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Low-Temperature Specific Heat of Spin-1/2 Anisotropic Heisenberg Ring

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Low-temperature specific heat of Heisenberg-Ising ring at $|\Delta|<1$ is calculated by the method of non-linear integral equations. Specific heat in constant magnetic field (C_H) is proportional to temperature (T) at $|2\mu_0 H|<(1+\Delta)J$. In particular $\lim_{H\to 0}\lim_{T\to 0} C_H/T$ is $2\theta/(3J\sin\theta)$ where $\theta=\cos^{-1}\Delta$, $0\leq\theta<\pi$. It is conjectured that $\lim_{T\to 0}\lim_{H\to 0} C_H/T=\lim_{H\to 0}\lim_{T\to 0} C_H/T$ from the result of numerical calculation. Low-temperature specific heat of the one-dimensional X-Y-Z model is proportional to $T^{-3/2}\exp(-\alpha/T)$ in the case of $J_z>J_y>0, J_y\geqq J_x>-J_y$.

§ 1. Introduction

The Heisenberg-Ising model plays a very important role in the theory of magnetism and quantum fluids. The one-dimensional system of this model can be treated by the method of Bethe's hypothesis. The Hamiltonian is

$$\mathcal{H}=J\sum_{i=1}^{N}\{S_i^xS_{i+1}^x+S_i^yS_{i+1}^y+\Delta(S_i^zS_{i+1}^z-\tfrac{1}{4})\}-2\mu_0H\sum_{i=1}^{N}S_i^z,\quad S_{N+1}=S_1. \tag{1·1}$$

The ground state energy, magnetic susceptibility ($\lim_{H\to 0}\lim_{T\to 0}\chi$) and elementary excitations were obtained by solving linear-integral equations.

We gave a set of non-linear integral equations for the free energy at $|\Delta|<1$ in previous papers,[1)] where we used some assumptions about the distribution of quasi-momenta on the complex plane. Though we cannot prove their validity, the high-temperature expansion of free energy is calculated through the second term and we find that it coincides with a known expansion. In Appendix A we will calculate the third term, by which the coincidence is supported. One obtains that the free energy per site is

$$f(T,H)=-T\log\left(2\operatorname{ch}\frac{\mu_0H}{T}\right)-\frac{J\Delta}{4}\left(\frac{1}{\operatorname{ch}^2(\mu_0H/T)}\right)-\frac{J^2}{4T}\left\{\frac{1+2\Delta^2}{4\operatorname{ch}^2(\mu_0H/T)}-\frac{3\Delta^2}{8\operatorname{ch}^4(\mu_0H/T)}\right\}+O\left(\frac{1}{T^2}\right) \tag{1·2}$$

by calculating $\lim_{N\to\infty}\{-T\log(\operatorname{Tr}\exp(-\mathcal{H}/T))/N\}$. It was shown in Ref. 1) that this set of equations gives known exact results at zero temperature. So it is quite possible that our assumptions about the distribution of quasi-momenta are correct and that the set of equations gives the exact free energy of one-dimensional Heisenberg-Ising model.

In § 2 we give the set of integral equations and some other representations of this set. In § 3 we obtain a set of linear integral equations for $\lim_{T\to 0} C_H/T$ at $J(1+\Delta)>2\mu_0 H>0$. We will obtain

$$\lim_{H\to 0}\lim_{T\to 0} C_H/T=\frac{2\theta}{3J\sin\theta}, \quad \theta=\cos^{-1}\Delta, \quad 0\leq\theta<\pi \qquad \text{for } J>0\,.$$

In § 4 we give a set of non-linear equations for the free energy at $T\ll J$ and $T\sim\mu_0 H$. We calculate numerically the value of $\lim_{T\to 0}\lim_{H\to 0} C_H/T$ and find $\lim_{H\to 0}\lim_{T\to 0} C_H/T=\lim_{T\to 0}\lim_{H\to 0} C_H/T$ within the error of numerical calculation in the case $\Delta=\cos(\pi/3)$, $\cos(\pi/4)$, $\cdots$, $\cos(\pi/10)$. In § 5 the low-temperature free energy of the spin-$\frac{1}{2}$ X-Y-Z ring is calculated through the use of integral equations given in Ref. 1).

§ 2. Mathematical formulation

In this section we give notations of series of numbers and functions which will be used in this paper:

$$\begin{aligned}
&\cos\theta=\Delta, \quad p_0\equiv\pi/\theta, \quad p_1=1\,,\\
&p_i=p_{i-2}-[p_{i-2}/p_{i-1}]p_{i-1}, \quad \nu_i=[p_{i-1}/p_i]\,,\\
&m_i=\sum_{l=1}^{i}\nu_l\,,\\
&y_{-1}=0, \quad y_0=1, \quad y_1=\nu_1, \quad y_i=\nu_i y_{i-1}+y_{i-2}\,,\\
&n_j=y_{i-2}+(j-m_i)y_{i-1} \qquad \text{for } m_{i-1}\leq j<m_i\,,\\
&q_j=(-1)^{i-1}(p_{i-1}+(m_{i-1}-j)p_i) \qquad \text{for } m_{i-1}\leq j<m_i\,,
\end{aligned} \tag{2·1}$$

$$\begin{aligned}
&s_i(x)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega x}}{2\,\mathrm{ch}\,p_i\omega}=\frac{1}{4p_i}\,\mathrm{sech}\frac{\pi x}{2p_i}\,,\\
&d_i(x)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\cdot\frac{\mathrm{ch}(p_i-p_{i+1})\omega\cdot e^{i\omega x}}{2\mathrm{ch}\,p_i\omega\,\mathrm{ch}\,p_{i+1}\omega}\,.
\end{aligned} \tag{2·2}$$

We define the following functions,

$$D_{jl}(x)=\begin{cases}(1-2\delta_{j,m_{i-1}})\delta_{j,l+1}s_i(x)+\delta_{j,l-1}s_i(x) & \text{for } m_{i-1}\leq j\leq m_i-2\,,\\ (1-2\delta_{j,m_{i-1}})\delta_{j,l+1}s_i(x)+\delta_{jl}d_i(x)+\delta_{j,l-1}s_{i+1}(x) & \text{for } j=m_i-1\,,\end{cases} \tag{2·3}$$

$$\begin{aligned}
\alpha_j(x)=&\int_{-\infty}^{\infty}\frac{\mathrm{ch}(p_0-1-j)\omega}{\mathrm{ch}(p_0-1)\omega}e^{i\omega x}\frac{d\omega}{2\pi}=\frac{1}{p_0-1}\left\{\mathrm{ch}\frac{\pi x}{2(p_0-1)}\sin\frac{\pi(p_0-1-j)}{2(p_0-1)}\right\}\\
&\times\left\{\mathrm{ch}\frac{\pi x}{p_0-1}+\cos\frac{\pi(p_0-1-j)}{p_0-1}\right\}^{-1} \qquad \text{for } j=1,2,\cdots,\nu_1-1\,,
\end{aligned} \tag{2·4a}$$

$$\beta(x)=\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{2\mathrm{ch}(p_0-1)\omega}\cdot\frac{d\omega}{2\pi}=\frac{1}{4(p_0-1)}\mathrm{sech}\frac{\pi x}{2(p_0-1)}\,, \tag{2·4b}$$

$$B_{jl}(x)=\begin{cases}\alpha_{|j-l|}+2\alpha_{|j-l|+2}+\cdots+2\alpha_{j+l-2}+\alpha_{j+l} & \text{for } j\neq l\,,\\ 2\alpha_2+2\alpha_4+\cdots+2\alpha_{2l-2}+\alpha_{2l} & \text{for } j=l\,,\end{cases} \tag{2·4c}$$

$$C_j(x)=s_2{}^*B_{j,\nu_1-1}(x)+\delta_{j,\nu_1-1}s_2(x)\,, \tag{2·4d}$$

$$a_j(x)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\sinh q_j\omega}{\sinh p_0\omega}e^{i\omega x}=\frac{\sin(\pi q_j/p_0)}{2p_0(\mathrm{ch}(\pi x/p_0)+\cos(\pi q_j/p_0))}\,. \tag{2·4e}$$

