We find all the spatially flat Robertson-Walker type solutions of Einstein's equations with the cosmological constant and quantum fluctuations of conformally invariant matter fields, which include effects of the conformal anomaly. We show that the relevant back-reaction equation is interpreted as an equation of motion for a classical one-dimensional potential problem with (anti-)dissipation. By considering the potential form, we can easily classify all the solutions and understand their qualitative behaviour, such as monotonic expansion (contraction), oscillation, contraction followed by expansion. Only two kinds of asymptotic behaviour are allowed, both in the past and in the future. They are (1) exponential expansion or contraction (asymptotically de Sitter) and (2) singularity (zero or infinity of the scale factor) at a finite cosmic time. Stability of de Sitter solutions (singular solutions) is also discussed. The Minkowski solution in the case of the vanishing cosmological constant is shown to be unstable, irrespectively of the sign of a parameter in the equation.

§ 1. Introduction

In the curved space the vacuum expectation value of the energy momentum tensor for quantum matter fields, \( \langle T_{\mu \nu} \rangle \), is nonvanishing even after renormalization. Several people have studied its effects on cosmological solutions of Einstein's equations. Their works, however, are mostly limited to the case of the vanishing cosmological constant (with a few exceptions in which modification of asymptotically Friedmann solutions by a small cosmological constant is discussed). From the viewpoint of the inflationary scenario, it is urgent to study the nonvanishing general case.

When \( \langle T_{\mu \nu} \rangle \) is ignored, the vacuum solution of Einstein's equations with the cosmological constant \( \Lambda \) is a de Sitter metric which is specified by its constant curvature \( R \). In Ref. 4 we found that, though a de Sitter metric is still a solution when \( \langle T_{\mu \nu} \rangle \) is included, the relation between \( R \) and \( \Lambda \) changes qualitatively. There are two or three solutions of \( R \) for some \( \Lambda \), while there is none for some others.

Though a de Sitter metric is a unique solution of the Robertson-Walker type for the equations without \( \langle T_{\mu \nu} \rangle \), there are wider classes of general and singular solutions when \( \langle T_{\mu \nu} \rangle \) is included. The purpose of this paper is to classify all the solutions and to present some numerical results. Our analysis is limited to free conformally invariant theories (gauge bosons, massless fermions and conformal scalars) because \( \langle T_{\mu \nu} \rangle \) in an arbitrary conformal flat metric is known only for them. Our study is also limited to the spatially flat metric and we do not include classical matters.

In § 2 we present our basic equation. By choosing an appropriate variable, the equation is written as an equation of motion for a classical potential system with dissipation or with anti-dissipation. The potential, which depends on three parameters, is classified into six types. In § 3 we discuss some features of the solutions which are rather
difficult to see directly from the potential picture. In § 4 we classify all the solutions, both general solutions and singular solutions, for each type of the potential. The singular solutions include previously obtained de Sitter solutions. Some numerical calculation is also presented. In § 5 we discuss a case where the cosmological constant vanishes. Minkowski metric emerges as a singular solution and is shown to be unstable against homogeneous contraction at least, whatever the sign of the parameter \(m\) in Eq. (4) is. (Instability when \(m > 0\) is shown in Ref. 6.) Section 6 is for conclusion and discussion. In the Appendix we deal with nonconformal scalar theories. Because \(\langle T_{\mu \nu}\rangle\) in such theories is known only for the metric with constant curvature, \(^7\) we cannot discuss general solutions. However, gross feature of the potential can be discussed.

\section{2. Cosmological equation with \(\langle T_{\mu \nu}\rangle\) for conformally invariant quantum matter fields}

When the action has a form

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi G} (R - 2\Lambda) + aR^2 + \beta R^2_{\mu \nu} + \mathcal{L}_{\text{matter}} \right\},
\]  

(1)

Einstein's equations become\(^*\)

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + g_{\mu \nu} \Lambda + 16\pi G (\alpha^{(1)} H_{\mu \nu} + \beta^{(2)} H_{\mu \nu}) = -8\pi G \langle T_{\mu \nu} \rangle.
\]

(2)

Here we replaced the energy momentum tensor operator for matter fields with its vacuum expectation value and treated the metric tensor as a classical field. One may call this equation semi-classical approximation or the leading order of a \(1/N\) expansion.\(^6\) As for \(\mathcal{L}_{\text{matter}}\) or \(\langle T_{\mu \nu}\rangle\), we deal only with free conformally invariant fields because only for them we can write down an explicit form of \(\langle T_{\mu \nu}\rangle\) in an arbitrary conformally flat metric (we discuss the case of nonconformal scalars in the Appendix). In fact we have

\[
\langle T_{\mu \nu}\rangle = \gamma^{(1)} H_{\mu \nu} + \delta^{(2)} H_{\mu \nu},
\]

