Progress of Theoretical Physics, Vol. 64, No. 1, July 1980

Rational Construction and Physical Signification of the Quantal Time Operator

Izuru FUJIWARA

Department of Mathematical Sciences
University of Osaka Prefecture, Sakai 591

(Received February 29, 1980)

First a general expression for an eigenstate $|t\rangle$ of the time operator T defined by the commutation relation $[T,H]=i\hbar$ is given in terms of the eigenvectors of the Hamiltonian H. Then the time operator itself is constructed as $T=\int dt\, t\,|t\rangle\langle t|$. When a compound state corresponding to transitions is introduced, it leads directly to the time-energy uncertainty relationship formulated previously by the same author.

§ 1. Introduction

The time operator T defined by the commutation relation

$$[T,H]=i\hbar \tag{1·1}$$

has been a subject of much debate for many years[1] in relation to the supposed time-energy uncertainty relation

$$\Delta t\cdot\Delta E\geq\hbar/2\,. \tag{1·2}$$

For the simplest case of a nonrelativistic free particle of mass m in one dimension with the Hamiltonian $H=p^2/2m$ one can conceive quite formally of a time operator

$$T=m(qp^{-1}+p^{-1}q)/2\,. \tag{1·3}$$

Because of the nonnegative character of the energy spectrum it is not, however, believed to be well defined in the Hilbert space.[2] On the other hand, as was stressed recently by the author,[3] an undoubtedly well defined time operator

$$T=-p/k \tag{1·4}$$

corresponds to the nonrelativistic Hamiltonian

$$H=p^2/2m+kq=k\exp(iup^3)\,q\exp(-iup^3) \tag{1·5}$$

with the constant $u=1/6m\hbar k$ for a one-dimensional motion of a particle of mass m in a constant uniform field of force specified by the force constant k. Nevertheless, no rational recipe for constructing the time operator T corresponding to a general Hamiltonian H has been put forward up to the present. Only the sojourn-time operator[4]

$$\theta = \int_0^\infty dt\, U_H^+(t)\, P(0)\, U_H(t) \tag{1·6}$$

with the evolution operator $U_H(t) = \exp(-itH/\hbar)$ was proposed in 1971 by Ekstein and Siegert as a substitute for the nonexistent time operator.

In 1945 Mandelstamm and Tamm[5] suggested to base the time-energy uncertainty relation (1·2) on the Heisenberg equation of motion $i\hbar dA/dt = [A, H]$ for a certain observable A. Especially if this is to be equivalent to the time parameter t, then the condition

$$dT/dt = 1 \tag{1·7}$$

defines the time operator T, which is subject to the commutation relation (1·1). Therefore, in one single normalized state $|\psi\rangle$ of the system one can construct at once a kind of uncertainty relation

$$(\Delta T)_\psi \cdot (\Delta H)_\psi \geq \hbar/2 \tag{1·8}$$

with the definitions

$$(\Delta T)_\psi^2 = \langle\psi|T^2|\psi\rangle - \langle\psi|T|\psi\rangle^2, \tag{1·9}$$

$$(\Delta H)_\psi^2 = \langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2. \tag{1·10}$$

However the meaning of Eq. (1·8) is quite obscure, for the state $|\psi\rangle$ is thought of at a definite instant of time, so that the scatter $(\Delta T)_\psi$ must vanish identically, as was pointed out in 1970 by Bunge.[6] Then should we conclude that the welcome commutation relation (1·1) never leads to the desired relationship (1·2)?

