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Generalized Dirac Equation with Four Orthogonal Families of Spin 1/2 Solutions

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For the description of a family of leptons—and that of quarks—a new concept of fused system with attribute both of the elementary particle and of the composite system is introduced and a new fusion theory is developed. The fused system is postulated to be characterized by a Dirac-like wave-equation with coefficients belonging to an algebra consisting of triple-direct-products of γ-matrices. Four Lorentz invariant manifolds of solutions of the wave equation represent the spin 1/2 particles with the same internal quantum number and different masses such as e, μ, τ and the fourth charged lepton all of which have the Dirac g-value of 2. In sharp contrast to the conventional composite model of leptons and quarks no spin 3/2 particle appears in this scheme.

§ 1. Introduction

The discovery of the tau lepton and the determination of its properties have revived the old electron-muon puzzle in an intensified form. Without the difference among their masses the three charged leptons e, μ and τ are confirmed to be indistinguishable.[1] This fact—along with the similar increase of quark generations and the parallelism between leptons and quarks—has naturally led to the hypothesis that leptons—and quarks—are composite and have intrinsic spacetime structure.[2] On the other hand, however, the precise agreement between theory[3] and experiment[4] for the gyromagnetic ratios of electron and muon has demonstrated definitely the validity of perturbative quantum electrodynamics, in which the electron and muon are postulated to be indivisible point-like particles obeying the Dirac equation. It seems very hard for the composite models of leptons[5] to give a satisfactory reason for the accurate experimental result that the lepton g-values, except for the QED correction calculated up to the third power of the fine structure constant, are identical with the Dirac value of 2.

In order to harmonize these apparently incompatible features of leptons, i. e., their elementarity and multiplicity, we propose in this paper a new viewpoint in which the leptons—similarly the quarks—carrying the same internal quantum numbers and different masses are identified with orthogonal eigenmodes of certain entity whose behaviour is described by a single generalized-wave-equation of Dirac type. Such an entity, which will be called a *fused system*, is considered to have intermediate properties in between the elementary particle and the composite system. To make the wave equation characterizing the fused system have plural Lorentz-invariant-manifolds of solutions representing the particle states with "a spin angular momentum of half a quantum and a magnetic moment of unit magneton",[6] we generalize the Dirac equation so that coefficient matrices belong to a new algebra larger than that of usual γ-matrices. We call it a *fusion algebra* and construct it explicitly from the triple-direct-products of γ-matrices.

In § 2 we clarify the distinction between the well-known algebra for the composite system and the fusion algebra which includes four sets of quartette product-matrices satisfying the anticommutation relations of Dirac type. The fusion algebra is proved to have 64 dimensions as a vector space and the order of its centre is shown to be 4 in § 3. For the description of the fused system we postulate a wave equation with coefficients belonging to the fusion algebra, prove its covariance under the proper and improper Lorentz transformations, and carry out the quantization in § 4. Corresponding to the order of the centre of fusion algebra the wave equation has four irreducible manifolds of solutions representing spin 1/2 particles. Section 5 is assigned to derive explicitly such solutions in the rest frame of fused system. In contrast to the composite system consisting of three Dirac particles the fused system has no particle-mode with spin 3/2. We make the interpretation that four orthogonal solutions describe the fundamental fermions with the same internal quantum number such as electron, muon, τ-lepton and another charged lepton whose existence is predicted in this paper, and give further comments on the unification of leptons and quarks in § 6.

§ 2. Fusion matrices

To distinguish our approach from the standard theory of composite system, let us consider first a system of three Dirac particles. The total spin angular momentum of the system is the vector sum of spin operators of constituents. In terms of relativity this means that the generators of spacetime rotation of the composite system $M_{\mu\nu}^{(c)}$ are given by the sums of direct-products of the component generators $\sigma_{\mu\nu}=i/2[\gamma_\mu, \gamma_\nu]$ as follows:*)

$$M_{\mu\nu}^{(c)}=\frac{1}{2}\ \Sigma_{\mu\nu}^{(c)}=\frac{1}{2}\sigma_{\mu\nu}\times 1\times 1+1\times\frac{1}{2}\sigma_{\mu\nu}\times 1+1\times 1\times\frac{1}{2}\sigma_{\mu\nu}\ , \tag{1}$$

where 1 symbolizes the 4×4 unit matrix. It is straightforward to verify that the resultant operators $M_{\mu\nu}^{(c)}$ satisfy the general commutation rules

$$[\mathcal{M}_{\kappa\lambda},\mathcal{M}_{\mu\nu}]=-ig_{\kappa\mu}\mathcal{M}_{\lambda\nu}+ig_{\kappa\nu}\mathcal{M}_{\lambda\mu}-ig_{\lambda\nu}\mathcal{M}_{\kappa\mu}+ig_{\lambda\mu}\mathcal{M}_{\kappa\nu} \tag{2}$$

for the generators of Lorentz group $\mathcal{M}_{\mu\nu}$.

*) We adopt the metric $(g_{\mu\nu})=(+---)$ and the representation of γ-matrices $\gamma^0=\gamma_0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\gamma^j=-\gamma_j=\begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix}$ for $j=1$, 2 and 3.