When θ/π is an irrational number and $0<\theta<\pi/2$, the set of integral equations is

$$\ln\eta_j=-As_1(x)\delta_{j1}/T+\sum_{l=1}^{\infty}D_{jl}^*\ln(1+\eta_l)\,,\quad j=1,2,\cdots,$$

$$\lim_{j\to\infty}\frac{\ln\eta_j}{n_j}=\frac{2\mu_0H}{T}\,,\quad A=2\pi J\sin\theta/\theta\,, \tag{2·5a}$$

and the free energy per site is

$$f(T,H)=-A\int_{-\infty}^{\infty}s_1(x)a_1(x)dx-T\int_{-\infty}^{\infty}s_1(x)\ln(1+\eta_1(x))dx\,. \tag{2·5b}$$

Equations (2·5a) can be transformed as follows:

$$\ln\eta_j=-\frac{A}{T}\alpha_j(x)+\sum_{l=1}^{\nu_1-1}B_{jl}^*\ln(1+\eta_l{}^{-1})+C_j{}^*\ln(1+\eta_{\nu_1})\qquad \text{for } 1\leq j<\nu_1\,,$$

$$\ln\eta_{\nu_1}=\frac{A}{T}\beta(x)-\sum_{l=1}^{\nu_1-1}C_l{}^*\ln(1+\eta_l{}^{-1})+\sum_{l>\nu_1}D_{\nu_1 l}^*\ln(1+\eta_l)-s_2{}^*C_{\nu_1-1}^*\ln(1+\eta_{\nu_1})\,,$$

$$\ln\eta_j=\sum_{l=\nu_1}^{\infty}D_{jl}^*\ln(1+\eta_l)\qquad \text{for } j>\nu_1\,,$$

$$\lim_{j\to\infty}\frac{\ln\eta_j}{n_j}=\frac{2\mu_0H}{T}\,. \tag{2·6a}$$

The expression for free energy (2·5b) can be transformed as follows:

$$f(T,H)=-A\int_{-\infty}^{\infty}s_1(x)(a_1(x)-\alpha_1(x))dx-T\sum_{j=1}^{\nu_1-1}\int_{-\infty}^{\infty}\alpha_j(x)\ln(1+\eta_j{}^{-1}(x))dx$$
$$-T\int_{-\infty}^{\infty}\beta(x)\ln(1+\eta_{\nu_1}(x))dx\,. \tag{2·6b}$$

By taking the limit in Eqs. (2·5), we obtain integral equations, when θ/π is a rational number. In this case the number of unknown functions becomes finite. Equations (2·5) and (2·6) are useful for the discussion of the low-temperature properties in the antiferromagnetic case ($J>0$) and ferromagnetic case ($J<0$), respectively.

§ 3. Low-temperature expansion of the free energy at $\mu_0H\gg T$

Assume that $\ln\eta_l\gg1$ for $J>0$, $l\geq2$ and $2\mu_0H\gg T$. The term $\ln(1+\eta_l)$ in

(2·5a) is replaced by $\ln \eta_l$ and one obtains

$$\ln \eta_j(x) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\operatorname{sh} q_j\omega}{\operatorname{sh} q_1\omega} e^{i\omega(x-x')} \ln(1+\eta_1(x'))$$
$$+\left(n_j - \frac{q_j}{q_1}\right)\frac{2\mu_0 H}{T} + \sum_{l=2}^{\infty} O(\eta_l^{-1}), \quad j=2,3,\cdots. \tag{3·1a}$$

Substituting the case $j=2$ of this equation into the case $j=1$ of (2·5a), we have

$$T\ln \eta_1 = -\frac{2\pi J \sin\theta}{\theta} s_1(x) + \frac{\pi\mu_0 H}{\pi-\theta} + \int_{-\infty}^{\infty} R_+(x-x')\, T\ln(1+\eta_1(x'))\,dx'$$
$$+O\left(T\exp\left(-\frac{2\pi\mu_0 H}{T(\pi-\theta)}\right)\right), \tag{3·1b}$$

where

$$R_+(x) \equiv \int_{-\infty}^{\infty} \frac{\operatorname{sh}(p_0-2)\omega e^{i\omega x}}{2\operatorname{ch}\omega \operatorname{sh}(p_0-1)\omega} \cdot \frac{d\omega}{2\pi}. \tag{3·1c}$$

At $J<0$, $1>\Delta\geq 0$ we assume that $\ln \eta_j \gg 1$ for $j>\nu_1$. Then we have

$$T\ln \eta_j(x) = T\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dx' \frac{\operatorname{sh} q_j\omega}{\operatorname{sh} q_{\nu_1}\omega} e^{i\omega(x-x')} \ln(1+\eta_{\nu_1}(x'))$$
$$+\left(n_j - \frac{q_j}{q_{\nu_1}}\right) 2\mu_0 H + \sum_{l>\nu_1}^{\infty} O(T\eta_l^{-1}) \qquad \text{for } j>\nu_1. \tag{3·2a}$$

In the same way as is used in the preceding paragraph, we obtain

$$\ln \eta_j = -\frac{A}{T}\alpha_j(x) + \sum_{l=1}^{\nu_1-1} B_{jl}^* \ln(1+\eta_l^{-1}) + C_j^* \ln(1+\eta_{\nu_1}), \tag{3·2b}$$

$$\ln\eta_{\nu_1} = +\frac{A}{T}\beta(x) - \sum_{l=1}^{\nu_1-1} C_l^* \ln(1+\eta_l^{-1}) + \int_{-\infty}^{\infty} R_-(x-x')\ln(1+\eta_{\nu_1})\,dx' + \frac{\pi\mu_0 H}{\theta T}, \tag{3·2c}$$

where

$$R_-(x) = \int_{-\infty}^{\infty} \frac{\operatorname{sh}(2-p_0)\omega e^{i\omega x}}{2\operatorname{ch}(p_0-1)\omega\cdot \operatorname{sh}\omega} \cdot \frac{d\omega}{2\pi}. \tag{3·2d}$$

Considering $-A/T \gg 1$ and $C_j^* \ln(1+\eta_{\nu_1}) \sim \mu_0 H/T$, we have $\ln \eta_j \gg 1$, $j=1,2,\cdots,\nu_1-1$.

From Eq. (2·6b) we have

$$f(T,H) - f(0,0) = -T\int_{-\infty}^{\infty} \beta(x)\ln(1+\eta_{\nu_1}(x))\,dx + \sum_{j=1}^{\nu_1-1} O(T\eta_j^{-1}). \tag{3·2e}$$

Considering that $f(T,H)$ is invariant under the transformation $(J,\Delta)\to(-J,-\Delta)$, we have from Eqs. (3·1), (3·2) and (2·5b)

$$\varepsilon_1(x) = \frac{2\pi J \sin\theta}{\theta}\frac{1}{4}\operatorname{sech}\frac{\pi x}{2} + \frac{\pi\mu_0 H}{\pi-\theta} + T\int_{-\infty}^{\infty} R(x-x')\ln(1+\exp(\varepsilon_1(x')/T))dx' , \tag{3·3a}$$

$$R(x) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega x}\frac{\operatorname{sh}(\pi-2\theta)\omega}{2\operatorname{ch}\omega\operatorname{sh}(\pi-\theta)\omega}, \tag{3·3b}$$

$$f(T,H) = -\frac{2\pi J\sin\theta}{\theta}\int_{-\infty}^{\infty}\frac{1}{4}\operatorname{sech}\frac{\pi x}{2}\cdot\frac{\theta}{2\pi}\frac{\sin\theta}{\cosh\theta x - \cos\theta}dx$$
$$-T\int_{-\infty}^{\infty}\frac{1}{4}\operatorname{sech}\frac{\pi x}{2}\ln(1+\exp(\varepsilon_1(x)/T)) + O\Big(T\exp\Big(-\frac{2\pi\mu_0 H}{(\pi-\theta)T}\Big)\Big), \tag{3·3c}$$