The present article is aimed first at giving a general recipe for constructing an eigenstate $|t\rangle$ of the time operator T defined by Eq. (1·1) for an arbitrarily prescribed Hamiltonian H, and then at showing that the time operator itself is defined simply by

$$T = \int dt\, t|t\rangle\langle t|\,. \tag{1·11}$$

The genuine uncertainty relation (1·2) is thus seen to be nothing but Eq. (1·8) reformulated on the basis of transitions of the system in just the same way as was argued by the same author.[7]

§ 2. Algorism for position and momentum operators

First of all, we shall recapitulate the algorism for position and momentum operators subject to the fundamental commutation relation

$$[q, p] = i\hbar\,. \tag{2·1}$$

In the momentum space we have, as is well known,

$$\langle p'|p''\rangle=\delta(p'-p'') \quad \text{and} \quad \int dp'|p'\rangle\langle p'|=1\,, \tag{2·2}$$

and the standard ket,[8] such that $\langle p'|S(p)\rangle=1$ for an arbitrary eigenvalue p', is defined by

$$|S(p)\rangle=\int dp'|p'\rangle\,. \tag{2·3}$$

Completely the same relationships hold also for the position operator q. The commutation relation (2·1) is equivalent to the Fourier transform:

$$\begin{aligned}|q'\rangle&=\int dp'|p'\rangle\langle p'|q'\rangle\\&=\int dp'|p'\rangle\exp(-ip'q'/\hbar)/\sqrt{h}\\&=U_p(q')\,|S(p)\rangle/\sqrt{h}\,,\end{aligned} \tag{2·4}$$

wherein the displacement operator $U_B(a)=\exp(-iaB/\hbar)$ is introduced for an eigenvalue a of an observable A such that $[A,B]=i\hbar$. If $|0(q)\rangle$ denotes a position eigenket $|q'\rangle$ with $q'=0$, then Eq. (2·4) reduces to

$$\sqrt{h}\,|0(q)\rangle=|S(p)\rangle\,, \tag{2·5}$$

so that

$$|q'\rangle=U_p(q')\,|0(q)\rangle\,. \tag{2·6}$$

Thus

$$\begin{aligned}q&=\int dq'q'|q'\rangle\langle q'|\\&=\int dq'q'U_p(q')\,|S(p)\rangle\langle S(p)|\,U_p^{+}(q')/h\,.\end{aligned} \tag{2·7}$$

In the second line the eigenvalue q' of the position operator enters simply as a c-number.

§ 3. Construction of the time operator

In order to pursue the parallelism between the commutation relations (1·1) and (2·1) we have to assume that the energy eigenvectors $|E'\rangle$ defined by $H|E'\rangle=E'|E'\rangle$ form a complete orthonormal set over a continuous and unbounded energy spectrum in the same way as in Eq. (2·2):

$$\langle E'|E''\rangle=\delta(E'-E'') \quad \text{and} \quad \int dE'|E'\rangle\langle E'|=1\,. \tag{3·1}$$

If the time parameter t is regarded as one of the eigenvalues of the time operator T in such a way that

$$T|t\rangle = t|t\rangle , \tag{3·2}$$

and moreover if the eigenket $|t\rangle$ with $t=0$ is denoted by $|0(T)\rangle$, one obtains, corresponding to Eqs. (2·3), (2·4), (2·5) and (2·6),

$$|S(H)\rangle = \int dE'|E'\rangle = \sqrt{h}|0(T)\rangle , \tag{3·3}$$

$$|t\rangle = U_H(t)\,|0(T)\rangle = U_H(t)\,|S(H)\rangle/\sqrt{h}. \tag{3·4}$$

Thus we have

$$H|t\rangle = i\hbar(\partial/\partial t)\,|t\rangle , \tag{3·5}$$

and just as for the position eigenket $|q'\rangle$

$$\langle t'|t''\rangle = \delta(t'-t'') \quad \text{and} \quad \int dt|t\rangle\langle t| = 1 . \tag{3·6}$$