(3)

where \(\gamma\) and \(\delta\) are determined from the conformal anomaly. In the spatially flat Robertson-Walker metric

\[
ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2),
\]

the \((0,0)\) component of Eq. (2) becomes

\[
\left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{3} \Lambda = l \left( \frac{\dot{a}}{a} \right)^4 + m \left( 2 \frac{a^2}{a^2} + a^2 \dot{a}^2 + 2 \frac{a^2}{a^2} \ddot{a}^2 + \frac{3}{a} \right),
\]

(4)

where the dot denotes differentiation in \(t\). In this equation \(m\) is equal to \(-48\pi G (2a + 2\beta/3 + \gamma)\) and will be taken as a free parameter which can be both positive and negative, though the convergence of the Euclidean path integral requires \(m > 0\). The constant \(l\) is proportional to \(\delta\) and is given by

\[
l = \frac{G}{360\pi} \left( N_9 + \frac{11}{2} N_{1;2} + 62 N_1 \right),
\]

(5)

\(^*\) Higher derivative terms \(\alpha^{(1)} H_{\mu \nu}\) and \(\alpha^{(2)} H_{\mu \nu}\) are necessary to renormalize \(\langle T_{\mu \nu}\rangle\). See Ref. 1) for their definition.
where \( N_{0(1/2)} \) is the number of conformally invariant fields with spin \( 0(1/2, 1) \). Equation (5) implies \( l > 0 \).(*)

In terms of a variable \( r \) defined by

\[ r = \frac{\dot{a}}{a} \]

(6)

Eq. (4) reduces to the second order differential equation

\[ m (2 r \ddot{r} - \dot{r}^2 + 6 r^2 \dot{r}) = -\frac{1}{3} \Lambda + r^2 - 1 r^4. \]

(7)

By using \( p = |r|^{1/2} \), we can rewrite it as

\[ 4 m p^3 (\dot{p} + 3 \varepsilon p^2 \dot{p}) = -\frac{1}{3} \Lambda + p^4 - l p^8, \]

where \( \varepsilon = \text{sgn}(r) \) and we ignored some terms proportional to \( \delta(p) \) and its derivatives.

Then we get

\[ \frac{d}{dt} \left( \frac{1}{2} p^2 + V(p^2) \right) = -3 \varepsilon p^2 \dot{p}^2, \]

\[ V(p^2) = \frac{1}{4 m} \left( -\frac{1}{6} \Lambda p^2 - \frac{1}{2} p^2 + \frac{1}{6} l p^6 \right). \]

(8)

This is nothing but the equation of motion of a (anti-) dissipative system with a unit mass and a potential \( V(p^2) \). When \( r > 0 \) (expansion) the system is dissipative, and when \( r < 0 \) (contraction) the system is ‘anti-dissipative’ (by which we mean the ‘energy’ increases as the time goes on).

§ 3. General features of the solutions

In the previous section we showed that the cosmological equation, Eq. (4) reduces to the classical potential problem Eq. (8). The potential \( V \) depends, among others, on the signs of \( \Lambda, m \) and

\[ D = 1 - \frac{4 l \Lambda}{3} \]

(9)

(the discriminant of the right-hand side of Eq. (7)). Here and in the next section we assume that \( l > 0 \) because of Eq. (5). Six types of the potential \( V \) are depicted in Fig. 1. In Figs. 1(b), (d)~(f), there are one or two extrema. They correspond to a de Sitter metric \( (r = \text{const}) \) and its constant curvature is given by \( R = 12 r^2 \). An interesting feature is that there can be two de Sitter solutions for some positive cosmological constant, while there is one de Sitter (not anti-de Sitter) solution even for the negative cosmological constant (Figs. 1(e) and (f)). This feature is what we found in Ref. 4) and is illustrated in Fig. 2. This figure also shows that, when \( \Lambda \) is larger than the critical value \( \Lambda_0 (= 3/4 l) \), a de Sitter solution ceases to exist (Figs. 1(a) and (c)).

The above de Sitter solutions are singular solutions of Eq. (8). The potentials in Fig. 1 also suggest qualitative behaviour of general solutions. However, there are some

(*) For nonconformal theories, the value which corresponds to \( l \) can become negative. See the Appendix.
Fig. 1. Six types of the potential Eq. (8). They are distinguished by the signs of \( A, m \) and \( D \) (Eq. (9)). (a) \((A, m, D) = (+-+), \) (b) \((+++), \) (c) \((+--), \) (d) \((+-+), \) (e) \((-++), \) and (f) \((-+-).\)

Fig. 2. The relation between the cosmological constant \( \Lambda \) and the constant scalar curvature \( R \) of de Sitter solutions. \( A, B \) and \( C \) correspond to the extrema of the same symbols in Fig. 1.

features of the solutions which are rather difficult to see intuitively from the potential picture. In the rest of this section we discuss three such problems.