We must notice, however, that it is possible to construct another set of product-operators obeying the rules (2) in the following form:

$$M_{\mu\nu}=\frac{1}{2}\Sigma_{\mu\nu}=-\frac{1}{2}\sigma_{\mu\nu}\times\sigma_{\mu\nu}\times\sigma_{\mu\nu}\,. \tag{3}$$

The comparison between the space parts of both operators $M_{\mu\nu}^{(c)}$ and $M_{\mu\nu}$ shows that, while the former embodies faithfully the addition law of spins, the latter does not allow the conventional interpretation of spin synthesis. In fact, in contrast to the spin quantum numbers 1/2 and 3/2 which the resultant spin operators $(M_{23}^{(c)}, M_{31}^{(c)}, M_{12}^{(c)})$ can take, the new operators (M_{23}, M_{31}, M_{12}) are directly confirmed to have the spin quantum number 1/2 exclusively.*) Therefore, $M_{\mu\nu}$ are not the Lorentz generators of composite system in the usual sense. Nevertheless, $M_{\mu\nu}$ constitute the representation of the Lie algebra of Lorentz group, and it is this peculiar representation with which we are going to formulate the theory for the fused system.

In order to develop the new fusion theory so that the operators $M_{\mu\nu}$ receive a proper interpretation, it is adequate to follow the method exploited in the ordinary fusion theory.[7] Just alike $\sigma_{\mu\nu}$ is generated out of γ-matrices, the composite operators $\Sigma_{\mu\nu}^{(c)}$ are expressed by the commutators

$$\Sigma_{\mu\nu}^{(c)}=\frac{i}{2}[\Gamma_\mu{}^{(c)},\Gamma_\nu{}^{(c)}] \tag{4}$$

provided that $\Gamma_\mu{}^{(c)}$ is defined by

$$\Gamma_\mu{}^{(c)}=\gamma_\mu\times 1\times 1+1\times\gamma_\mu\times 1+1\times 1\times\gamma_\mu\,. \tag{5}$$

We must find such kinds of product-matrices, which generate not the conventional Lorentz operators $M_{\mu\nu}^{(c)}$ but the new Lorentz operators $M_{\mu\nu}$, in the $(16)^3=4096$ dimensional linear space consisting of triple-direct-products of 1, γ_μ, $\sigma_{\mu\nu}$, $\gamma_5\gamma_\mu$ and γ_5. Four sets**) of product-matrices satisfying this requirement are hunted down as follows:

$$\begin{cases}\Gamma_0(0)\equiv\Gamma_0=\gamma_0\times\gamma_0\times\gamma_0\,,\\ \Gamma_1(0)\equiv\Gamma_1=\gamma_1\times\gamma_1\times\gamma_1\,,\\ \Gamma_2(0)\equiv\Gamma_2=\gamma_2\times\gamma_2\times\gamma_2\,,\\ \Gamma_3(0)\equiv\Gamma_3=\gamma_3\times\gamma_3\times\gamma_3\,,\end{cases} \tag{6}$$

*) While the operator $M_{(c)}^2=(M_{23}^{(c)})^2+(M_{31}^{(c)})^2+(M_{12}^{(c)})^2$ obeys the (minimal) equation

$$(M_{(c)}^2-\frac{3}{4}I)(M_{(c)}^2-\frac{15}{4}I)=0\,,$$

the operator $M^2=(M_{23})^2+(M_{31})^2+(M_{12})^2$ satisfies

$$M^2-\frac{3}{4}I=0\,.$$

**) As for the problem whether more than four sets of product-matrices generating $M_{\mu\nu}$ exist or not, see the comment (3) given in § 6.

$$\begin{cases} \Gamma_0(1)=\frac{1}{\sqrt{3}}(\gamma_5\gamma_1\times\gamma_5\gamma_2\times\gamma_5\gamma_3+\gamma_5\gamma_2\times\gamma_5\gamma_3\times\gamma_5\gamma_1+\gamma_5\gamma_3\times\gamma_5\gamma_1\times\gamma_5\gamma_2), \\ \Gamma_1(1)=\frac{1}{\sqrt{3}}(\gamma_5\gamma_0\times\gamma_3\times\gamma_2+\gamma_3\times\gamma_2\times\gamma_5\gamma_0+\gamma_2\times\gamma_5\gamma_0\times\gamma_3), \\ \Gamma_2(1)=\frac{1}{\sqrt{3}}(\gamma_3\times\gamma_5\gamma_0\times\gamma_1+\gamma_5\gamma_0\times\gamma_1\times\gamma_3+\gamma_1\times\gamma_3\times\gamma_5\gamma_0), \\ \Gamma_3(1)=\frac{1}{\sqrt{3}}(\gamma_2\times\gamma_1\times\gamma_5\gamma_0+\gamma_1\times\gamma_5\gamma_0\times\gamma_2+\gamma_5\gamma_0\times\gamma_2\times\gamma_1), \end{cases} \tag{7}$$

$$\begin{cases} \Gamma_0(3)=\frac{1}{\sqrt{3}}(\gamma_5\gamma_3\times\gamma_5\gamma_2\times\gamma_5\gamma_1+\gamma_5\gamma_1\times\gamma_5\gamma_3\times\gamma_5\gamma_2+\gamma_5\gamma_2\times\gamma_5\gamma_1\times\gamma_5\gamma_3), \\ \Gamma_1(3)=\frac{1}{\sqrt{3}}(\gamma_2\times\gamma_3\times\gamma_5\gamma_0+\gamma_5\gamma_0\times\gamma_2\times\gamma_3+\gamma_3\times\gamma_5\gamma_0\times\gamma_2), \\ \Gamma_2(3)=\frac{1}{\sqrt{3}}(\gamma_1\times\gamma_5\gamma_0\times\gamma_3+\gamma_3\times\gamma_1\times\gamma_5\gamma_0+\gamma_5\gamma_0\times\gamma_3\times\gamma_1), \\ \Gamma_3(3)=\frac{1}{\sqrt{3}}(\gamma_5\gamma_0\times\gamma_1\times\gamma_2+\gamma_2\times\gamma_5\gamma_0\times\gamma_1+\gamma_1\times\gamma_2\times\gamma_5\gamma_0) \end{cases} \tag{8}$$

and

$$\begin{cases} \Gamma_0(2)=\Gamma_0(0)\Gamma_0(1)\Gamma_0(3), \\ \Gamma_1(2)=-\Gamma_1(0)\Gamma_1(1)\Gamma_1(3), \\ \Gamma_2(2)=-\Gamma_2(0)\Gamma_2(1)\Gamma_2(3), \\ \Gamma_3(2)=-\Gamma_3(0)\Gamma_3(1)\Gamma_3(3). \end{cases} \tag{9}$$

