} = - \int S_\mu(x,x_0)D(x-x',x_0-x_0')dx.
$$

Integrating this with regard to $x_0$, we obtain

$$
\varphi_\mu'(x',x_0) = - \int \lim_{x_0 \to x_0'} \int S_\mu(x_\tau)D(x-x', x_\tau-x_0')dx_\tau + \varphi_\mu^{(0)}(x'x_0'),
$$

where $\varphi_\mu^{(0)}(x'x_0')$ is independent on $x_0$, and further from the field equation

$$
\Box' \varphi_\mu'(x'x_0') = 0
$$

and the identity

$$
\Box' D(x-x', x_0-x_0') = 0,
$$

$\varphi_\mu^{(0)}$ has necessarily to satisfy

$$
\Box' \varphi_\mu^{(0)}(x'x_0') = 0.
$$
As $\varphi_{\mu}^{\alpha}$ is independent on the field quantities of the electron, it is the four potential to Maxwell field in vacuum, and so we can assume the Lorentz condition for $\varphi_{\mu}^{\alpha}$ as follows,

$$\frac{\partial \varphi_{\mu}^{\alpha}(x', x'')}{\partial x'_\mu} = 0$$

(28)

Following Dirac's paper we can interpret this $\varphi_{\mu}^{\alpha}$ as the potential of the incident field.

Now we can also give the expression of the Lorentz condition more easily than in the case of (18). Using (27), (28) and (7), we obtain

$$X = \varphi_{\mu}'(x', x'_0) = - \int_{-\infty}^{\infty} d\tau \int \frac{\partial S_\mu'(x, \tau)}{\partial x'_\mu} D(x - x', \tau - x'_0) dx$$

$$+ \int S_\mu'(x, x_0) D(x - x', x_0 - x'_0) dx$$

$$= - \int S_\mu'(x, x_0) D(x' - x, x'_0 - x_0) dx.$$

This result is completely identical with (18). The contents of the above mentioned argument are completely the same with those given by Wentzel in quantum theory.

(to be continued.)