The first problem is to see what will happen when the system passes \( \dot{a} = r = \dot{p} = 0 \) (in Figs. 1(a), (b) and (f)). Though \( r \) and \( \dot{a} \) change the signs, \( \dot{p} \) remains positive by its definition. It is useful here to study the trajectories of the solutions in the \( r-\dot{r} \) plane, which are determined by
Classification of Spatially Flat Cosmological Solutions

\[ \frac{d\dot{r}}{dr} = -3r + \frac{1}{2mr} \left( m\dot{r}^2 - \frac{\Lambda}{3} + r^2 - lr^4 \right). \] 

This equation tells that \( r = 0 \) requires \( \dot{r} = \pm (\Lambda/3m)^{1/2} \). These are singular points (nodes) in the \( r - \dot{r} \) plane. In their neighbourhood, solutions of Eq. (10) have a form

\[ \dot{r} = \sqrt{\frac{\Lambda}{3m}} + E_+ r + O(r^2) \] 

or

\[ \dot{r} = -\sqrt{\frac{\Lambda}{3m}} - E_- r + O(r^2), \]

where \( E_\pm \) is an arbitrary constant. The signs in front of \( E_\pm \) are chosen for convenience.

Now we calculate the energy of the solutions defined by the formula inside the bracket of Eq. (8). From Eq. (11) we find that

\[ \frac{1}{2} \dot{p}^2 + V(p^2) \approx \begin{cases} \frac{1}{4} \sqrt{\frac{\Lambda}{3m}} E_+ , & \text{when } r > 0 , \\ -\frac{1}{4} \sqrt{\frac{\Lambda}{3m}} E_- , & \text{when } r < 0 . \end{cases} \]

This means that the energy changes its sign at the point \( p = r = 0 \), viz, at the transition from expansion to contraction or vice versa. The implication of this fact will become clear in the next section.

The second problem is the behaviour of the solutions around the local minimum of the potential (in Figs. 1(b), (d) and (e)). When \( r > 0 \), the system is dissipative. Any small motion around the minimum settles at the bottom as \( t \to \infty \). The solutions become an expanding de Sitter metric asymptotically. There are two different ways for the approach to a de Sitter metric. For small motion around the minimum, it is sufficient to consider a linearized version of Eq. (8). This is easily solved and integrated to get the scale factor \( a \). The linearized equation is a second order differential equation and qualitative behaviour of the solutions depends on its discriminant. When it is negative we get

\[ a \sim a_0 e^{\tau \dot{a}t} (1 + a_1 e^{-a_1 \tau} \sin a_2 (t - t_0)) . \] 

The system oscillates around \( r = r_0 \). The oscillation gradually attenuates due to dissipation. (Above and also in Eq. (13') \( a_1 \) and \( t_0 \) are integration constants, \( r_0 \) is the value of \( r \) at the bottom of the potential and \( a_i \)'s are real positive numbers determined by \( l, m \) and \( D \).) When the discriminant is positive, on the other hand, we get

\[ a \sim a_0 e^{\tau \dot{a}t} (1 + a_2 e^{-a_2 \tau} + a_3 e^{-a_3 \tau}) . \] 

The system approaches \( r = r_0 \) monotonically because dissipation is so strong that it cannot oscillate at all.

When \( r < 0 \), on the other hand, the system is anti-dissipative. A motion around the bottom of the potential accelerates, and the system is eventually released from the hollow, however slow the initial motion is. Its behaviour for \( t \to -\infty \) is given by Eqs. (13) and (13') with \( t \to -t \).
The last problem of this section is the behaviour of the solution for $p \rightarrow \infty$, which is related to initial and final singularities of a universe. For $m < 0$ (Figs. 1(c), (d) and (f)), it is easy to see that the system can start from or end at $p \rightarrow \infty \ (r \rightarrow \pm \infty)$. Its initial behaviour is, from Eq. (4),

$$a \approx \begin{cases} a_0(t-t_i)^{k_+}, & \text{(expansion)} \\ a_0(t-t_i)^{k_-}, & \text{(contraction)} \end{cases}$$

(14)

and its final behaviour is

$$a \approx \begin{cases} a_0'(t_f-t)^{k_+}, & \text{(expansion)} \\ a_0'(t_f-t)^{k_-}, & \text{(contraction)} \end{cases}$$

(14')

where

$$k_\pm = \frac{3m}{l} \left\{ 1 \pm \left( 1 - \frac{l}{3m} \right)^{1/2} \right\}.$$ 

In Eqs. (14) and (14'), $a_0$ and $a_0'$ are arbitrary and $t_i(t_f)$ represents the initial (final) time of a universe.

What is difficult to see intuitively is that, for $3m > l > 0$, the system can start from $p \rightarrow \infty \ (r \rightarrow +\infty)$ and end at $p \rightarrow \infty \ (r \rightarrow -\infty)$ because of the (anti-) dissipation. The scale factor near the initial or final singularity behaves as

$$a \approx \begin{cases} a_0(t-t_i)^{k_+}, & \text{(expansion)} \\ a_0'(t_f-t)^{k_-}. & \text{(contraction)} \end{cases}$$

(14'')

Note that both $k_+$ and $k_-$ are positive here. Some consideration (e.g., in the $r$-$r$ plane) tells that the solution with $k_+$ is stable (a general solution) and that the one with $k_-$ is unstable (a singular solution).