In fact it can be readily demonstrated that all these four sets of fusion matrices $\Gamma_\mu(a)$ build up the common Lorentz generators $\Sigma_{\mu\nu}$ in Eq. (3), i.e.,

$$\Sigma_{\mu\nu}=-\sigma_{\mu\nu}\times\sigma_{\mu\nu}\times\sigma_{\mu\nu}=\frac{i}{2}[\Gamma_\mu(a),\Gamma_\nu(a)] \tag{10}$$

for $a=0$, 1, 2 and 3. These fusion matrices turn out to satisfy the following clear-cut algebraic relations as

$$\Gamma_\mu(a)\Gamma_\nu(a)+\Gamma_\nu(a)\Gamma_\mu(a)=2g_{\mu\nu}I, \tag{11}$$

$$\Gamma_\mu(a)\Gamma_\nu(b)+\Gamma_\nu(b)\Gamma_\mu(a)=0 \qquad \text{for } \mu\neq\nu \tag{12}$$

and

$$\Gamma_\mu(a)\Gamma_\nu(b)-\Gamma_\mu(b)\Gamma_\nu(a)=0, \tag{13}$$

where $I=1\times1\times1$.

Notice that, in spite of the simple form (5), the algebra of composite matrices $\Gamma_\mu{}^{(c)}$ is known to be very complicated. This is nothing but the direct reflection of the composite structure of system described by $\Gamma_\mu{}^{(c)}$. On the other hand, despite the ostensible complexity of the representations (6)~(9), the algebra of fusion matrices has the remarkable simplicity. In particular, the four subsets of generators $\{\Gamma_\mu(a): \mu=0, 1, 2, 3\}$ for $a=0, 1, 2$ and 3 have exactly the same structure with the basic γ-matrices. This fact seems to support our design that the fusion matrices $\Gamma_\mu(a)$ and consequently the new Lorentz generators $M_{\mu\nu}=\frac{1}{2}\Sigma_{\mu\nu}$ describe, not the composite system, but the physical entity retaining the elementarity which is considered to be characterized solely by the Dirac equation as well as the multiplicity which is not describable by that too genuine equation.

At this stage we should recognize definitely the different roles played respectively by the fusion Γ-matrices and the component γ-matrices. Since $\Sigma_{\mu\nu}$ and $\Gamma_\lambda(a)$ satisfy the relations

$$[\Sigma_{\mu\nu}, \Gamma_\lambda(a)]=2ig_{\lambda\nu}\Gamma_\mu(a)-2ig_{\lambda\mu}\Gamma_\nu(a), \tag{14}$$

every quartette $(\Gamma_0(a), \Gamma_1(a), \Gamma_2(a), \Gamma_3(a))$ for $a=0, 1, 2$ and 3 forms separately the Lorentz invariant set (see § 4). Therefore, the index μ of $\Gamma_\mu(a)$ discriminates covariantly the component of spacetime coordinates of the Minkowskian continuum in which the fused system exists. On the other hand, as is recognized from the explicit forms of $\Gamma_\mu(a)$ in Eqs. (6)~(9), γ-matrices play no more role than the building-blocks of fusion matrices and their indices should not be considered to be immediately related to the spacetime coordinates.*)

§ 3. Fusion algebra

The Dirac algebra spans the 16 dimensional linear space with the standard basis $\{I, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5\}$ generated from the products of γ-matrices. Similarly, by forming products of Γ-matrices and selecting out linearly-independent elements, we are able to generate the fusion algebra.

First of all there are four linearly-independent scalar**) matrices. Namely, in addition to the identity $I\equiv I(0)$, we have

$$\begin{aligned} I(1)&=\frac{1}{4}\Gamma^\mu(2)\Gamma_\mu(3)=\frac{1}{4}\Gamma^\mu(0)\Gamma_\mu(1) \\ &=\frac{1}{\sqrt{3}}(\sigma^{12}\times\sigma^{23}\times\sigma^{31}+\sigma^{23}\times\sigma^{31}\times\sigma^{12}+\sigma^{31}\times\sigma^{12}\times\sigma^{23}), \end{aligned} \tag{15}$$

$$\begin{aligned} I(3)&=\frac{1}{4}\Gamma^\mu(1)\Gamma_\mu(2)=\frac{1}{4}\Gamma^\mu(0)\Gamma_\mu(3) \\ &=\frac{1}{\sqrt{3}}(\sigma^{31}\times\sigma^{23}\times\sigma^{12}+\sigma^{12}\times\sigma^{31}\times\sigma^{23}+\sigma^{23}\times\sigma^{12}\times\sigma^{31}) \end{aligned} \tag{16}$$

*) Nevertheless it is interesting to observe that, if the operation of direct product in the expressions (6)~(9) is replaced by the ordinary multiplication of component γ-matrices, all the $\Gamma_\mu(a)$ reduce to γ_μ.