where θ is determined by

$$\Delta = \cos\theta , \qquad 0\leq\theta\leq\pi \tag{3·3d}$$

for $|\Delta|\leq 1$ and $J>0$, $2\mu_0 H \gg T$. This set of equations is equivalent to

$$\varepsilon_1(x) = -\frac{2\pi J\sin\theta}{\theta}a_1(x) + 2\mu_0 H$$
$$+T\int_{-\infty}^{\infty} a_2(x-x')\ln(1+\exp(-\varepsilon_1(x')/T))dx' , \tag{3·4a}$$

$$a_j(x) = \frac{\theta}{2\pi}\cdot\frac{\sin j\theta}{\operatorname{ch}\theta x - \cos j\theta}, \quad j=1,2, \tag{3·4b}$$

$$f(T,H) = -\mu_0 H - T\int_{-\infty}^{\infty} a_1(x)\ln(1+\exp(-\varepsilon_1(x)/T))dx . \tag{3·4c}$$

a) *Case* $J>0$, $J(1+\Delta)>2\mu_0 H$

We put the solution of (3·4a) at $T=0$ as $\varepsilon_1^{(0)}(x)$, which satisfies

$$\varepsilon_1^{(0)}(x) = -\frac{2\pi J\sin\theta}{\theta}a_1(x) + 2\mu_0 H - \int_{\varepsilon_1^{(0)}(x')<0} a_2(x-x')\varepsilon_1^{(0)}(x')dx' . \tag{3·5}$$

Function $\varepsilon_1^{(0)}$ is a monotonically increasing function of x^2 and is $2\mu_0 H$ in the limit $x\to\pm\infty$ because $a_1(\infty)$ and $a_2(\infty)$ are zero. It is clear that if $-(2\pi J\sin\theta/\theta)\times a_1(0)+2\mu_0 H<0$ (namely, $J(1+\Delta)>2\mu_0 H$), $\varepsilon_1^{(0)}(x)$ has necessarily two zeros $\pm K$, $K>0$. Using $\varepsilon_1^{(0)}(K)=0$ and (3·5), one can calculate $2\mu_0 H$ from K:

$$2\mu_0 H = \frac{2\pi J\sin\theta}{\theta}\cdot\frac{V(K)}{Z(K)}, \tag{3·6a}$$

where

$$V(x) + \int_{-K}^{K} a_2(x-x')V(x')dx' = a_1(x), \tag{3·6b}$$

$$Z(x) + \int_{-K}^{K} a_2(x-x')Z(x')dx' = 1 . \tag{3·6c}$$

If $T \ll J(1+\Delta)-2\mu_0 H$, $\varepsilon_1(x)$ has two zeros even at finite T and we put these zeros $\pm K'$, $K'>0$. From (3·4a) and (3·5) we have

$$\varepsilon_1(x)-\varepsilon_1^{(0)}(x)+\int_{-K}^{K} a_2(x-x')(\varepsilon_1-\varepsilon_1^{(0)})(x')dx'$$
$$=\int_{K'}^{K} a_2(x-x')\varepsilon_1(x')dx'+\int_{-K}^{-K'}\cdots$$
$$+T\int_{-\infty}^{\infty} a_2(x-x')\ln\left(1+\exp\left(-\frac{|\varepsilon_1(x')|}{T}\right)\right)dx'. \qquad (3\cdot7)$$

The first and the second terms of r.h.s. are of the order of $(K-K')^2$. The third term is

$$\frac{\pi^2T^2}{6\varepsilon_1'(K')}(a_2(x-K')+a_2(x+K'))(1+O(T)). \qquad (3\cdot8)$$

From Eq. (3·7) one can see

$$K-K'=O(T^2).$$

Equation (3·7) can then be written as

$$\varepsilon_1(x)-\varepsilon_1^{(0)}(x)+\int_{-K}^{K} a_2(x-x')(\varepsilon_1(x')-\varepsilon_1^{(0)}(x'))dx'$$
$$=\frac{\pi^2T^2}{6\varepsilon_1^{(0)\prime}(K)}(a_2(x-K)+a_2(x+K))+O(T^3), \qquad (3\cdot9a)$$

or

$$\varepsilon_1(x)-\varepsilon_1^{(0)}(x)=\frac{\pi^2T^2}{6\varepsilon_1^{(0)\prime}(K)}U(x)+O(T^3), \qquad (3\cdot9b)$$

where $U(x)$ is defined by

$$U(x)+\int_{-K}^{K} a_2(x-x')U(x')dx'=a_2(x-K)+a_2(x+K). \qquad (3\cdot9c)$$

Using Eq. (3·4c) we have

$$f(T,H)-f(0,H)=-T\int_{-\infty}^{\infty} a_1(x)\ln(1+\exp(-|\varepsilon_1(x)|/T))$$
$$+\int_{-K'}^{K'} a_1(x)\varepsilon_1(x)dx'-\int_{-K}^{K} a_1(x)\varepsilon_1^{(0)}(x)dx$$
$$=-\frac{\pi^2T^2}{6\varepsilon_1^{(0)\prime}(K)}\left[2a_1(K)-\int_{-K}^{K} a_1(x)U(x)dx\right]+O(T^3).$$
$$(3\cdot10a)$$

The bracket of the r.h.s. is $2V(K)$, where $V(x)$ is defined in (3·6b). Differentiating (3·5) and using partial integration, we have

$$\varepsilon_1^{(0)\prime}(x)=\frac{2\pi J\sin\theta}{\theta}\cdot W(x), \qquad (3\cdot10b)$$

$$W(x)+\int_{-K}^{K}a_2(x-x')W(x')dx'=-\frac{d}{dx}a_1(x). \tag{3·10c}$$

Then we have

$$f(T,H)=f(0,H)-\frac{\pi\theta T^2}{6J\sin\theta}\cdot\frac{V(K)}{W(K)}+O(T^3). \tag{3·11}$$

In the limit $K\to\infty\,(H\to 0)$ we can calculate $V(K)/W(K)$.

By the Fourier transformation (3·6b) and (3·10c), we have

$$V(x)-\int_K^{\infty}(R(x-x')+R(x+x'))V(x')dx'=\frac{1}{4}\operatorname{sech}\frac{\pi x}{2}, \tag{3·12a}$$

$$W(x)-\int_K^{\infty}(R(x-x')+R(x+x'))V(x')dx'=\frac{d}{dx}\Big(\frac{1}{4}\operatorname{sech}\frac{\pi x}{2}\Big). \tag{3·12b}$$

Considering that the inhomogeneous terms are $\frac{1}{2}e^{-\pi x/2}$ and $e^{-\pi x/2}\pi/4$ at $x\gg 1$ and that $R(x)$ becomes zero at $x\to\infty$, we have

$$\frac{V(K)}{W(K)}=\begin{cases}\dfrac{2}{\pi}+O(e^{-\pi K})+O(e^{-2\pi\theta K/(\pi-\theta)})=\dfrac{2}{\pi}+O\Big(\Big(\dfrac{\mu_0 H}{J}\Big)^2\Big) & \\ \qquad +O\Big(\Big(\dfrac{2\mu_0 H}{J}\Big)^{4\theta/(\pi-\theta)}\Big) \quad \text{for } -1<\Delta<1, & \\ \dfrac{2}{\pi}+O(e^{-\pi K})+O(K^{-2})=\dfrac{2}{\pi}+O\Big(\Big(\dfrac{\mu_0 H}{J}\Big)^2\Big)+O\Big(\Big(\ln\Big(\dfrac{\mu_0 H}{J}\Big)\Big)^{-2}\Big) & \\ \qquad \text{for } \Delta=1. & \end{cases} \tag{3·13}$$