For $l > 3m > 0$, the system cannot go to (or come from) $p \rightarrow \infty$ monotonically, because the (anti-) dissipation effect is not large enough compared to the potential force. Note that the potential for large $p$ is proportional to $l/m$. The system which moves to $p \rightarrow \infty$ necessarily returns to small $p$. In such a case asymptotic behaviour depends on the potential at small $p$. This will be discussed for each type of the potential below.

§ 4. Classification of the solutions

We are now ready to carry out the classification of all the solutions for each type of the potential, which we call Type I~VI in the order in Fig. 1 (see also Table I in § 6).

4.1. Type I ($(A, m, D) = (+, +, -)$)

There is only one class of solutions, i.e., nonperiodic oscillation of $r$ around $r=0$. Because $r > 0$ and $r < 0$ mean expansion and contraction, respectively (Eq. (6)), the process is also an oscillation of the scale factor $a$.

Below we study how the oscillation looks like by using the analysis in the $r$-$\dot{r}$ plane. As was explained in the previous section, there are two singular points (nodes) in this plane,
Classification of Spatially Flat Cosmological Solutions

\[(r, \dot{r}) = (0, \pm \sqrt{\frac{A}{3m}}), \]  

which we denote by P and Q (Fig. 3(a)). All the trajectories pass through P(Q) when they cross the \(\dot{r}\)-axis from left to right (from right to left). An example of the trajectories is shown in Fig. 3(a). The values of the parameters in the caption are given in an arbitrary unit. This example is also shown in Fig. 3(b) in terms of \(a\). Obviously Eq. (4) is scale-invariant, and any multiple of the solution is also a solution.

Only a part of the solution is shown in Figs. 3(a) and (b). The whole trajectory in the \(r\)-\(\dot{r}\) plane is a spiral which passes P and Q once in every rotation. The centre of the rotation shifts from \(r=\infty\) to \(r=-\infty\). In terms of \(a\) the overall trend is an expansion followed by a contraction as is shown in Fig. 3(b). This trend is common not only for all the solutions in Type I but also for other solutions which oscillate around \(r=0\) ([II 1], [IV 4] etc.). Below we prove it by using the interpretation of Eq. (8) as a (anti-) dissipative system.

First remember that the energy of the system at P and Q is given by the slope of the trajectories (Eqs. (11) and (12)). We follow how it changes for each rotation. When the trajectory moves from P to Q, the system is dissipative because \(\dot{r}>0\). Therefore we get

\[E_+ > E_- .\]

When the trajectory passes the singularity Q, the energy changes the sign (Eq. (12)) and it increases until the trajectory reaches P,

\[-E_- < -E_+ . \quad (i.e., \ E_- > E_+)\]

Therefore, when the trajectory rotates once from P to P,

\[E_+ (\text{beginning}) > E_+ (\text{end}) .\]

Likewise the rotation from Q to Q results in

\[E_- (\text{beginning}) > E_- (\text{end}) .\]

Combining this with the fact that the trajectory can intersect only at the nodes P and Q, we conclude that the centre of the spiral moves from right to left.

This feature is illustrated in Fig. 3(c). The line \(E_+ vs E_-\) shows the decrease of the energy when the system moves from P to Q. The line \(-E_+ vs -E_-\) shows its increase when the system moves from Q to P. Then the whole history of a universe is expressed by a zigzag between the two as is shown in Fig. 3(c).

Now we should discuss how a universe begins and how it ends. The answer is easy when \(l \leq 3m\). The universe starts from \(r=+\infty\) and ends at \(r=-\infty\). Its asymptotic behaviour is given by Eq. (14”). In the \(r\)-\(\dot{r}\) plane the asymptotic behaviour of the trajectories is given by

\[\dot{r} \simeq -\frac{1}{k_\pm} r^2 , \quad (r \to \pm \infty)\]

The trajectories with \(k_-(k_+)\) represent general (singular) solutions. In any case, a trajectory which starts from \(r=+\infty\) reaches Q, and rotates between P and Q several times. Then it runs away from Q to \(r=-\infty\). The lifetime of the universe is finite. A zigzag
Fig. 3. Various aspects of the solutions for Type I potential.

(a) A typical trajectory in the $r$-$\dot{r}$ space ($m=0.2$, $A=0.9$, $l=5$). The logarithmic scale is used when $|r|, |\dot{r}| > 10$.

(b) Time-dependence of the scale factor $a(t)$ of the solution in (a).

(c) The relation between $E_+$ and $E_-$ ($-E_-$ and $-E_+$) when the trajectories rotate from P to Q (from Q to P). The two lines are rotationally symmetric around the origin. A zigzag between the two represents the history of a universe.

(d) The relation between $E_+$ and the increase of the scale factor $a$.

(e) The relation between $E_+$ and the time. The system needs to go from P to Q.

like the one in Fig. 3(c) can represent the history of a universe. However, a difference from Fig. 3(c) is that the $E_+$ vs $E_-$ line for $l\leq 3m$ approaches a finite value of $E_-$ as $E_+ \to \infty$. This value corresponds to the slope of the trajectory at Q of the above singular solutions.