**) For the sake of brevity we use the concepts (pseudo) scalar, (axial) vector and tensor in advance of the analysis of Lorentz transformation made in detail in the next section.

and

$$I(2)=\frac{1}{4}\Gamma^{\mu}(3)\Gamma_{\mu}(1)=\frac{1}{4}\Gamma^{\mu}(0)\Gamma_{\mu}(2)=I(3)I(1) \tag{17}$$

which are commutative with all $\Gamma_{\mu}(a)$ and involutorial as

$$I(a)^2=I\,, \qquad a=1,\,2,\,3. \tag{18}$$

In the defining equations (15)~(17) the summation convention, according to which repeated covariant and contravariant indices are summed, is used provided that the index of fusion matrices $\Gamma_{\mu}(a)$ is raised by $\Gamma^{\mu}(a)=g^{\mu\nu}\Gamma_{\nu}(a)$. Correspondingly we have four pseudoscalars

$$\Gamma_5(0)\equiv\Gamma_5=i\Gamma^0(a)\Gamma^1(a)\Gamma^2(a)\Gamma^3(a)=-\gamma_5\times\gamma_5\times\gamma_5 \tag{19}$$

and

$$\Gamma_5(a)=\Gamma_5 I(a), \qquad a=1,\,2,\,3\,, \tag{20}$$

which are commutative with each other and anticommutative with all $\Gamma_{\mu}(a)$, and satisfy the relations $\Gamma_5(1)\Gamma_5(2)\Gamma_5(3)=\Gamma_5$ and $\Gamma_5(a)^2=I$ for $a=0$, 1, 2 and 3.

A similar quadruplicate structure is found also for the axial-vector and tensor matrices. Four linearly-independent axial-vector matrices are synthesized by

$$\Gamma_5\Gamma_{\mu}(a)=\Gamma_5(a)\Gamma_{\mu}\,, \qquad a=0,\,1,\,2,\,3. \tag{21}$$

Over and above the original Lorentz generators $\Sigma_{\mu\nu}\equiv\Sigma_{\mu\nu}(0)$ in Eq. (10), we have three sets of tensor matrices such as

$$\Sigma_{\mu\nu}(a)=\delta_{abc}\frac{i}{2}[\Gamma_{\mu}(b),\,\Gamma_{\nu}(c)]=\Sigma_{\mu\nu}I(a), \quad a=1,\,2,\,3, \tag{22}$$

where $\delta_{abc}=1$ if $(a,\,b,\,c)$ is a permutation of $(1,2,3)$ and $\delta_{abc}=0$ otherwise. Representing one kind of matrices among $I(a)$, $\Gamma_{\mu}(a)$, $\Sigma_{\mu\nu}(a)$, $\Gamma_5\Gamma_{\mu}(a)$ and $\Gamma_5(a)$ by the symbol $\Lambda(a)$, we find generally the relations

$$\begin{aligned} I(a)\Lambda(b)=&(\delta_{ab}-2\delta_{a0}\delta_{b0})\Lambda(0)\\ &+\delta_{a0}\Lambda(b)+\delta_{b0}\Lambda(a)+\delta_{b0}\Lambda(a)+\delta_{abc}\Lambda(c). \end{aligned} \tag{23}$$

In this way it is confirmed that the six generators Γ_0, Γ_1, Γ_2, Γ_3, $I(1)$ and $I(3)$ generate the fusion algebra $R[\Gamma]=\{I(a),\Gamma_{\mu}(a),\Sigma_{\mu\nu}(a),\Gamma_5\Gamma_{\mu}(a),\Gamma_5(a)$: μ, ν, $a=0,1,2,3\}$, the dimension of which is 64. The centre $Z[\Gamma]$ of this algebra, i. e., $Z[\Gamma]=\{z\epsilon R[\Gamma]$: $xz=zx$, $^{\forall}x\epsilon R[\Gamma]\}$, is $\{I(a)$: $a=0,1,2,3\}$ and its order is 4. The identity I has the decomposition

$$I=\varpi_1+\varpi_2+\varpi_3+\varpi_4 \tag{24}$$

in terms of the projection operators defined by

$$\varpi_1 = \frac{1}{4}[I - I(1)][I + I(3)], \tag{25}$$

$$\varpi_2 = \frac{1}{4}[I + I(1)][I - I(3)], \tag{26}$$

$$\varpi_3 = \frac{1}{4}[I + I(1)][I + I(3)] \tag{27}$$

and

$$\varpi_4 = \frac{1}{4}[I - I(1)][I - I(3)] \tag{28}$$

which belong to the centre and satisfy the relations $\varpi_r \varpi_{r'} = \delta_{rr'} \varpi_r$. Similarly the semisimple algebra $R[\Gamma]$ is decomposed into the direct sum of ideals as

$$R = R_1 + R_2 + R_3 + R_4\,, \tag{29}$$

where $R_r = R\varpi_r = \varpi_r R$ are ideals of $R[\Gamma]$. The physical implications of these decompositions are expounded in the later sections.

The transpose of fusion matrices is defined by the direct product of transposed-component-matrices, and likewise the complex conjugate and the Hermite conjugate of fusion matrices. Immediately we get

$$\begin{cases} \Gamma_0{}^T(a) = \varepsilon_a \Gamma_0(a), & \Gamma_1{}^T(a) = -\varepsilon_a \Gamma_1(a), \\ \Gamma_2{}^T(a) = \varepsilon_a \Gamma_2(a), & \Gamma_3{}^T(a) = -\varepsilon_a \Gamma_3(a), \end{cases} \tag{30}$$

$$\begin{cases} \Gamma_0{}^*(a) = \varepsilon_a \Gamma_0(a), & \Gamma_1{}^*(a) = \varepsilon_a \Gamma_1(a), \\ \Gamma_2{}^*(a) = -\varepsilon_a \Gamma_2(a), & \Gamma_3{}^*(a) = \varepsilon_a \Gamma_3(a) \end{cases} \tag{31}$$

and

$$\Gamma_\mu{}^\dagger(a) = \Gamma_0(b) \Gamma_\mu(a) \Gamma_0(b) \tag{32}$$

for a, $b = 0$, 1, 2 and 3, where $\varepsilon_a = (-1)^a$ are sign factors.