Using the relation $C_H=-\partial/\partial T(T^2(\partial/\partial T)(f/T))$, we have

$$\lim_{H\to 0}\lim_{T\to 0}C_H/T=\frac{2\theta}{3J\sin\theta}. \tag{3·14}$$

At $\Delta=1$, $\lim_{H\to 0}\lim_{T\to 0}C_H/T$ is $2/3J$. This coincides with Yamada's result calculated from the Fermi liquid theory, and with Johnson and McCoy's[3] calculation. At $\Delta=0$, (3·14) is $\pi/3J$. This coincides with the known result for the X-Y model.[4] In the limit $\Delta\to -1$, (3·14) becomes infinity.

b) *Case* $J>0$, $2\mu_0 H\geq J(1+\Delta)$, $1\geq\Delta>-1$

The third term of (3·4a) is of the order of $T^{3/2}\exp((-2\mu_0 H+J(1+\Delta))/T)$. Substituting (3·4a) into (3·4c), we have

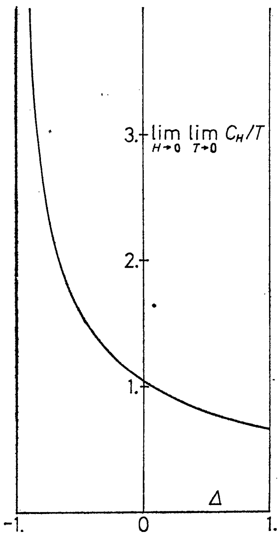

Fig. 1. The coefficient of T-linear low-temperature specific heat in the limit $T\to 0$ and $H\to 0$. In the limit $\Delta\to -1$, $\lim_{H\to 0}\lim_{T\to 0}C_H/T$ diverges as $(1+\Delta)^{-1/2}$.

$$f(T,H)=-\mu_0 H-\frac{T^{3/2}}{\pi}\sqrt{\frac{2}{J}}\int_0^\infty \ln\left(1+\exp\left(\frac{-2\mu_0 H+J(1+\Delta)}{T}\right)\right.$$
$$\left.\times\exp(-x^2)\right)dx(1+O(T))+O\left(T^{1/2}\exp\left(\frac{-2\mu_0 H+J(1+\Delta)}{T}\right)\right). \tag{3·15a}$$

At $2\mu_0 H-J(1+\Delta)\gg T$, (3·15a) is

$$f(T,H)=-\mu_0 H-T^{3/2}\frac{1}{\sqrt{2\pi J}}\exp\left(-\frac{2\mu_0 H-J(1+\Delta)}{T}\right)[1+O(T)], \tag{3·15b}$$

and we have

$$C_H=\frac{1}{\sqrt{2\pi J}}\frac{(2\mu_0 H-J(1+\Delta))^2}{T^{3/2}}\exp\left(-\frac{2\mu_0 H-J(1+\Delta)}{T}\right)[1+O(T)]. \tag{3·15c}$$

At $2\mu_0 H=J(1+\Delta)$ one obtains from (3·15a)

$$f(T,H)=-\mu_0 H-T^{3/2}\frac{1}{\sqrt{2\pi J}}\zeta\left(\frac{3}{2}\right)\left(1-\frac{1}{\sqrt{2}}\right)(1+O(T^{1/2})), \tag{3·15d}$$

$$C_H=\frac{3}{4}\frac{1}{\sqrt{2\pi}}\left(1-\frac{1}{\sqrt{2}}\right)\zeta\left(\frac{3}{2}\right)\sqrt{\frac{T}{J}}(1+O(T^{1/2})), \tag{3·15e}$$

where ζ is Riemann's zeta function. It is remarkable that the low-temperature specific heat is proportional to $T^{1/2}$ and that the coefficient does not depend on Δ at $2\mu_0 H=J(1+\Delta)$.

§ 4. Case $\mu_0 H\sim T$ and $T\ll J$

a) *Case* $J>0$, $1\geq\Delta\geq 0$

Equations (2·5) are transformed as

$$\ln\eta_j(x)=-\frac{A}{4T}\operatorname{sech}\frac{\pi x}{2}\delta_{j1}+\int_0^\infty\{D_{jl}(x-x')+D_{jl}(x+x')\}\ln(1+\eta_l(x'))dx',$$

$$\lim_{j\to\infty}\frac{\ln\eta_j}{n_j}=\frac{2\mu_0 H}{T},$$

$$f(T,H)-f(0,0)=-2T\int_0^\infty\frac{1}{4}\operatorname{sech}\frac{\pi x}{2}\ln(1+\eta_1(x))dx.$$

Putting $x\to x+(2/\pi)\ln(A/2T)$ and $T\to 0$, we have a set of non-linear integral equations which determines the T^2 order term of free energy as follows:

$$\ln\eta_j=-\delta_{j1}e^{-\pi x/2}+\sum_{l=1}^\infty D^*_{jl}\ln(1+\eta_l),\qquad j=1,2,\cdots, \tag{4·1a}$$

$$\lim_{j\to\infty}\frac{\ln\eta_j}{n_j}=y, \tag{4·1b}$$

$$C(y) = \int_{-\infty}^{\infty} e^{-\pi x/2} \ln(1+\eta_1(x))\, dx\,, \tag{4·1c}$$

$$f(T,H) - f(0,0) = -\frac{2\theta T^2}{\pi J \sin\theta} C\left(\frac{2\mu_0 H}{T}\right) + O(T^4). \tag{4·1d}$$

When π/θ is a rational number, one can reduce this set of equations to that with finite unknowns. Therefore the numerical method is available for the calculation of $C(x)$. Especially when $\pi/\theta=3$, the set of integral equations is

$$\ln\eta_1 = -e^{-\pi x/2} + s_1 * \ln(1+2\mathrm{ch}(3y/2)\cdot\kappa+\kappa^2),$$

$$\ln\kappa = s_1 * \ln(1+\eta_1).$$

In order to determine $\lim_{T\to 0}\lim_{H\to 0} C_H/T$, it is sufficient to calculate $C(0)$. From the numerical calculation, we have $C(0)=0.5235378$. This value is very near to $\pi/6=0.5235987\cdots$. When $\pi/\theta=\nu$ is an integer larger than 3, we have

$$\ln\eta_1 = -e^{-\pi x/2} + s_1 * \ln(1+\eta_2),$$

$$\ln\eta_j = s_1 * \ln(1+\eta_{j-1})(1+\eta_{j+1}), \quad 2\le j<\nu-2\,,$$

$$\ln\eta_{\nu-2} = s_1 * \ln(1+\eta_{\nu-3})\left(1+2\ \mathrm{ch}\left(\frac{\nu y}{2}\right)\cdot\kappa+\kappa^2\right),$$

$$\ln\kappa = s_1 * \ln(1+\eta_{\nu-2}).$$

The results of our numerical calculation of $C(0)$ at $\Delta=\cos(\pi/\nu)$, $\nu=3, 4,\cdots, 10$ are shown in Table I. In this calculation the number of unknown functions is $\nu-1$. An unknown function is represented by 40 unknown numbers. This set of non-linear equations is solved by the method of iteration through the use of computer NEAC 500. These values are very near to $\pi/6$. We can regard the small deviations from $\pi/6$ as numerical errors. It is very plausible that $C(0)$ is $\pi/6$ for any value of Δ and that $\lim_{T\to 0}\lim_{H\to 0} C_H/T$ is $2\theta/3J\sin\theta$, that is, $\lim_{T\to 0}\lim_{H\to 0} C_H/T = \lim_{H\to 0}\lim_{T\to 0} C_H/T$.

Table I. Numerical calculation of $C(0)$.