We now turn to the other case $l>3m$. The system cannot start from or end at $r \to \pm \infty$. The oscillation around $r=0$ (the rotation between P and Q) takes place infinite times. However, we argue that the lifetime of a universe is still finite. We prove this statement by treating the oscillation adiabatically, which is valid when $l \gg m$. We believe that this argument is enough, because the lifetime becomes larger when the ratio $l/m$
increases (viz., when the (anti-) dissipation effect becomes relatively small). Anyway, the lifetime is already proved to be finite when \( l \leq 3m \).

First we note that the time \( T(E) \) and the energy gain/loss \( \Delta(E) \) for each oscillation with energy \( E \) are given by

\[
T(E) \propto \int \frac{dp}{\sqrt{E-V}}, \quad (\propto |E|^{1+a/2a})
\]

\[
\Delta(E) \propto \int p^2 \sqrt{E-V} \, dp, \quad (\propto |E|^{3+a/2a})
\]

if the change in \( E \) during each oscillation is ignored (\( \Delta(E) \gg E \)). The expressions inside the parentheses are the results when \( V \propto p^2 \). Now consider the \( n \)-th oscillation after the maximum of a universe. (The process before is essentially time-reversed one.) If \( n \gg 1 \), the energy when \( \dot{r} < 0 \) is large positive, \( E_n \), and the energy when \( \dot{r} > 0 \) is large negative, \( -E_n \). \( V \) can then be approximated by \( (l/24m)p^2 \) when \( \dot{r} < 0 \) and by \( -(\lambda/24m)p^2 \) when \( \dot{r} > 0 \). Putting these into Eq. (16), we find

\[
T(E_n) \propto \xi^{1/3}E_n^{-1/3},
\]

\[
\Delta(E_n) \approx \xi E_n,
\]

where \( \xi \) is some positive number proportional to \( \sqrt{m/l} \). This means that \( E_n \) increases as \( (1+\xi)^n \) and then

\[
\sum_{n=1}^{\infty} T(E_n) \propto \xi^{1/3} \sum_n E_n^{-1/3} \propto \xi^{1/3} \sum_n (1+\xi)^{-n/3} \sim 3\xi^{-2/3} < \infty.
\]

The lifetime of a universe is finite. We can also estimate the change in the scale factor during each oscillation from

\[
\Delta(\log a(E_n)) \propto \int \frac{p^2 \, dp}{\sqrt{E_n-V}},
\]

which tends to a constant as \( E_n \to \infty \). This means

\[
\sum_n \Delta(\log a(E_n)) = \infty,
\]

and therefore a universe starts from \( a=0 \) and collapses to \( a=0 \).

Figure 3(d) shows the increase in the scale factor \( a \) when the trajectory goes from \( P \) to \( Q \) as a function of \( E_+ \). Only the ratio of \( a \) is determined. Its absolute value is arbitrary. Figure 3(e) shows the time it takes during the same process (\( P \to Q \)), also as a function of \( E_+ \). The values for the process \( Q \to P \) are the same except that the increase in \( a \) should be replaced with the decrease in \( a \). These numerical results are in agreement with the above qualitative analysis about behaviour for large \( |E| \).

4.2. Type II \(((A, m, D) = (+, +, +))\)

In the previous section we explained that, if the motion of \( r \) around \( B \) is small enough, the system gradually settles at \( B \) as \( t \to \infty \), as is shown in Eq. (13) or (13'). On the contrary, the motion of \( r \) around \( B' \) accelerates because of the anti-dissipation, and the system is eventually released from the hollow around \( B' \). From these considerations we classify the solutions into four general classes (II 1~4) and two special classes (II 5~6).
(By ‘special’ we mean that the measure of the solutions in the function space of all the solutions is zero.)

[II 1] \( r \) oscillates around \( r=0 \). Qualitative behaviour is the same as Type I above. The oscillations in the early period go over the right peak A and those in the late period go over the left peak A'. The lifetime of a universe is finite by the same reason as in Type I case.

[II 2] The same behaviour as [II 1] in the early stage. However, \( r \) is eventually trapped in the right hollow around B and gradually stabilizes to the de Sitter solution B.

[II 3] \( r \) moves inside the left hollow in the early stage, but the motion gradually amplifies. The system is released from the left hollow and joins class [II 1].

[II 4] This class starts from [II 3] and ends up in [II 2].

[II 5] There are two expanding de Sitter solutions A and B, and two contracting ones A' and B'. B is stable to the future ([II 2]) and B' is stable to the past ([II 3]). By ‘stable to the future (past)’ we mean that solutions of finite measure approach B(B') when \( t \rightarrow -\infty \) (\( +\infty \)).

[II 6] There are four pairs of solutions which either start from or end at A (or A').

All the solutions except [II 1] have infinite lifetime because they approach a de Sitter metric asymptotically, either in the past or in the future, or both. In such a case, the scale factor behaves like Eqs. (13) and (13') or their time reversal, asymptotically.