§ 4. Wave equation, proof of covariance and quantization

We postulate that the fused system is described at a very good approximation by a local field obeying the Dirac-like wave-equation with coefficients belonging to the fusion algebra $R[\Gamma]$. There are four candidates for such a differential equation for free field as follows:

$$[i\Gamma_\mu(a)\partial^\mu - M(a)]\psi(x : a) = 0, \quad a = 0,\ 1,\ 2,\ 3\,. \tag{33}$$

In order that the free fused system has an appropriate energy-momentum relation, the solution $\psi(x : a)$ of these equations should satisfy the conditions of Klein-

Gordon type

$$[\partial_\mu \partial^\mu + M(a)^2]\psi(x:a)=0, \quad a=0,\ 1,\ 2,\ 3\,. \tag{34}$$

This natural requirement demands the mass operators $M(a)$ to be commutative with the generating matrices $\Gamma_\mu(a)$. Accordingly $M(a)$ must belong to the centre $Z[\Gamma]$ and have the general form $M(a)=\sum_{r=1}^{4} m_r(a)\varpi_r$ with unknown real parameters $m_r(a)$.

To select the most suitable equation ruling the behaviour of the fused system out of the four candidates (33), we must find their mutual relationships. For that purpose let us operate the Lorentz scalar $I(a)$ to Eq. (33) with fixed a. Noting the identity $I(a)\Gamma_\mu(a)=\Gamma_\mu(0)\equiv\Gamma_\mu$ and the fact that $I(a)M(a)$ can be equated to $M(0)\equiv M$ if the unspecified parameters $m_r(a)$ are properly adjusted, we recognize that the three equations of (33) for $a=1$, 2 and 3 are equivalent to the one equation of (33) for $a=0$. In other words, all the equations (33) are equivalent with each other granting the adjustment of parameters in mass operators. Taking account of the facts that the quartette $\Gamma_\mu(0)\equiv\Gamma_\mu$ has the simplest γ-structure, we postulate without loss of generality that the field $\psi(x:0)\equiv\psi(x)$ of free fused system obeys Eq. (33) with $a=0$ which is rewritten here in the concise form

$$(i\Gamma_\mu\partial^\mu - M)\psi(x)=0 \tag{35}$$

with

$$M = m_1\varpi_1 + m_2\varpi_2 + m_3\varpi_3 + m_4\varpi_4\,. \tag{36}$$

Under the proper Lorentz transformation $x'_\mu = \Lambda_{\mu\nu}x^\nu$ where $\Lambda_{\lambda\mu}\Lambda^\lambda{}_\nu = g_{\mu\nu}$ and det $|\Lambda|=1$, we assume the field $\psi(x)$ to be transformed as

$$\psi'(x')=S(\Lambda)\psi(x) \tag{37}$$

as usual. The form invariance of Eq. (35) leads to

$$S^{-1}(\Lambda)\Gamma_\mu S(\Lambda)=\Lambda_{\mu\nu}\Gamma^\nu \quad \text{and} \quad S^{-1}(\Lambda)MS(\Lambda)=M \tag{38}$$

which result in the transformation matrix

$$S(\Lambda)=\exp\left(-\frac{i}{4}\Sigma_{\mu\nu}\omega^{\mu\nu}\right) \tag{39}$$

for a spacetime rotation through a covariant angle $\omega_{\mu\nu}$ in the μ-ν plane. Since $S(\Lambda)$ has the property

$$S^{-1}(\Lambda)=\Gamma_0 S^\dagger(\Lambda)\Gamma_0\,, \tag{40}$$

the adjoint field defined by

$$\bar{\psi}(x)=\psi^\dagger(x)\Gamma_0 \tag{41}$$

is subject to the Lorentz transformation

$$\bar{\psi}'(x')=\bar{\psi}(x)S^{-1}(\Lambda). \tag{42}$$

Next let us turn our attention to the discrete symmetry. For its purpose it is adequate to examine the field equation in the presence of electromagnetic fields $A_\mu(x)$ as

$$[\Gamma_\mu\{i\partial^\mu-eA^\mu(x)\}-M]\psi(x)=0 \tag{43}$$

by postulating the minimal-coupling substitution $p_\mu\to p_\mu-eA_\mu(x)$ for the fused system which is assumed to have an electric charge e. In order that Eq. (43) is form-invariant under the space inversion $\boldsymbol{x}'=-\boldsymbol{x}$ and $t'=t$, i.e.,

$$[\Gamma_\mu\{i\partial'^\mu-eA'^\mu(x')\}-M]\psi'(x')=0\,, \tag{44}$$

where $A'_\mu(x')$ are related to $A_\mu(x)$ by $\boldsymbol{A}'(x')=-\boldsymbol{A}(x)$ and $A'_0(x')=A_0(x)$, the field $\psi(x)$ must have the transformation property

$$\psi'(x')=P\psi(x)=\Gamma^0\psi(x). \tag{45}$$

As far as the proper Lorentz transformation and the space inversion are concerned, the present scheme has been shown to retain the same structure with the Dirac theory.*) The literal parallelism, however, is lost for the time-reversal and charge conjugation transformations, since the Γ-matrices have different properties from the component γ-matrices with respect to the complex conjugate and transpose operations. The time-reversal transformation $\boldsymbol{x}'=\boldsymbol{x}$ and $t'=-t$ brings the electromagnetic fields $A_\mu(x)$ to $\boldsymbol{A}'(x')=-\boldsymbol{A}(x)$ and $A'_0(x')=A_0(x)$ and Eq. (43) is converted into

$$[\Gamma_\mu\{i\partial'^\mu-eA'^\mu(x')\}-M]\psi'(x')=0 \tag{46}$$

in the time-reversed frame, provided that the field $\psi(x)$ has the transformation property

$$\psi'(x')=\mathscr{T}\psi(x)=T\psi^*(x)=i\Gamma^1\Gamma^3\Xi\psi^*(x). \tag{47}$$