ν	Δ	$C(0)$
3	0.500000	0.5235378
4	0.707107	0.5233042
5	0.809017	0.5230769
6	0.866025	0.5229287
7	0.900969	0.5228421
8	0.923880	0.5227903
9	0.939693	0.5227562
10	0.951057	0.5227310

b) *Case* $J<0$, $1>\Delta\ge 0$

At $x\gg p_0-1$, it is clear that

$$\alpha_j(x) = \frac{1}{p_0-1}\sin\frac{\pi(p_0-1-j)}{2(p_0-1)}\cdot e^{-\pi x/2(p_0-1)}\,,$$

$$\beta(x) = \frac{1}{2(p_0-1)} e^{-\pi x/2(p_0-1)}\,,$$

from (2·4a) and (2·4b). Using these equations, we have the following integral equations for T^2 order term of free energy at $\mu_0 H \sim T$ and $T \ll |J|$:

$$\ln \eta_j = 2 \sin \frac{\pi(p_0-1-j)}{2(p_0-1)} e^{-\pi x/2(p_0-1)} + \sum_{l=1}^{\nu_1-1} B_{jl}^* \ln(1+\eta_l^{-1})$$
$$+ C_j^* n(1+\eta_{\nu_1}) \qquad \text{for } 1 \le j \le \nu_1 - 1, \tag{4·2a}$$

$$\ln \eta_{\nu_1} = -e^{-\pi x/2(p_0-1)} - \sum_{j=1}^{\nu_1-1} C_j^* \ln(1+\eta_j^{-1}) - s_2^* C_{\nu_1-1}^* \ln(1+\eta_{\nu_1})$$
$$+ \sum_{l>\nu_1} D_{\nu_1 l}^* \ln(1+\eta_l), \tag{4·2b}$$

$$\ln \eta_j = \sum_{l=\nu_1}^{\infty} D_{jl}^* \ln(1+\eta_l) \qquad \text{for } j > \nu_1, \tag{4·2c}$$

$$\lim_{j\to\infty} \frac{\ln \eta_j}{n_j} = y, \tag{4·2d}$$

$$C(y) = \sum_{\alpha=1}^{\nu_1-1} \sin \frac{\pi(p_0-1-j)}{2(p_0-1)} \int_{-\infty}^{\infty} e^{-\pi x/2(p_0-1)} \ln(1+\eta_j^{-1}) dx$$
$$+ \int_{-\infty}^{\infty} e^{-\pi x/2(p_0-1)} \ln(1+\eta_{\nu_1}) dx, \tag{4·2e}$$

$$f(T,H) - f(0,0) = -\frac{2(\pi-\theta)T^2}{\pi |J| \sin\theta} C\left(\frac{2\mu_0 H}{T}\right) + O(T^4). \tag{4·2f}$$

The author believes that $C(0)$ is $\pi/6$ in this case also, though numerical calculation is not yet done.

§ 5. The X-Y-Z model in zero field

In this section we consider the low-temperature specific heat of the X-Y-Z model in zero field. The Hamiltonian is

$$\mathcal{H} = \sum_{i=1}^{N} J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z, \quad \boldsymbol{S}_{N+1} = \boldsymbol{S}_1. \tag{5·1}$$

a) *Case* $0 \le J_x \le J_y < J_z$

Equations (4·10), (4·13) and (4·16) of Ref. 1) are written as

$$\varepsilon_1 = -A\boldsymbol{s}_1(x) + T\boldsymbol{D}_{12}^* \ln(1+\eta_2) + T\boldsymbol{D}_{11}^* \ln(1+\exp(\varepsilon_1/T)), \tag{5·2a}$$

$$\ln \eta_j = \boldsymbol{D}_{j1}^* \ln(1+\exp(\varepsilon_1/T)) + \sum_{l=2}^{\infty} \boldsymbol{D}_{jl}^* \ln(1+\eta_l), \quad j \ge 2, \tag{5·2b}$$

$$\lim_{j=\infty} \frac{\ln \eta_j}{n_j} = 0, \tag{5·2c}$$

$$f(T) = f(0) - T \int_{-Q}^{Q} \ln(1+\exp(\varepsilon_1(x)/T)) \boldsymbol{s}_1(x) dx, \tag{5·2d}$$

$$A = \pi J_z \operatorname{sn} 2\zeta/\zeta, \tag{5·2e}$$

$$\boldsymbol{D}_{jl}(x)=\begin{cases}(1-2\delta_{j,m_{i-1}})\delta_{j,l+1}\boldsymbol{s}_i(x)+\delta_{j,l-1}\boldsymbol{s}_i(x) & \text{for } m_{i-1}\leq j<m_i-1\,,\\ (1-2\delta_{j,m_{i-1}})\delta_{j,l+1}\boldsymbol{s}_i(x)+\delta_{jl}\boldsymbol{d}_i(x)+\delta_{j,l-1}\boldsymbol{s}_{i+1}(x) & \text{for } j=m_i-1\,.\end{cases} \tag{5·2f}$$

The first term of the r.h.s. of (5·2b) is of the order of $T\exp(-A\boldsymbol{s}_1(Q)/T)$. From Eqs. (5·2b) and (5·2c) we have

$$\eta_2=3+O(\sqrt{T}\exp(-A\boldsymbol{s}_1(Q)/T)) \tag{5·3a}$$

and from (5·2a)

$$\varepsilon_1(x)=-A\boldsymbol{s}_1(x)+T\ln 2+O(T^{3/2}\exp(-A\boldsymbol{s}_1(Q)/T)). \tag{5·3b}$$

Substituting this into (5·2d), we have

$$f(T)=f(0)-2T^{3/2}\boldsymbol{s}_1(Q)\sqrt{\frac{2\pi}{A\boldsymbol{s}_1''(Q)}}\exp(-A\boldsymbol{s}_1(Q)/T)\{1+O(T)\}. \tag{5·4a}$$

Using

$$\boldsymbol{s}_1(x)=\frac{K_{k'}}{2\pi}\operatorname{dn}(K_{k'}x;\,k),$$

we have

$$\boldsymbol{s}_1(Q)=\frac{K_{k'}}{2\pi}k',\quad \boldsymbol{s}_1''(Q)=\frac{K_{k'}^3}{2\pi}k^2k'\,, \tag{5·4b}$$

where k' is $\sqrt{1-k^2}$ and $K_{k'}$ is the complete elliptic integral of the first kind with modulus k', k being determined by

$$\frac{K_k}{K_{k'}}=\frac{K_{l'}}{\zeta},\quad \frac{J_y}{J_z}=\operatorname{dn}(2\zeta,l),\quad \frac{J_x}{J_z}=\operatorname{cn}(2\zeta,l), \tag{5·4c}$$

with $0\leq\zeta\leq K_l/2$.

b) *Case* $0\leq -J_x<-J_y<-J_z$

In the same way as Eqs. (2·5) were transformed into Eqs. (2·6), Eqs. (5·2) are transformed as follows:

$$\varepsilon_j(x)=-A\boldsymbol{\alpha}_j(x)+\sum_{l=1}^{\nu_1-1}\boldsymbol{B}_{jl}^*T\ln(1+\exp(-\varepsilon_l/T))+\boldsymbol{C}_j{}^*T\ln(1+\exp(\varepsilon_{\nu_1}/T))$$
$$\text{for } 1\leq j<\nu_1\,, \tag{5·5a}$$

$$\varepsilon_{\nu_1}(x)=A\boldsymbol{\beta}(x)-\sum_{l=1}^{\nu_1-1}\boldsymbol{C}_l{}^*T\ln(1+\exp(-\varepsilon_l/T))+\sum_{l>\nu_1}\boldsymbol{D}_{\nu_1 l}^*\,T\ln(1+\eta_l)$$
$$-\boldsymbol{s}_2{}^*\boldsymbol{C}_{\nu_1-1}^*T\ln(1+\exp(\varepsilon_{\nu_1}/T)), \tag{5·5b}$$

$$\ln\eta_j=\sum_{l>\nu_1}^{\infty}\boldsymbol{D}_{jl}^*\ln(1+\eta_l)+\boldsymbol{D}_{j,\nu_1}^*\ln(1+\exp(\varepsilon_{\nu_1}/T))\qquad\text{for } j>\nu_1\,, \tag{5·5c}$$

$$\lim_{j\to\infty}\frac{\ln\eta_j}{n_j}=0\,, \tag{5·5d}$$

$$f(T)=f(0)-T\sum_{j=1}^{\nu_1-1}\int_{-Q}^{Q}\boldsymbol{\alpha}_j(x)\ln(1+\exp(-\varepsilon_j/T))dx$$

