Letters to the Editor

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On Quadrupole Effects on Planetary Orbits

C. HOENSELAERS

Research Institute for Fundamental Physics
Kyoto University, Kyoto

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In a recently published paper[1] a relativistic approach to quadrupole effects on planetary orbits was tried. As this paper is not free from inaccuracies, the relativistic approach has to be done again with the correct arguments for power-series expansion of the geodesic equation and with the results one would expect looking at the Curzon-metric.

The main result is a justification of the intuitively obvious procedure of adding the classical quadrupole-effect to the Schwarzschild perihelion precession. The first relativistic quadrupole effects is also calculated, which is very small for mercury but may come into the range of observation for astronomical objects like binary neutron stars with highly eccentric orbits.

The metric for the Weyl-solutions is in polar Weyl-coordinates[2]

$$ds^2 = e^{2(\mu-\delta)}(dr^2 + r^2 d\theta^2) + e^{-2\delta} r^2 \sin^2\theta d\phi^2 - e^{+2\delta} dt^2 \qquad (1)$$

and the functions $\delta(r,\theta)$, $\mu(r,\theta)$ are given by[3]

$$\delta = \sum_{n=0}^{\infty} a_n r^{-(n+1)} P_n ,$$

$$\mu = \sum_{n'm=0}^{\infty} a_n a_m \frac{(n+1)(m+1)}{n+m+2} \times r^{-(n+m+2)} [P_{n+1}P_{m+1} - P_n P_m], \qquad (2)$$

where δ is the classical potential of some mass-distribution in (r, θ, ϕ)-coordinates,[2] which means

$$\delta(r, \theta) \leq 0 . \qquad (3)$$

Due to the two Killing-vectors of the Weyl-solutions we immediately have two first integrals of the geodesic-equation:

$$e^{-2\delta} r^2 \sin^2\theta \dot{\phi} = l ,$$

$$e^{2\delta} \dot{t} = E , \qquad (4)$$

where the dot denotes differentiation with respect to ds. Choosing now a solution of (2) with vanishing odd a_n, from the symmetry of δ and μ we can argue that

$$\theta = \frac{\pi}{2}$$

is also a solution of the geodesic equation.

Setting $R=1/r$, $\theta=\pi/2$, replacing $\dot{\phi}$, $\dot{t}$ by (4) and writing $R' = \dot{R}(\dot{\phi})^{-1}$ we get from (1)

$$R'^2 + R^2 e^{-2\mu} - \frac{1}{l^2} e^{-2(\mu+\delta)} (E^2 e^{-2\delta} - 1) = 0 . \qquad (5)$$

The solutions of this equation have to satisfy

$$e^{-2\delta}(E^2 e^{-2\delta} - 1) - l^2 R^2 =: F(R) \geq 0 .$$

As it can be seen from (2) and (3) we have (for $a_0 < 0$)

$$F(0) = E^2 - 1 .$$

$e^{-2\delta}$ is a monotonously increasing function; so by a proper choice of E and l we can make $F(0) < 0$ and $F(R)$ have zeros at $R_1 < R_2 < R_3$. This means that there is a solution of (5) confined to $R_1 \leq R \leq R_2$.

Expanding $e^{-2\mu} \cdot F(R)$ now in a power series

$$e^{-2\mu} \cdot F(R) = \sum_{n=0}^{\infty} \lambda_n R^n,$$

we can make an approximation of the order n if we assume the constants being chosen such that

$$\sup_{R_1 \leq R \leq R_2} | \sum_{m=n+1}^{\infty} \lambda_m R^m | \ll 1 .$$

We are now interested in the effect of the quadrupole moment on planetary orbits. Therefore we take the solution

$$\delta = -\frac{m}{r} + \frac{q}{2r^3}(3\cos^2\theta - 1),$$

$$\mu = -\frac{1}{2}\sin^2\theta \left[\frac{m^2}{r^2} - \frac{3}{2}\frac{mq}{r^4}(5\cos^2\theta - 1) + \frac{3}{4}\frac{q^2}{r^6}(25\cos^4\theta - 14\cos^2\theta + 1)\right] \quad (6)$$

of (2). To get a first impression of what we have to expect, we calculate the quadrupole moment of this solution.[4),*] As this Weyl-solution has the rotational symmetry about the axis $\theta = 0$ and the reflections symmetry on the equational plane $\theta = \pi/2$, the multipole-moments must be multiples of the symmetrized trace-free outer products of the axis-vector with itself. However the monopole and quadrupole are given by

$$M = -m ,$$

$$Q = q + \tfrac{1}{3} m^3. \quad (7)$$

Hence the quadrupole-moment will vanish iff

$$q = -\tfrac{1}{3} m^3$$

in accord with the well-known fact that the Schwarzschild-solution appears in Weyl-coordinates as the potential of a rod of length $2m$ and mass density $-\frac{1}{2}$.