The charge conjugation transmutes Eq. (43) into

$$[\Gamma_\mu\{i\partial^\mu+eA^\mu(x)\}-M]\psi_c(x)=0\,, \tag{48}$$

where the charge conjugate of $\psi(x)$ is defined by

$$\psi_c(x)=\mathscr{C}\psi(x)=C\psi^*(x)=i\Gamma^2\Gamma^0\Xi\psi^*(x), \tag{49}$$

where C satisfies

*) Since $I(a)$ for $a=1$, 2 and 3 are commutative with P and $S(\Lambda)$, it is legitimated to term $I(a)$, $\Gamma_\mu(a)=\Gamma_\mu I(a)$, $\Sigma_{\mu\nu}(a)=\Sigma_{\mu\nu}I(a)$, $\Gamma_5\Gamma_\mu(a)=\Gamma_5\Gamma_\mu I(a)$ and $\Gamma_5(a)=\Gamma_5 I(a)$, respectively, Lorentz scalars, vectors, tensors, axial vectors and pseudoscalars.

$$C^{-1}\Gamma_\mu C = -\Gamma_\mu{}^T . \tag{50}$$

The operator Ξ in the formulae (47) and (49) has the form

$$\Xi = \frac{1}{2}[I - I(2)]R + \frac{1}{2}[I + I(2)]\cdot\frac{1}{2}[I - (1\times\sigma_{12}\times\sigma_{12} + \sigma_{12}\times 1\times\sigma_{12} + \sigma_{12}\times\sigma_{12}\times 1)], \tag{51}$$

where R works to reverse the ordering in the direct-product of undors. The involution Ξ being commutative with Γ_μ induces the interchanges among the projection operators as follows:

$$\begin{aligned} &\Xi\varpi_1\Xi^{-1} = \varpi_2 , \qquad \Xi\varpi_2\Xi^{-1} = \varpi_1 , \\ &\Xi\varpi_3\Xi^{-1} = \varpi_4 \quad \text{and} \quad \Xi\varpi_4\Xi^{-1} = \varpi_3 . \end{aligned} \tag{52}$$

By this function which reduces the complex-conjugated mass-operator M^* back to M, i.e.,

$$\Xi M^* \Xi^{-1} = M , \tag{53}$$

Ξ plays an indispensable role to preserve the form invariance of the theory under the time-reversal and charge-conjugation transformations.*)

Combining Eqs. (44)~(49), we find that the $P\mathscr{C}\mathscr{T}$ operation transforms $\psi(x)$ into

$$\psi_{\mathrm{PCT}}(x') = P\mathscr{C}\mathscr{T}\psi(x) = i\Gamma^5\psi(x) \tag{54}$$

and Eq. (43) into

$$[\Gamma_\mu\{i\partial'^\mu + eA'^\mu(x')\} - M]\psi_{\mathrm{PCT}}(x') = 0 . \tag{55}$$

Therefore, the Stückelberg-Feynman interpretation on antiparticles holds true in this scheme just like in the conventional Dirac theory.

There is no obstacle to the canonical quantization of fusion field $\psi(x)$. The Lorentz covariance and the Fermi-Dirac statistics require the anticommutation relations

$$\begin{cases} \{\psi(x), \bar{\psi}(y)\} = -iS(x-y), \\ \{\psi(x), \psi(y)\} = \{\bar{\psi}(x), \bar{\psi}(y)\} = 0 , \end{cases} \tag{56}$$

where

$$S(x-y) = \frac{i}{(2\pi)^3}\int d^{(4)}k(\Gamma_\mu k^\mu + M)\delta(k^2 - M^2)\varepsilon(k_0)e^{-ik(x-y)} . \tag{57}$$

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*) Although Ξ does not belong to $R[\Gamma]$, it is not necessary at all to enlarge the fusion algebra so as to include it since the $\mathscr{C}$ and $\mathscr{T}$ transformations are discrete symmetries.

The causal propagator is defined by

$$S_F(x-y)=i\langle 0|\,T[\psi(x)\,\bar{\psi}(y)]|0\rangle$$
$$=\frac{1}{(2\pi)^4}\int d^{(4)}k\,(\Gamma_\mu k^\mu-M+i\varepsilon)^{-1}e^{-ik(x-y)}\,. \tag{58}$$

In the integral representations (57) and (58) the kernel function and distribution are defind on the fusion algebra $R[\Gamma]$ and have the decompositions

$$(k^2-M^2+i\varepsilon)^{-1}=\sum_{r=1}^{4}(k^2-m_r{}^2+i\varepsilon)^{-1}\varpi_r \tag{59}$$

and

$$\delta(k^2-M^2)=\sum_{r=1}^{4}\delta(k^2-m_r{}^2)\varpi_r\,. \tag{60}$$

§ 5. Solutions of free wave equation

The best way to elucidate the characteristic features of the present scheme is to solve Eq. (35) explicitly. In the rest frame of the fused system let us derive the solutions of the simultaneous eigenvalue problems