$$-T\int_{-Q}^{Q}\boldsymbol{\beta}(x)\ln(1+\exp(\varepsilon_{\nu_1}/T))dx, \tag{5·5e}$$

where $\boldsymbol{\alpha}_j$, $\boldsymbol{\beta}$, $\boldsymbol{B}_{jl}$, $\boldsymbol{C}_j$ and $\boldsymbol{D}_{jl}$ are defined by

$$\boldsymbol{\alpha}_j(x)=\sum_{j=-\infty}^{\infty}\alpha_j(x+2jQ),\quad \boldsymbol{\beta}(x)=\sum_{j=-\infty}^{\infty}\beta(x+2jQ),\cdots$$

and $f(0)$ is the ground state energy per site. One can show

$$\boldsymbol{\beta}(x)=\frac{K_{k'}}{2\pi(p_0-1)}\mathrm{dn}\left(\frac{K_{k'}x}{p_0-1},\ k\right),\quad \boldsymbol{\alpha}_j(x)=\boldsymbol{\beta}(x+i(p_0-1-j))$$

$$+\boldsymbol{\beta}(x-i(p_0-1-j))=\frac{K_{k'}}{\pi(p_0-1)}\mathrm{dn}\left(\frac{K_{k'}x}{p_0-1},\ k\right)\mathrm{dn}\left(\frac{K_{k'}(p_0-1-j)i}{p_0-1},\ k\right)$$

$$\times\left(1-k^2\,\mathrm{sn}^2\left(\frac{K_{k'}x}{p_0-1},\ k\right)\mathrm{sn}^2\left(\frac{K_{k'}(p_0-1-j)i}{p_0-1},\ k\right)\right)^{-1},$$

where k is defined by

$$\frac{K_k}{K_{k'}}=\frac{K_{l'}}{K_l-\zeta},\quad \frac{J_y}{J_z}=\mathrm{dn}(2\zeta,l),\quad \frac{J_x}{J_z}=\mathrm{cn}(2\zeta,l). \tag{5·6}$$

Functions $\boldsymbol{\alpha}_j(x)$ and $\boldsymbol{\beta}(x)$ have the minimum at $x=Q$ and

$$\boldsymbol{\alpha}_j(Q)=\frac{K_{k'}k'}{\pi(p_0-1)}\mathrm{sn}\left(\frac{K_{k'}j}{p_0-1},\ k'\right),$$

$$\boldsymbol{\alpha}_j''(Q)=\frac{K_{k'}^3k'}{\pi(p_0-1)^3}\mathrm{sn}\left(\frac{K_{k'}j}{p_0-1},\ k'\right)\left(1+k'^2-2k'^2\,\mathrm{sn}^2\left(\frac{K_{k'}j}{p_0-1},\ k'\right)\right),$$

$$\boldsymbol{\beta}(Q)=\frac{K_{k'}k'}{2\pi(p_0-1)},\quad \boldsymbol{\beta}''(Q)=\frac{K_{k'}^3k^2k'}{2\pi(p_0-1)^3}. \tag{5·7}$$

From (5·5c) and (5·5b) we have

$$\eta_{\nu_1+1}=3+O(\exp(A\boldsymbol{\beta}(Q)/T)). \tag{5·8}$$

Substituting this into (5·5a) and (5·5b), we have

$$\varepsilon_j(x)=-A\boldsymbol{\alpha}_j(x)+O(T\exp(A\boldsymbol{\beta}(Q)))+O(T\exp(A\boldsymbol{\alpha}_1(Q)))$$
$$\text{for } j=1,2,\cdots,\nu_1-1,$$

$$\varepsilon_{\nu_1}(x)=A\boldsymbol{\beta}(x)+T\ln 2+O(T\exp(A\boldsymbol{\beta}(Q)))+O(T\exp(A\boldsymbol{\alpha}_1(Q))). \tag{5·9}$$

From (5·5e) one has

$$f(T)=f(0)-2T^{3/2}\boldsymbol{\beta}(Q)\sqrt{\frac{2\pi}{-A\boldsymbol{\beta}''(Q)}}\exp(A\boldsymbol{\beta}(Q)/T)\{1+O(T)\}$$

$$-T^{3/2}\alpha_1(Q)\sqrt{\frac{2\pi}{-A\alpha_1''(Q)}}\exp(A\alpha_1(Q)/T)\{1+O(T)\}. \qquad (5\cdot10)$$

At $\mathrm{sn}(K_{k'}/(p_0-1),\ k')>\frac{1}{2}$ the second term of (5·10) is dominant because of $\beta(Q)<\alpha_1(Q)$. It is interesting that the energy gap appearing in specific heat cannot be analytically continued as a function of $p_0=K_l/\zeta$ at $p_0=1+K_{k'}/(\mathrm{sn}^{-1}(\frac{1}{2},k'))$.

It is well known that $f(T)$ is invariant under the transformation $(J_x, J_y, J_z)\to(J_x, -J_y, -J_z)$. Then we restrict ourselves to the case $J_z>J_y\geq|J_x|$. The energy gap appearing in specific heat is

$$\frac{J_z\,\mathrm{sn}(2\zeta, l)K_{k'}k'}{2\zeta} \quad \text{for } \frac{K_l-\zeta}{\zeta}\geq\frac{\mathrm{sn}^{-1}(\frac{1}{2},\ k')}{K_{k'}}, \qquad (5\cdot11\text{a})$$

$$\frac{J_z\,\mathrm{sn}(2\zeta, l)K_{k'}k'}{\zeta}\,\mathrm{sn}\left(\frac{K_{k'}(K_l-\zeta)}{\zeta},\ k'\right) \quad \text{for } \frac{K_l-\zeta}{\zeta}\leq\frac{\mathrm{sn}^{-1}(\frac{1}{2},\ k')}{K_{k'}}, \qquad (5\cdot11\text{b})$$

where k, l and ζ are defined by (5·4c) in the condition $0\leq\zeta\leq K_l$.

The low-temperature free energy, at $J_z=J_y>0$ and $J_y\geq J_x>-J_y$, is obtained if we put $\mu_0H=0$ in Eqs. (4·1) and (4·2). Then the specific heat is proportional to T at low-temperature. But we cannot obtain low-temperature free energy at $J_z\geq J_y=-J_x>0$.

§ 6. Summary and discussion

For the Heisenberg-Ising ring, $f(0,0)$ and $\lim_{H\to0}\lim_{T\to0}\chi$ were obtained by Yang and Yang,[5] and des Cloizeaux and Gaudin.[6] In Table II these quantities are given as functions of $\theta(\Delta=\cos\theta)$ at $J>0$. An interesting feature is that these quantities are analytic as functions of Δ at $-1<\Delta<1$.

Table II. Analytic expression of physical quantities at $J>0$ and $0\leq\theta<\pi$.

$\lim_{H\to0}\lim_{T\to0}\chi$	$\lim_{H\to0}\lim_{T\to0}C_H/T$	The ground state energy per site
$\dfrac{4\theta\mu_0^2}{J(\pi-\theta)\pi\sin\theta}$	$\dfrac{2\theta}{3J\sin\theta}$	$-J\sin\theta\displaystyle\int_{-\infty}^{\infty}\frac{\mathrm{sh}(\pi-\theta)\omega\,d\omega}{2\,\mathrm{ch}\,\theta\omega\,\mathrm{sh}\,\pi\omega}$

We can discuss the low-temperature properties by considering a few kinds of excitations when $\mu_0H\gg T$, because other excitations have a finite energy gap. On the other hand in the case $H=0$, we must consider all kinds of excitations, because the energy gap vanishes. Therefore, integral equations for low-temperature thermodynamics have many unknown functions.

a) *Relations between the results for* $|\Delta|\geq1$ *and those for* $|\Delta|<1$

Gaudin[7] gave a set of integral equations for H-I ring at $\Delta\geq1$ which included author's set for $\Delta=1$[8] as a special case. His set of equations is equivalent to the following set:

$$\varepsilon_1(x) = -As_1(x) + s_1 * T\ln(1+\exp(\varepsilon_2(x)/T)), \qquad (6\cdot1a)$$

$$\varepsilon_j(x) = s_1 * T\ln(1+\exp(\varepsilon_{j-1}/T))(1+\exp(\varepsilon_{j+1}/T)), \qquad (6\cdot1b)$$

$$\lim_{j\to\infty}\frac{\varepsilon_j(x)}{j} = 2\mu_0 H, \qquad (6\cdot1c)$$

$$f(T,H) = f(0,0) - T\int_{-Q}^{Q} s_1(x)\ln(1+\exp(\varepsilon_1(x)/T))dx, \qquad (6\cdot1d)$$

where

$$s_1(x) = \sum_{j=-\infty}^{\infty}\frac{1}{4}\operatorname{sech}\frac{\pi}{2}(x-2jQ) = \frac{K_{k'}}{2\pi}\operatorname{dn}(K_{k'}x;\ k), \qquad (6\cdot1e)$$

$$Q = \frac{\pi}{\Phi},\ \frac{K_k}{K_{k'}} = Q, \qquad (6\cdot1f)$$

and Φ is determined by $\operatorname{ch}\Phi = \Delta$. Johnson and McCoy[8] calculated the low-temperature free energy using these equations.