Then Eq. (2·7) is transcribed straightforwardly to give

$$\begin{aligned} T &= \int_{-\infty}^{\infty} dt\; t|t\rangle\langle t| \\ &= \int dt\; tU_H(t)\,|S(H)\rangle\langle S(H)|\,U_H{}^{+}(t)/h , \end{aligned} \tag{3·7}$$

and Eq. (3·5) leads after a formal integration by parts to

$$[T, H]/i\hbar = -\int dt\, t(\partial/\partial t)\,|t\rangle\langle t| = 1 \tag{3·8}$$

in complete agreement with Eq. (1·1). Furthermore we have

$$T|E'\rangle = -i\hbar(\partial/\partial E')\,|E'\rangle \tag{3·9}$$

corresponding to Eq. (3·5) and

$$U_H{}^{+}(t)\,TU_H(t) = T + t , \tag{3·10}$$

as is required by Eq. (1·7). It is important to note in the second line of Eq. (3·7) that the eigenvalue t may be regarded as a simple time parameter.

For the Hamiltonian (1·5) the orthonormal energy eigenket reads

$$|E'\rangle = |k|^{-1/2}\exp(iup^3)\,|q'\rangle \tag{3·11}$$

with $q' = E'/k$. Thus Eq. (3·3) is evaluated as

$$|k|^{-1/2}|S(H)\rangle = \exp(iup^3)\,|S(q)\rangle = \sqrt{h}\,|0(p)\rangle \tag{3·12}$$

with $\sqrt{h}\,|0(p)\rangle = |S(q)\rangle = \int dq'\,|q'\rangle$. In view of the unitary transformation (1·5) we see that

$$U_H(t) = \exp(iup^3)\,U_q(kt)\exp(-iup^3), \tag{3·13}$$

so that Eq. (3·4) is rewritten with $|p'\rangle = U_q(-p')\,|0(p)\rangle$ as

$$|t\rangle = |k|^{1/2}U_H(t)\,|0(p)\rangle = |k|^{1/2}|p'\rangle\exp(iup'^3) \tag{3·14}$$

with $p' = -kt$. Hence Eq. (3·7) leads at once to the desired result (1·4) as

$$T = -\int dp'p'\,|p'\rangle\langle p'|/k = -p/k. \tag{3·15}$$

§ 4. Time-energy uncertainty relation

The foregoing discussion enables one to show how the time operator T defined generally by Eq. (3·7) leads to the time-energy uncertainty (1·2). As was pointed out by the present author,[7),9)] the relationship (1·2) is established on the basis of the transition amplitude

$$\begin{aligned}\Psi(t) &= N\langle\psi|U_H^+(t)\,|\phi\rangle/\sqrt{h}\\ &= \int dE'\Phi(E')\exp(itE'/\hbar)/\sqrt{h}\end{aligned} \tag{4·1}$$

with the energy amplitude

$$\Phi(E') = N\langle\psi|E'\rangle\langle E'|\phi\rangle \tag{4·2}$$

and a normalization constant N being positive and adjusted so that

$$\int dt\,|\Psi(t)|^2 = \int dE'\,|\Phi(E')|^2 = 1. \tag{4·3}$$

Since Eqs. (3·1) and (3·3) give

$$|\psi\rangle = \int dE'\,|E'\rangle\langle E'|\psi\rangle = \psi(H)\,|S(H)\rangle \tag{4·4}$$

with $\psi(E') = \langle E'|\psi\rangle$, the transition amplitude (4·1) is rewritten with the aid of Eq. (3·4) as

$$\Psi(t) = N\langle S(H)\,|\psi^*(H)\,U_H^+(t)\,|\phi\rangle/\sqrt{h} = \langle t|\Psi\rangle \tag{4·5}$$

for a compound state corresponding to transitions:

$$|\Psi\rangle = |\psi^*\phi\rangle = N\psi^*(H)\,|\phi\rangle. \tag{4·6}$$

In the same way, Eq. (4·2) is transcribed as

$$\Phi(E') = \langle E'|\Psi\rangle, \tag{4·7}$$

and Eq. (4·3) is nothing but the normalization condition:

$$\langle\Psi|\Psi\rangle=1\,. \tag{4·8}$$

The pair of transition amplitudes $\Psi(t)$ and $\Phi(E')$ respectively in time and energy are related with each other by the Fourier transform (4·1). This is exactly of the same form as that derived from Eq. (2·4):