Analysis in the \( r-\dot{r} \) plane is useful here again. Besides the same singular points as in Type I, there are four others, which correspond to A'(saddle points) and B'(foci) in Fig. 1(b). In Fig. 4(a) we show some characteristic trajectories in the \( r-\dot{r} \) plane. Behaviour of the trajectories near P and Q are the same as in Type I (Eq. (11)). The difference is that, if \( E_+ \) is in some region (dubbed ‘trapped’ in Fig. 4(a)), the trajectory cannot reach Q but is trapped around B. On the left side of the plane there is another new type of trajectories which rotate around B' first and are released later.

In Fig. 4(b) we show the \( E_+ \) vs \( E_- \) and \(-E_+ \) vs \(-E_- \) lines of Type II. There are

![Fig. 4. (a) Some trajectories in the \( r-\dot{r} \) plane for Type II potential \((m=0.2, A=0.09, \ell=5)\). The logarithmic scale is used when \(|r|, |\dot{r}|>10\).
(b) The relation between \( E_+ \) and \( E_- \) \((-E_- \) and \(-E_+ \) \) when the trajectories rotate from P to Q (from Q to P). Three typical solutions ([II 1~3]) are shown by zigzag lines.](https://academic.oup.com/ptp/article-abstract/75/4/845/1858517)
discontinuous parts, and the zigzags which go through the gaps represent trapped or released processes. Relative position of the two gaps depends on the values of the parameters, and for some cases the general class [II 1] or [II 4] may not exist. *)

4.3. Type III ((\(\Lambda, m, D\)) = (+, -, -))

There are two classes of solutions; monotonic expansion and monotonic contraction (from zero to \(\infty\) and vice versa). In terms of \(|r|\), it starts from \(|r| = \infty\), reaches some minimum value \(r_{\text{min}}\) and returns to \(\infty\). The lifetime of a universe is finite. In the expanding case, for example, the initial behaviour is \(a \propto (t-t_i)^{k-}\) and the final behaviour is \(a \propto (t_f-t)^{k+}\) (see Eq. (14)).

An example of the solutions is shown in Fig. 5. After the initial singularity, there is a period when the expansion of the scale factor is very slow (i.e., \(r\) is small). Such a feature resembles our Friedmann universe. In reality, however, it is difficult, without the fine tuning of \(\Lambda\), to make the lifetime even as long as a second, because of the astronomical constraint \(|m| < 10^{14}\) cm\(^{28}\) from the modification of Schwarzschild's exterior solution.

4.4. Type IV ((\(\Lambda, m, D\)) = (+, -, +))

The same consideration as the one for Type II applies here about the motion inside the hollows. We classify the solutions into three general classes and two special classes.

[IV 1] \(r \geq 0\). Solutions similar to those in Type III. The lifetime is finite.

[IV 2] \(r > 0\). \(r\) starts from \(\infty\), decreases and is trapped in the hollow. The motion inside the hollow damps gradually. The solutions approach the expanding de Sitter metric \(A\).

[IV 3] \(r < 0\). The system starts from infinitesimal oscillation around (or infinitesimal deviation from) \(A\) at \(t = -\infty\). The motion amplifies and the system is released from the hollow. In a finite time it goes to \(r = -\infty\).

[IV 4] There are four de Sitter solutions. One \((r > 0)\) is stable to the future and another \((r < 0)\) is stable to the past.

[IV 5] There are four pairs of solutions which either start from or end at \(B\) \((r > 0\) and \(r < 0)\).

4.5. Type V ((\(\Lambda, m, D\)) = (-, +, +))

Monotonically expanding solutions start from \(a = 0\) at a finite cosmic time and approaches the de Sitter metric \(C\) asymptotically. Initial behaviour is the same as in Type I. When \(l \leq 3m\), it is given by Eq. (14\(^*\)). The system starts from \(r = +\infty\).

*) For example, when the whole gap of the \(E_+ \) vs \(E_-\) line is on the right-hand side of that of the \(-E_+ \) vs \(-E_-\) line, we cannot write a zigzag which enters from the latter and goes out from the former. Then [II 4] does not exist. We can also imagine a graph in which we cannot write a zigzag for [II 1]. However, we have not checked for what values of the parameters these cases are actually realized.
When $l > 3m$, initial behaviour is the oscillation between $0 < r < +\infty$ similar to that of Type I. Though the potentials are different for small $r$, we can apply essentially the same analysis. When $t \to \infty$, asymptotic behaviour is given by either Eq. (13) or Eq. (13'). Monotonically contracting solutions are the time reversal of the above.

4.6. Type VI ($(A, m, D) = (-, -, +)$)

There are four general classes and two special classes. The lifetime of a universe in the general classes is finite. Asymptotic behaviour is one of Eqs. (14) and (14').