At $J>0$, $0<2\mu_0 H<As_1(Q)$, $\Delta>1$ one obtains

$$\varepsilon_1(x) = -As_1(x) + T\ln(2\operatorname{ch}(\mu_0 H/T)) + O(T\exp(-\varepsilon_1(Q)/T)),$$

$$\varepsilon_j(x) = T\ln\left(\left(\frac{\operatorname{sh}(j\mu_0 H/T)}{\operatorname{sh}(\mu_0 H/T)}\right)^2 - 1\right) + O(T\exp(-\varepsilon_1(Q)/T)) \quad \text{for } j>1. \qquad (6\cdot2a)$$

Then one has

$$f(T,H) = f(0,0) - T^{3/2}s_1(Q)\sqrt{\frac{2\pi}{As_1''(Q)}}\exp(-As_1(Q)/T)\,2\operatorname{ch}\frac{\mu_0 H}{T}(1+O(T)). \qquad (6\cdot2b)$$

At $J>0$, $J(1+\Delta)>2\mu_0 H>As_1(Q)$, $\Delta\geq 1$ we can treat the problem in the same way as § 3, and obtain

$$\varepsilon_1(x) = -\frac{2\pi J\operatorname{sh}\Phi}{\Phi}a_1(x) + 2\mu_0 H + \int_{-Q}^{Q} a_2(x-x')T\ln(1+\exp(-\varepsilon_1(x')/T))dx', \qquad (6\cdot3a)$$

where

$$a_j(x) = \frac{\Phi}{2\pi}\cdot\frac{\operatorname{sh} j\Phi}{\operatorname{ch} j\Phi - \cos\Phi x}, \quad j=1,2. \qquad (6\cdot3b)$$

These correspond to Eqs. (3·4a) and (3·4b). The free energy is given by

$$f(T,H) - f(0,H) = -\frac{\pi\Phi T^2}{6J\operatorname{sh}\Phi}\frac{V(K)}{W(K)} + O(T^3) \qquad (6\cdot4a)$$

and K is determined by

$$2\mu_0 H = \frac{2\pi J\operatorname{sh}\Phi}{\Phi}\cdot\frac{V(K)}{Z(K)}, \qquad (6\cdot4b)$$

where

$$W(x)+\int_{-K}^{K}a_2(x-x')\,W(x')\,dx'=a_1(x),\tag{6·4c}$$

$$V(x)+\int_{-K}^{K}a_2(x-x')\,V(x')\,dx'=-\frac{d}{dx}a_1(x),\tag{6·4d}$$

$$Z(x)+\int_{-K}^{K}a_2(x-x')Z(x')\,dx'=1\,.\tag{6·4e}$$

The second term of (3·11) and (6·4a) are analytic functions of θ^2 and Φ^2, respectively. Putting $\Phi^2\to-\theta^2$ one finds that (6·4a), (6·4b) and $\Delta=\mathrm{ch}\,\Phi$ analytically continue to (3·11), (3·6a) and $\Delta=\cos\theta$ as functions of θ^2, respectively. Then one finds that $\lim_{T\to0}C_H/T$ is analytic as a function of Δ at $\Delta=1, H\neq0$.

At $J>0$, $\Delta\geq1, 2\mu_0H\geq J(1+\Delta)$ one has

$$\varepsilon_j(x)=-\frac{J\,\mathrm{sh}\,\Phi\,\mathrm{sh}\,j\Phi}{\mathrm{ch}\,j\Phi-\cos\Phi x}+2j\mu_0H+O(T^{-3/2}\lambda),\tag{6·5a}$$

$$\lambda=\exp\left(\frac{-2\mu_0H+J(1+\Delta)}{T}\right),\tag{6·5b}$$

therefore

$$f(T,H)=-\mu_0H-\frac{T^{3/2}}{\pi}\sqrt{\frac{2}{J}}\int_0^\infty\ln(1+\lambda\exp(-x^2))\,dx+O(\lambda T^2).\tag{6·5c}$$

In the case $J<0$, $\Delta>1$ and $\mu_0H\gg T$ one has

$$\varepsilon_j(x)=\frac{J\,\mathrm{sh}\,\Phi\,\mathrm{sh}\,j\Phi}{\mathrm{ch}\,j\Phi-\cos\Phi x}+2j\mu_0H+O(T^{-3/2}\lambda),\tag{6·6a}$$

$$\lambda=\exp\left(\frac{-2\mu_0H-J(1-\Delta)}{T}\right),\tag{6·6b}$$

$$f(T,H)=-\mu_0H-T^{3/2}\frac{1}{\sqrt{2\pi|J|}}\exp\left(\frac{-2\mu_0H-J(1-\Delta)}{T}\right).\tag{6·6c}$$

As $f(T,H)$ is invariant under the transformation $(J,\Delta)\to(-J,-\Delta)$, one has

$$f(T,H)=-\mu_0H-T^{3/2}\frac{1}{\sqrt{2\pi J}}\exp\left(\frac{-2\mu_0H+J(1+\Delta)}{T}\right),\tag{6·7}$$

for $J>0$, $\Delta\leq-1$ and $\mu_0H\gg T$. The results, (6·5c) and (6·7), analytically continue as functions of Δ to those of § 3 b) at $\Delta=\pm1$.

The results for the Heisenberg-Ising ring are summarized in Fig. 2. In the shaded region and on the segment d, the system shows T-linear low-temperature specific heat. On the line a, C_H is proportional to $T^{1/2}$. In the region $2\mu_0H>(1+\Delta)$, specific heat is proportional to $T^{-3/2}\exp(-\alpha/T)$.

Putting $l=1$ in Eq. (5·2a), one can derive the low-temperature free energy

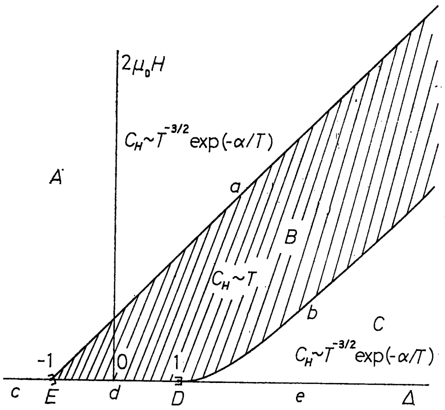

Fig. 2.

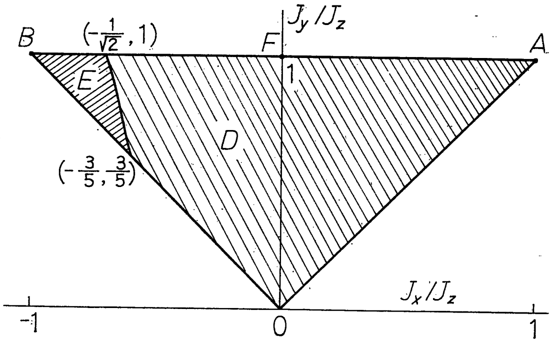

Fig. 3.