$$\langle q'|\psi\rangle=\int dp'\langle p'|\psi\rangle\exp(ip'q'/\hbar)/\sqrt{h} \tag{4·9}$$

for the amplitudes $\langle q'|\psi\rangle$ and $\langle p'|\psi\rangle$ that are defined just in one single normalized state $|\psi\rangle$ of a system contemplated at a definite instant of time. The compound state $|\Psi\rangle=|\psi^*\phi\rangle$ refers to a transition between a pair of normalized states $|\psi\rangle$ and $|\phi\rangle$ separated by a finite interval of time, so that the expectation value

$$\langle\Psi|T|\Psi\rangle=\int dt\, t|\Psi(t)|^2 \tag{4·10}$$

defines the average transition time or equivalently the arrival time investigated extensively in 1969 by Allcock.[10] On the other hand,

$$\langle\Psi|H|\Psi\rangle=\int dE'E'|\Phi(E')|^2 \tag{4·11}$$

specifies the central energy value of the energy channel of the transition, which is determined statistically by the probability $|\Phi(E')|^2$. Equations (1·9) and (1·10), wherein the one single state $|\psi\rangle$ is replaced by our new compound state $|\Psi\rangle$, give the scatters

$$(\Delta t)_{\phi,\psi}=(\Delta T)_\Psi \quad \text{and} \quad (\Delta E)_{\phi,\psi}=(\Delta H)_\Psi\,, \tag{4·12}$$

to which applies quite automatically the inequality

$$(\Delta T)_\Psi\cdot(\Delta H)_\Psi\geq\hbar/2\,. \tag{4·13}$$

This is nothing but the desired time-energy uncertainty relation (1·2). Here the uncertainty in time measures the sojourn time,[4] which should not vanish in general.[6] On the other hand, the uncertainty in energy measures the width of the energy channel of transition.

§ 5. Explicit demonstrations

An explicit construction of the time-energy uncertainty relation has been given recently by the present author in collaboration with Wakita and Yoro[9] for the system specified by the Hamiltonian (1·5). It is based on the principles and interpretations proposed by the author[7] in 1970 and 1979. The content of that article may be reformulated with the aid of the time operator (3·15) in the following way.*)

A normalized state $|\psi\rangle$ rewritten here as $|\alpha\rangle$ is defined conveniently in the energy-space by

$$\langle E'|\alpha\rangle = N_\alpha \exp(ip_A E'/\hbar k)\,[1+(E'-E_0)^2/c^2]^{-\alpha} \tag{5·1}$$

with the constants c, p_A, E_0 and the normalization constant

$$N_\alpha = (\pi c^2)^{-1/4}[\Gamma(2\alpha)/\Gamma(2\alpha-1/2)]^{1/2}. \tag{5·2}$$

Here the continuous parameter α is assumed to be larger than 3/4. Then one obtains the expectation value $\langle\alpha|H|\alpha\rangle = E_0$ and

$$(\Delta H)_\alpha^2 = \langle\alpha|H^2|\alpha\rangle - \langle\alpha|H|\alpha\rangle^2 = c^2/(4\alpha-3). \tag{5·3}$$

On the other hand, for the momentum we have the expectation value $\langle\alpha|p|\alpha\rangle = p_A$ and

$$\begin{aligned}(\Delta p)_\alpha^2 &= \langle\alpha|p^2|\alpha\rangle - \langle\alpha|p|\alpha\rangle^2\\ &= \alpha(\alpha-1/4)(\hbar k/c)^2/(\alpha+1/2),\end{aligned} \tag{5·4}$$

so that corresponding to the commutation relation $[H,p]=i\hbar k$ we arrive at the uncertainty relation

$$(\Delta p\cdot\Delta H)_\alpha = [1+3/(2\alpha+1)(4\alpha-3)]^{1/2}\hbar|k|/2. \tag{5·5}$$

All the above results are concerned with one single state $|\psi\rangle = |\alpha\rangle$ contemplated at a definite instant of time.