![Diagram](https://example.com/diagram.png)

Fig. 6. Some trajectories in the $r$-$\dot{r}$ plane for Type VI potential. Critical trajectories are shown by the bold lines. (a) $(m = -0.2, A = -0.9, l = 20)$, (b) $(m = -0.2, A = -0.15, l = 5)$ and (c) $(m = -0.2, A = -0.03, l = 5)$. 
Monotonically expanding solutions start at $r=\infty$. Then $r$ decreases to a minimum value larger than $r_0$ and returns to $\infty$. Symmetrical monotonically contracting solutions exist.

Monotonically expanding solutions start at $r=\infty$. Then $r$ decreases to a minimum value larger than $-r_0$. Symmetrical solutions in $-\infty < r < r_0$ exist.

$r$ monotonically decreases from $\infty$ to $-\infty$ (or increases from $-\infty$ to $\infty$).

$r$ decreases from $\infty$ and oscillates around $r=0$. Then it goes to $-\infty$.

There are two de Sitter solutions, $C$ and $C'$.

There are four pairs of solutions which either start from or end at $C^\prime$.

For certain ranges of the parameters, some of [VI 3, 4] do not exist. This is easily understood from the analysis in the $r$-$\dot{r}$ plane. There are two nodes $P$ and $Q$, and two saddle points $C$ and $C'$ (Fig. 6). They are connected by four “critical” trajectories of the solutions [VI 6], and there are three types of the shape for the region ($PCQC'$) surrounded by them (Figs. 6(a)$\sim$(c)). They are distinguished by the angles at $P$ and $Q$. If a trajectory stays inside or goes into this region then it passes $P(Q)$, it returns to $Q(P)$. Otherwise it goes to $r \rightarrow \pm \infty$. Therefore [VI 4] does not exist in Fig. 6(a) because we cannot write trajectories at $P$ and $Q$ which stay inside this region. Likewise [VI 3] does not exist in Fig. 6(c) because we cannot write trajectories at $P$ and $Q$ which stay outside this region. Such a difference is related to the height of the maximum $C^\prime$ of the potential, because the slopes at $P$ and $Q$ of the trajectories are proportional to the energy (see Eq. (12)).

§ 5. The $\Lambda=0$ case

In this short section we discuss a special case in which the cosmological constant vanishes. We have two types of the potential (Fig. 7) which are distinguished by the sign of $m$. First we discuss the case $m>0$ (Fig. 7(a)). The trajectories in the $r$-$\dot{r}$ plane are shown in Fig. 8(a). Behaviour of the trajectories which pass the origin is, for $r \approx 0$,

$$r \approx k \dot{r}^2. \quad (k; \text{an arbitrary constant})$$

(17)

The trajectories in $r>0$ and those in $r<0$ decouple. All the trajectories in $r>0$ stay there and, in general, approach $B$ asymptotically. This means that they are an asymptotically de Sitter metric. Their initial behaviour is the same as that of Type I (or II 1) solutions except that the present ones never go to $r<0$ even if $l > 3m$. There are four critical trajectories which behave like

---

*1 When the potential is large positive at $C^\prime$ (e.g., $|\Lambda|<1$ and $|l/m|>1$), the system which starts from $C^\prime$ has positive energy when it reaches $r=0$. Then the case in Fig. 6(c) is realized. Likewise when the potential at $C^\prime$ is large negative (e.g., $|\Lambda|>1$ and $|l/m|>1$), the case in Fig. 6(a) is realized. Figure 6(b) corresponds to the case in between.
Fig. 8. (a) Two critical trajectories (bold lines) and one general trajectory for the potential Fig. 9(a) \( (m=0.2, \lambda=5) \).
(b) Four critical trajectories which start from or end at B for the potential Fig. 9(b) \( (m=-0.2, \lambda=5) \).

\[ \dot{r} \approx \pm \frac{1}{\sqrt{m}} r, \]

near the origin (one pair for each side of the \( \dot{r} \) axis). The trajectories with the \( + \) sign start from the origin at \( t=-\infty \) and the others approach it as \( t \to \infty \). Both have exactly the zero energy at \( r=0 \). Note that the potential is flat at \( r=0 \) in terms of \( \dot{p} \).

Next we discuss the case \( m<0 \) (Fig. 7(b)). The trajectories in the \( r-\dot{r} \) plane are shown in Fig. 8(b). The trajectories in \( r>0 \) and those in \( r<0 \) decouple. The spirals in \( r>0 \) shrink to the origin (Minkowski metric), while the spirals in \( r<0 \), which is the time reversal of the above, swell. Therefore, Minkowski metric (a singular solution) is unstable against contraction, although the origin is the local minimum of the potential.*1

Behaviour of the solutions for \( r \to \pm \infty \) is given by Eqs. (14) and (14').