$$(i\Gamma_0\partial_0-M)\,\psi(x)=0\,, \tag{61}$$

$$M\psi(x)=m\psi(x) \tag{62}$$

and

$$\Sigma_{12}\psi(x)=\pm\psi(x) \tag{63}$$

in the form $\psi(x:\varepsilon,s)=v(\varepsilon,s)\exp(-i\varepsilon mt)$ where $\varepsilon=\pm 1$ and $s=\uparrow,\ \downarrow$ signify, respectively, the positive or negative frequency solution and the up or down spin solution. Table I summarizes the eigenvalues of Γ_0, Σ_{12}, M and $\varpi_r(r=1,2,3,4)$ and the simultaneous eigenvectors. In this table we adopt the symbols

$$\overset{\circ}{\mid}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\qquad \underset{\circ}{\mid}=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\qquad \overset{\bullet}{\mid}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\quad\text{and}\quad \underset{\bullet}{\mid}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \tag{64}$$

and $\omega=\exp(i\,2\pi/3)$ is a cubic root of unity. Eigenvectors are definite linear combinations of triple-direct-products of undors and satisfy the orthonormal relations

$$\bar{v}_r(\varepsilon,s)v_{r'}(\varepsilon',s')=\varepsilon\delta_{rr'}\delta_{\varepsilon\varepsilon'}\delta_{ss'} \tag{65}$$

Table I. Explicit solutions of the free wave equation. Eigenvectors for the down-spin states $v_r(\pm,\downarrow)$ are obtained by making the replacements ⫯ ↔ ⫰ and ╿ ↔ ╽ in the up-spin solutions $v_r(\pm,\uparrow)$.

Simultaneous eigenvectors	Eigenvalues						
	Γ_0	Σ_{12}	$\tilde{\omega}_1$	$\tilde{\omega}_2$	$\tilde{\omega}_3$	$\tilde{\omega}_4$	M
$v_1(+,\uparrow)=(\omega^2$ ⫰⫯⫯ $+\omega$ ⫯⫰⫯ $+$ ⫯⫯⫰ $)/\sqrt{3}$	1	1	1	0	0	0	m_1
$v_1(-,\uparrow)=(\omega^2$ ╽╿╿ $+\omega$ ╿╽╿ $+$ ╿╿╽ $)/\sqrt{3}$	−1	1	1	0	0	0	m_1
$v_2(+,\uparrow)=($ ⫰⫯⫯ $+\omega$ ⫯⫰⫯ $+\omega^2$ ⫯⫯⫰ $)/\sqrt{3}$	1	1	0	1	0	0	m_2
$v_2(-,\uparrow)=($ ╽╿╿ $+\omega$ ╿╽╿ $+\omega^2$ ╿╿╽ $)/\sqrt{3}$	−1	1	0	1	0	0	m_2
$v_3(+,\uparrow)=($ ⫰⫯⫯ $+$ ⫯⫰⫯ $+$ ⫯⫯⫰ $-i\sqrt{3}$ ⫰⫰⫰ $)/\sqrt{6}$	1	1	0	0	1	0	m_3
$v_3(-,\uparrow)=($ ╽╿╿ $+$ ╿╽╿ $+$ ╿╿╽ $-i\sqrt{3}$ ╽╽╽ $)/\sqrt{6}$	−1	1	0	0	1	0	m_3
$v_4(+,\uparrow)=($ ⫰⫯⫯ $+$ ⫯⫰⫯ $+$ ⫯⫯⫰ $+i\sqrt{3}$ ⫰⫰⫰ $)/\sqrt{6}$	1	1	0	0	0	1	m_4
$v_4(-,\uparrow)=($ ╽╿╿ $+$ ╿╽╿ $+$ ╿╿╽ $+i\sqrt{3}$ ╽╽╽ $)/\sqrt{6}$	−1	1	0	0	0	1	m_4

for $r, r'=1, 2, 3, 4$; $\varepsilon, \varepsilon'=\pm 1$ and $s, s'=\uparrow, \downarrow$. Notice that such states as ⫯⫯⫰ and ⫰⫰⫰ coexist in the solutions v_3 and v_4. As a matter of fact, therefore, these solutions are not the eigenstates of the composite spin operator $\Sigma_{12}^{(c)}$ and it is not allowed to interpret the states ⫯, ⫰, ╿ and ╽ as the constituent spin eigenstates.

Applying the Lorentz booster (39), we find the plane-wave solutions with momentum $\boldsymbol{p}$ of free field equation (35) as

$$\psi_r(x:\varepsilon, s)=v_r(\boldsymbol{p}:\varepsilon, s)e^{-i\varepsilon px} \tag{66}$$

with

$$v_r(\boldsymbol{p}:\varepsilon, s)=\exp\left[-\frac{1}{2p}\Gamma_0\boldsymbol{\Gamma}\cdot\boldsymbol{p}\tanh^{-1}\left(\frac{p}{p_0+M}\right)\right]v_r(\varepsilon, s) \tag{67}$$

for $r=1, 2, 3, 4$, $\varepsilon=\pm 1$ and $s=\uparrow, \downarrow$. The manifold of general solutions of Eq. (35) is the direct sum of the submanifolds $\{v_r(\boldsymbol{p}:\varepsilon, s): \varepsilon=\pm 1; s=\uparrow, \downarrow; \boldsymbol{p}\varepsilon R^3\}$, every one of which is invariant under the Lorentz transformation and represents the particle and antiparticle states carrying the definite mass and spin 1/2.