In order to think of a transition of the system we have to assume another state $|\phi\rangle = |\beta\rangle$ defined by

$$\langle E'|\beta\rangle = N_\beta \exp(ip_B E'/\hbar k)\,[1+(E'-E_0)^2/c^2]^{-\beta} \tag{5·6}$$

with the normalization constant N_β which is obtained by substituting β for α in Eq. (5·2). In the compound state (4·6) we thus have $N = N_{\alpha+\beta}/N_\alpha N_\beta$ and

$$\psi^*(H) = N_\alpha \exp(-ip_A H/\hbar k)\,[1+(H-E_0)^2/c^2]^{-\alpha}. \tag{5·7}$$

Therefore the energy amplitude (4·7) is obtained at once as

$$\Phi(E') = N_{\alpha+\beta}\exp[i(p_B-p_A)E'/\hbar k]\,[1+(E'-E_0)^2/c^2]^{-\alpha-\beta}, \tag{5·8}$$

which is nothing but Eq. (5·1) wherein only α and p_A are replaced respectively by $\alpha+\beta$ and p_B-p_A. Hence $\langle\Psi|H|\Psi\rangle = E_0$ and Eq. (5·3) is rewritten as

$$(\Delta H)_\Psi^2 = c^2/(4\alpha+4\beta-3). \tag{5·9}$$

Since $T=-p/k$ in the present case, one obtains at once the arrival time $\langle\Psi|T|\Psi\rangle$

*) For the details of the computation, readers are requested to refer to Ref. 9).

$=(p_A-p_B)/k$, which is in complete agreement with the classical result. Moreover, Eq. (5·4) is transcribed as

$$(\Delta T)_{\Psi}{}^2=(\alpha+\beta)(\alpha+\beta-1/4)\hbar^2/(\alpha+\beta+1/2)c^2, \tag{5·10}$$

so that the inequality (4·13) is written explicitly as

$$(\Delta T)_{\Psi}\cdot(\Delta H)_{\Psi}=[1+3/(2\alpha+2\beta+1)(4\alpha+4\beta-3)]^{1/2}\hbar/2\,. \tag{5·11}$$

The right side is larger than $\hbar/2$ for finite values of α and β, but in the Gaussian limit[9] with α, $\beta\to\infty$ it reaches the minimum uncertainty product $\hbar/2$.

§ 6. Decay process and mean lifetime

If the normalized state $|\phi\rangle$ happens to be identical with $|\psi\rangle$, the time amplitude (4·1) reduces to

$$\Psi_\psi(t)=N\langle\psi|U_H{}^+(t)|\psi\rangle/\sqrt{h}=\Psi_\psi{}^*(-t), \tag{6·1}$$

of which the absolute square is an even function in t. Therefore in this particular case $\langle\Psi_\psi|T|\Psi_\psi\rangle=0$ in Eq. (4·10), and accordingly the arrival time vanishes as it should. If the probability $|\Psi_\psi(t)|^2$ decreases sufficiently rapidly starting with N^2/h at $t=0$, then we have a decay process investigated in 1947 by Fock and Krylov.[11] Now as a modified form of the sojourn time we can define the mean lifetime of the state $|\psi\rangle$ by

$$\tau_\psi=2\int_0^\infty dt\; t|\Psi_\psi(t)|^2 \tag{6·2}$$

subject in general to the condition

$$(\tau_\psi)^2\leq(\Delta t)^2_{\psi,\psi}=2\int_0^\infty dt\; t^2|\Psi_\psi(t)|^2, \tag{6·3}$$

which is obtained from the Schwarz inequality

$$\int_a^b dt[f(t)]^2\cdot\int_a^b dt[g(t)]^2\geq\left(\int_a^b dt f(t)g(t)\right)^2$$