The conformal transformation mapping this rod into the Schwarzschild-solution is

$$r^2 = \bar{r}^2 - 2m\bar{r} + m^2\cos^2\bar{\theta} ,$$

$$\cos\theta = \cos\bar{\theta}\,(\bar{r} - m) \times (\bar{r}^2 - 2m\bar{r} + m^2\cos^2\bar{\theta})^{-1/2} \quad (8)$$

or in the equatorial plane $\theta = \bar{\theta} = \pi/2$ with $u = 1/\bar{r}$

$$R = u(1 - 2mu)^{-1/2}. \quad (9)$$

Considering now the slow motion of a planet, thus taking $E^2 = 1$, substituting (9) in (6) and (5) and approximating until 4th-order we find after lengthy calculations

$$u'^2 + u^2 = 2mu^3 + \frac{1}{l^2}\left[2mu + \left(q + \frac{1}{3}m^3\right) \times u^3 + 5m\left(q + \frac{1}{3}m^3\right)u^4\right]. \quad (10)$$

A look in a standard textbook of classical mechanics shows that we may identify

$$\varepsilon := q + \tfrac{1}{3} m^3 \quad (11)$$

with the classical quadrupole-moment, which could be expected according to (7).

Differentiating Eq. (10) one arrives for $u' \neq 0$ at

$$u'' + u = \frac{m}{l^2} + 3mu^2 + \frac{3}{2}\frac{\varepsilon}{l^2}u^2 + 10\frac{m\varepsilon}{l^2}u^3 \quad (12)$$

or in dimensionless form

$$x'' + x = 1 + Ax^2 + Bx^3,$$

$$A = \frac{3m^2}{l^2} + \frac{3m\varepsilon}{2l^2} , \quad B = 10\frac{m^3\varepsilon}{l^6}$$

an equation which may be solved by various methods. Also (10) can be solved by the Weierstrass elliptic integrals.

If we assume that A is small of first and B small of second order, we may write

$$x_0'' + x_0 = 1 ,$$

$$x_1'' + x_1 = A x_0^2,$$

*) Note that each of the quoted expressions for the mass potential leads to the same multipole-moments for the Weyl-solutions.

$$x_2'' + x_2 = 2Ax_0x_1 + Bx_0^3$$

and enter with

$$x_0 = 1 + \alpha \cos \phi$$

into lengthy calculations, which finally give us a perihelion shift of

$$\Delta_\phi = 2\pi A + \pi\left[5A^2\left(1+\frac{\alpha^2}{6}\right) + 3B\left(1+\frac{\alpha^2}{4}\right)\right]. \tag{13}$$

Here one sees that neglecting second-order effects one may simply add the classical quadrupole-effect to the relativistic Schwarzschild-effect; moreover, that the coefficient (11) on which the quadrupole-effect depends, is just the same, as the quadrupole-moment calculated in (7) by quite another method.

The arguments following Eq. (5) give the rigorous justification of the admissibility of a power-series expansion of $F(R)$ in a certain region of the E-l phase-space of the particle.

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A Simple Composite Model Calculation for the Inelastic Structure Function

Masao NINOMIYA*) and Keiji WATANABE

Department of Physics, Nagoya University Nagoya

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It is the purpose of this paper to derive in a reliable way[1] the characteristic properties of the electroproduction structure functions from a composite model field theory. We take a super-renormalizable theory, which is known to reproduce[2] some of the features of the parton model.[3]

We consider a two-body bound state (hadron) of mass M where a charged spin 0 boson (quark) with a mass m and a neutral boson without the mass and spin are interacting via the interaction Lagrangian $\mathcal{L}_I = g\bar{\varphi}\varphi\phi$. To derive the qualitative features of the structure function we perform the ladder approximation; the bound state wave function $\Phi(P, q)$ is represented by the Wick-Cutkosky solution of the Bethe-Salpeter equation, $\Phi(P, q) = \int_{-1}^{1} dz \times (1-|z|)[-q^2 + Pqz + m^2 - M^2/4]^{-3}$.

To investigate the structure function $F(x)$ in the inelastic region, we apply the operator expansion near the light-cone for the product of currents.[4] $F(x)$ is then given by the following expression with standard notations

$$F(x) = \frac{1}{2\pi i}\int_{n_0 - i\infty}^{n_0 + i\infty} dn x^{-n+1} A(n), \tag{1}$$

where $n_0 > 0$. $A(n)$ is given by the reduced hadron matrix element of the expansion operator:

$$\langle P|\bar{\varphi}\partial_{\mu_1}\cdots\partial_{\mu_n}\varphi|P\rangle = A(n)\,[P_{\mu_1}, \cdots, P_{\mu_n}]. \tag{2}$$

The bracket denotes the totally symmetric

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