§ 6. Discussion

In the viewpoint of one-particle quantum mechanics Eq. (43) is regarded as describing the time development of the fused system in the presence of external electromagnetic field. The Hamiltonian has the form $H=\boldsymbol{\alpha}\cdot(\boldsymbol{p}-e\boldsymbol{A})+eA_0+\beta M$ with $\beta=\Gamma_0$ and $\alpha_i=\Gamma_0\Gamma^i$. Following the familiar procedure we arrive at the results that the fused system has the total angular momentum $\boldsymbol{J}=\boldsymbol{r}\times\boldsymbol{p}+\frac{1}{2}\boldsymbol{\Sigma}$

and the magnetic moment $\boldsymbol{\mu}=-\frac{1}{2}eM^{-1}\boldsymbol{\Sigma}$ where $M^{-1}=\sum_r m_r^{-1}\varpi_r$. Consequently the fused system turns out to have the four orthogonal particle-modes with spin 1/2 and g-value of 2. These results justify to interpret that the three particle-modes $\psi_1=\varpi_1\psi$, $\psi_2=\varpi_2\psi$ and $\psi_3=\varpi_3\psi$ represent electron, muon and τ-lepton, respectively, and that the remaining mode $\psi_4=\varpi_4\psi$ predicts the existence of the fourth lepton.

In order to establish the e-μ-τ universality it is necessary to presuppose the vector and axialvector currents being composed exclusively of Γ_μ and $\Gamma_5\Gamma_\mu$. Namely the matrices $\Gamma_\mu(a)$ and $\Gamma_5\Gamma_\mu(a)$ with $a\neq 0$ are forbidden to appear in those currents. Then the Lorentz invariant decompositions of the currents

$$\bar{\psi}\begin{Bmatrix}\Gamma_\mu\\ \Gamma_5\Gamma_\mu\end{Bmatrix}\psi'=\sum_{r=1}^{4}\bar{\psi}_r\begin{Bmatrix}\Gamma_\mu\\ \Gamma_5\Gamma_\mu\end{Bmatrix}\psi'_r \tag{68}$$

guarantee the universality of four kinds of particle modes.

In this way two paradoxical aspects of leptons——elementarity and horizontal multiplicity——have been successfully formulated in terms of the simple mathematical language of new fusion algebra. Although this theory has no power to explain the mass spectrum of leptons at the present stage of development, it has certainly cut one promising way toward the resolution of the e-μ-τ puzzle. In particular the non-appearance of the spin 3/2 particle-mode makes the present approach much prominent over the conventional composite model. In all the composite models where leptons and quarks are assumed to be made of three Dirac particles it is recognized[8] that no algebraic constraint works to remove the composite field of spin 3/2 while retaining all physical fields. The spin 3/2 state being repugnant in the composite model is reincarnated in our fusion theory as two orthogonal particle modes with spin 1/2 and carrying different masses which are represented by the eigenvectors $v_3(\varepsilon, s)$ and $v_4(\varepsilon, s)$ in Table I.

Here the emphasis should be laid on the fact that it is not necessary at all in this theory of leptons and quarks to challenge the dynamical problem of confinement in front, which must be solved inevitably at the level of hadrons regarded as the composite systems of quarks and also at the level of leptons and quarks if the composite model is adopted. Without doubt this is another advantage of fusion approach over the composite model of leptons and quarks. This kind of conversion of dynamical difficulty into a formal and kinematical problem seems to be unavoidable in order to understand the deeper levels of material world.

We conclude by making several comments for further developments of the fusion theory:

(1) So far the theory has been developed only in reference to one kind of fused system with integral charge e which has four orthogonal particle-modes interpreted as charged leptons. There are two different ways to generalize the theory in the vertical direction so as to attain the unification of leptons and quarks. The

simplest way is to assume the existence of eight kinds of fused systems with different electric and colour charges correspondingly to the first generation $\bar{e}$, u, $\bar{d}$ and ν_e of basic fermions. Describing the fourfold generations wholly in terms of eight independent fusion fields, we are able to formulate the standard gauge theory of strong and electroweak interaction based on the $SU_c(3)\times SU(2)\times U(1)$ group, and at the very high energy limit of the scale $\sim 10^{15}$ GeV we may further postulate the grand unification based on a simple group such as $SU(5)$.

(2) Another way to the unification is to apply our fusion scheme to the internal degrees of freedom, i. e., to melt the composite model of leptons and quarks totally into the fusion theory. To see this let us consider here the most economical model proposed by Harari and Shupe[9] in which leptons and quarks are assumed to be the three-body composite systems of two kinds of fundamental Dirac-like entities T and V. This "very radical model" attempts to explain not only the modes of existence of composite fermions but also to generate the symmetries of colour $SU_c(3)$ and weak-isospin at the level of composite fermions. As the Γ-matrices are defined by the triple-direct-products of building-block γ-matrices, our fusion theory suggests to construct the Lie algebra of $SU_c(3)\times SU(2)\times U(1)$ symmetry as the triple-direct-products of generators of basic $SU(2)$ group whose fundamental representation is the rishon doublet (T, V). Furthermore, if we succeed in "fusing" simultaneously the symmetries of spacetime and internal degrees of freedom, we will obtain the key to open the secret of the world of leptons and quarks.

(3) From the viewpoint of the composite model, the solutions of the free wave equation $v_r(+, s)$ and $v_r(-, s)$ listed in Table I are formally——except for the interpretation of spin——related to the composite systems of three fundamental particles and three fundamental antiparticles, respectively. Namely our fusion field describes exclusively the eigenmodes of fused system corresponding to the (fundamental particle)^3 and (fundamental antiparticle)^3 configurations. Theoretically speaking, however, there is no *a priori* reason to prohibit the fused system from having such mixed eigenmodes as corresponding to the configurations (fundamental particle)^2 (fundamental antiparticle) and (fundamental particle) (fundamental antiparticle)^2. In fact it is possible to find out a larger algebra which includes our fusion algebra $R[\Gamma]$ and allows to describe the mixed and unmixed eigenmodes in a unified way. The considerations on this problem will be left in the future task.

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