with $f(t)/t=g(t)=|\Psi_\psi(t)|$. For the Hamiltonian (1·5) and the state (5·1) the mean lifetime is evaluated to give

$$\tau_\alpha=\hbar\Gamma(2\alpha+1/2)^2\Gamma(4\alpha)/2c\sqrt{\pi}\alpha\Gamma(2\alpha)^2\Gamma(4\alpha-1/2), \tag{6·4}$$

which is always smaller than

$$(\Delta t)_{\alpha,\alpha}=\hbar[2\alpha(\alpha-1/8)/(\alpha+1/4)]^{1/2}/c \tag{6·5}$$

for $\alpha>1/8$. But both of them vanish at $\alpha=1/8$.

§ 7. Concluding remarks

All the above discussions based on the formal parallelism between the commutation relations $[q, p] = i\hbar$ and $[T, H] = i\hbar$ are valid only when the conditions (3·1) are guaranteed over a continuous and unbounded energy spectrum. Therefore the scheme established above can be applied safely to the Hamiltonians such as

$$H = p^2/2m - m\omega^2 q^2/2 \qquad (7\cdot1)$$

for a one-dimensional hyperbolic oscillator and then the Hamiltonian (1·5) extended directly to the case of an electron:

$$H = H_F + kq_3 , \qquad (7\cdot2)$$

wherein H_F denotes the free Dirac Hamiltonian, since for both of them the energy spectrum must be continuous and unbounded. Especially for the latter Hamiltonian we shall show that Eq. (3·7) leads to the time operator $T = -p_3 I/k$ instead of Eq. (3·15), wherein the 4×4 identity matrix is denoted by I.

On the other hand, one may conclude that the assumed time operator (1·3) for a nonrelativistic free particle does not fit into our present scheme, for the energy spectrum of the Hamiltonian $H = p^2/2m$ is believed to be nonnegative. However, as will be reported in the immediate future, our general recipe (3·7) reproduces it indeed when the Hilbert space is extended adequately to include the negative energy spectrum. Finally we may be allowed to add that the Hamiltonian

$$H = p^2/2m + m\omega^2 q^2/2 = \hbar\omega\,(a^+ a + 1/2) \qquad (7\cdot3)$$

for one-dimensional harmonic oscillator is now under investigation, corresponding to which the time operator

$$T = i\,(\log a - \log a^+)\,/2\omega \qquad (7\cdot4)$$

is very likely to result too from our equation (3·7). Here, of course, a and a^+ denote the construction operators.

References

1) M. Razavy, Am. J. Phys. **35** (1967), 955; Nuovo Cim. **63B** (1969), 271; Can. J. Phys. **49** (1971), 3075.
D. M. Rosenbaum, J. Math. Phys. **10** (1969), 1127.
Other references being numerous in number are found in R. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40** (1968), 411; *The Uncertainty Principle and Foundations of Quantum Mechanics*, edited by W. C. Price and S. S. Chissick (John Wiley and Sons, 1977).
2) W. Pauli, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, 1958), vol. 5/1, p. 60.
3) I. Fujiwara, Prog. Theor. Phys. **62** (1979), 1179.
4) H. Ekstein and A. J. F. Siegert, Ann. of Phys. **68** (1971), 509.

5) L. Mandelstamm and I. Tamm, J. of Phys. (USSR) **9** (1945), 249.
6) M. Bunge, Can. J. Phys. **48** (1970), 1410.
7) I. Fujiwara, Prog. Theor. Phys. **44** (1970), 1701; **62** (1979), 1438.
8) P. A. M. Dirac, *The Principles of Quantum Mechanics,* 4th ed. (Oxford University Press, 1958), p. 79.
9) I. Fujiwara, K. Wakita and H. Yoro, Prog. Theor. Phys. **64** (1980), No. 2.
10) G. R. Allcock, Ann. of Phys. **53** (1969), 235, 286, 311.
11) V. Fock and N. Krylov, J. of Phys. (USSR) **11** (1947), 112.