Fig. 2. Properties of the low-temperature specific heat of the Heisenberg-Ising model at $J=+1$. Regions A, B, C contain neither lines $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$ nor end points D, E. Point D is contained in line $\boldsymbol{d}$ and point E is contained in line $\boldsymbol{c}$.

Fig. 3. Properties of the low-temperature specific heat of the X-Y-Z model in zero field.

of Heisenberg-Ising model in zero field. This coincides with (6·2b) at $H=0$. We summarize the results for the X-Y-Z model in zero field in Fig. 3. On the line AFB except B, the specific heat (C) is proportional to T. In the shaded region and on lines AO except points A and O, C to $T^{-3/2}\exp(-\alpha/T)$. On the point O (Ising case), C to $T^{-2}\exp(-\alpha/T)$. In the regions D and E, the energy gap α is given by (5·11a) and (5·11b) respectively. One should note that the gap cannot be analytically continued from region D to E.

b) *Unsolved problems*

We can obtain neither low-temperature specific heat of H-I ring at $H=0$, $\Delta\leq -1$, $J>0$ nor that of X-Y-Z ring at $J_z\geq J_y=-J_z>0$.

If the magnetic susceptibility is invariant in reversing the order of the limits $T\to 0$, $H\to 0$, i.e.

$$\lim_{T\to 0}\lim_{H\to 0}\chi=\lim_{H\to 0}\lim_{T\to 0}\chi\,,$$

it is required

$$C''(0)=\frac{1}{2(\pi-\theta)}\qquad \text{for (4·1)},$$

and

$$C''(0)=\frac{1}{2\theta}\qquad \text{for (4·2)}.$$

But we can prove neither its validity nor $C(0)=\pi/6$ from Eqs. (4·1) or Eqs. (4·2).

Acknowledgment

The financial support of the Sakkokai Foundation is gratefully acknowledged.

Appendix A

——*Calculation of the third term of high-temperature expansion*——

Here we continue the calculation of Appendix C of Ref. 1). From Eqs. (C. 3) of Ref. 1) we have

$$\sum_{l=1}^{\infty}[\delta_{jl}(1+(\eta_j{}^{(0)})^{-1})-D_{jl}^{*}]c_l{}^{(2)}=\tfrac{1}{2}((\eta_j{}^{(0)})^{-1}+(\eta_j{}^{(0)})^{-2})(c_j{}^{(1)})^2. \quad (\mathrm{A}\cdot 1)$$

Then we have

$$\widehat{c_j^{(2)}}(\omega)=\sum_{l=1}^{\infty}\widehat{B}_{jl}(\omega)\frac{(\eta_l{}^{(0)})^{-1}+(\eta_l{}^{(0)})^{-2}}{2}\widehat{\{c_l{}^{(1)}\}^2}(\omega),$$

where $\widehat{B}_{jl}(\omega)$ is the jl element of inverse matrix of $\{\delta_{jl}(1+(\eta_j{}^{(0)})^{-1})-\widehat{D}_{jl}(\omega)\}$. One can show

$$\widehat{B}_{1l}(\omega)=-2\,\mathrm{ch}\,\omega\widehat{c_l^{(1)}}(\omega)(-1)^{r(l)+1},$$

where $r(j)$ is defined by $m_{r(j)}\leq j<m_{r(j)+1}$, therefore

$$\widehat{c_1^{(2)}}=\tfrac{1}{2}\sum_{l=1}^{\infty}(-1)^{r(l)}\{(\eta_l{}^{(0)})^{-1}+(\eta_l{}^{(0)})^{-2}\}2\,\mathrm{ch}\,\omega\widehat{c_l^{(1)}}(\omega)\widehat{(c_l{}^{(1)})^2}(\omega). \quad (\mathrm{A}\cdot 2)$$

Then the third term of $1/T$ expansion of f is

$$-A^2\int_{-\infty}^{\infty}s_1(x)c_1{}^{(2)}(x)dx=-\frac{A^2}{2}\sum_{l=1}^{\infty}(-1)^{r(l)}\{(\eta_l{}^{(0)})^{-1}+(\eta_l{}^{(0)})^{-2}\}\int_{-\infty}^{\infty}\{c_l{}^{(1)}(x)\}^3dx$$

$$=-\frac{A^2}{2}\left[\sum_{i=0}^{\infty}(-1)^i\left(\sum_{l=m_i+1}^{m_{i+1}-1}U_l\right)+\sum_{i=1}^{\infty}(-1)^iU_{m_i}\right], \quad (\mathrm{A}\cdot 3)$$

where

$$U_l=[(\eta_l{}^{(0)})^{-1}+(\eta_l{}^{(0)})^{-2}]\int_{-\infty}^{\infty}\{c_l{}^{(1)}(x)\}^3dx.$$

Substituting (C. 2) and (C. 6) of Ref. 1) into the above equation and carrying out integration, we obtain

$$U_l=\frac{1}{8\pi p_0{}^2f_{m_{i+1}}\{f(1)\}^3}\left[\frac{1}{f_lf_{l+1}}\{h_{m_{i+1}}^{(3)}-h_{l+1}^{(3)}-6\cot\psi_{m_{i+1}}(h_l{}^{(2)}-h_{l+1}^{(2)})+2(\psi_l+\psi_{l+1})\}\right.$$

$$+\frac{1}{f_{l+1}f_{l+2}}\{h_{m_{i+1}}^{(3)}+h_{l+1}^{(3)}+6\cot\psi_{m_{i+1}}(h_{l+1}^{(2)}-h_{l+2}^{(2)})+2(\psi_{l+1}+\psi_{l+2})\}$$

$$\left.+\frac{f_{l+2}}{f_l^2f_{l+1}}h_l{}^{(3)}-\frac{f_l}{f_{l+1}f_{l+2}^2}h_{l+2}^{(3)}\right] \qquad \text{for } m_i\leq l<m_{i+1},$$

where

$$\psi_l = q_l\pi/p_0, \quad \psi_{l+1} = \pi(q_l + (-1)^{i+1}p_{i+1})/p_0\,,$$
$$\psi_{l+2} = (q_l + 2(-1)^{i+1}p_{i+1})/p_0, \quad \psi_{m_{i+1}} = q_{m_{i+1}}\pi/p_0,$$
$$h_j^{(3)} = -3\cot\psi_j + (1+3\cot^2\psi_j)\psi_j\,,$$
$$h_j^{(2)} = 1 - \cot\psi_j\cdot\psi_j\,,$$
$$f_l = f(n_l - 1), \quad f_{l+1} = f(n_l + y_i - 1),$$
$$f_{l+2} = f(n_l + 2y_i - 1), \quad f_{m_{i+1}} = f(m_{i+1} - 1).$$

Using

$$\frac{f_{l+2}}{f_l^2 f_{l+1}} = -\frac{1}{f_l f_{l+1}} + \frac{2\,\mathrm{ch}\, y_i\mu_0 H}{f_l^2}\,,$$
$$\frac{f_l}{f_{l+1}f_{l+2}^2} = -\frac{1}{f_{l+1}f_{l+2}} + \frac{2\,\mathrm{ch}\, y_i\mu_0 H}{f_{l+2}^2}$$

and

$$\frac{1}{f_l f_{l+1}} = \frac{1}{f_{m_{i+1}}}\left(\frac{e^{-n_l\mu_0 H}}{f_l} - \frac{e^{-n_{l+1}\mu_0 H}}{f_{l+1}}\right),$$

one finds that almost all the terms in (A. 3) cancel out except a few terms, and obtains finally

$$(\mathrm{A}\cdot 3) = -\frac{J^2}{4}\left\{\frac{1+2\Delta^2}{(2\,\mathrm{ch}\,\mu_0 H)^2} - \frac{6\Delta^2}{(2\,\mathrm{ch}\,\mu_0 H)^4}\right\}. \qquad (\mathrm{A}\cdot 4)$$

This coincides with the third term of (1·2).

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