§ 6. Conclusion and discussion

In this paper we derived general behaviour of the spatially flat cosmological solutions of Einstein's equations with the cosmological constant and quantum effects of conformally invariant matter fields. The latter includes the conformal anomaly and other higher derivative terms. We showed that the equation is described as a one dimensional (anti-) dissipative system in a potential. From the form of the potential we could exhaust all the

*1 Note that a spiral in \( r>0 \) which shrinks to the origin does not become a Minkowski metric asymptotically. This is because \( r \) approaches zero as \( t \to \infty \), but only as

\[ r \approx \frac{2}{\epsilon=3t} \left[ 1 + \sin(t-t_0)/\sqrt{-m} \right], \]

which means \( a \propto t^{3/2} \), as was noticed first by Starobinski and was also shown in Ref. 3) through a different method. Likewise initial behaviour of the spirals in \( r<0 \) is not a Minkowski metric either. However, any time-dependent homogeneous perturbation in a Minkowski metric puts the system on a spiral, and therefore the above statement on the instability of a Minkowski metric is valid. (Time-independent fluctuation, which can be absorbed to the coordinate by its rescaling, is not physical fluctuation.)
Classification of Spatially Flat Cosmological Solutions

Table 1. Initial and final asymptotic behaviours of each solution. Solutions with parentheses are particular or singular solutions, which require fine tuning of the boundary conditions. The solutions which require, in addition, fine tuning of the parameters in the equation are listed with ( ).

<table>
<thead>
<tr>
<th>Initial</th>
<th>Singularity $a=0$</th>
<th>Singularity $a=\infty$</th>
<th>Contracting de Sitter</th>
<th>Expanding de Sitter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singularity $a=0$</td>
<td>I, III, VI3, VI4.</td>
<td>III, IV1, VI1, VI2.</td>
<td>II3, (II6), IV3, (IV5), V, (VI6).</td>
<td>(II6), (VI6).</td>
</tr>
<tr>
<td>Singularity $a=\infty$</td>
<td>III, IV1, VI1, VI2.</td>
<td>VI3, VI4.</td>
<td>(VI6).</td>
<td>(IV6), (VI6).</td>
</tr>
<tr>
<td>Expanding de Sitter</td>
<td>II2, (II6), IV2, (IV5), V, (VI6).</td>
<td>(VI6).</td>
<td>II4, (II6).</td>
<td>(II5), (IV4), (IV5), (VI5).</td>
</tr>
<tr>
<td>Contracting de Sitter</td>
<td>(II6), (VI6).</td>
<td>(IV6), (VI6).</td>
<td>(II5), (IV4), (IV5), (VI5).</td>
<td>(II6), (VI6).</td>
</tr>
</tbody>
</table>

Possible behaviours of the solutions, which include a de Sitter metric as special cases. Possible behaviours of the solutions are oscillation, contraction followed by expansion, oscillation followed by expansion, monotonic expansion (contraction), contraction-oscillation-expansion and some other variations. Their asymptotic behaviours were carefully studied. Unless a universe hits a singularity at a finite cosmic time and unless $\Lambda=0$, its asymptotic behaviour can only be a de Sitter metric; expanding de Sitter for $t \to \infty$ and contracting de Sitter for $t \to -\infty$, in general. Some particular solutions can have opposite behaviour. These are summarized in Table 1.

From the behaviour of the general solutions we can also discuss stability of de Sitter solutions which correspond to the extrema of the potential. Some of them are shown to be stable (to the future) in the classical sense. Minkowski metric becomes a singular solution when the cosmological constant vanishes. It is shown to be unstable whatever other parameters are.

The present analysis has been confined to the spatially flat Robertson-Walker metric. Its extension to the closed or open universe brings about some complexity but is possible. Effects of classical matters should also be considered. These were partially done in Ref. 3), where only asymptotically Friedmann (classical) and Friedmann-like solutions were studied.

Numerical calculations were performed partially by IBM 4341 of Center for Data Processing and Computer Science, Dokkyo University.

Appendix

--- Effective Potential for the Curvature in the Nonconformal Scalar Theories ---

All the discussion in the text is about effects of conformally invariant quantum fields, for which general expression for $\langle T_{\mu\nu} \rangle$ is available (Eq. (3)). For massive and massless free scalar theories in general, we know $\langle T_{\mu\nu} \rangle$ only when $r=\text{const.}$ We can discuss only de Sitter solutions and this was done in Ref. 4). We found that the relation between the cosmological constant and the curvature of the solution ($R=12 \rho$) crucially depends on the mass and the magnitude ($\xi$) of the coupling $\phi^2 R$ in the Lagrangian. One typical behaviour (an example of which is the conformal case) was shown before (Fig. 2), and this is easily understood in terms of the potential as was explained at the beginning of § 3.
Two qualitatively different behaviours are shown in Fig. 9. Below we show that we can also understand them in terms of the effective potential.

After the path integration of matter fields in $\mathcal{L}_{\text{matter}}$ (Eq. (1)) in a de Sitter metric, the action Eq. (1) should have a form (after the renormalization of $\Lambda$, $G$, $\alpha$ and $\beta$),

$$
S_{\text{eff}}(g_{\mu\nu}) = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi G} (R - 2\Lambda) + aR^2 + \beta R_{\mu\nu}^2 - V_m(R) \right\},
$$

where $V_m$ is the contribution of $\mathcal{L}_{\text{matter}}$. Now consider the global scale transformation

$$
g_{\mu\nu} \rightarrow g_{\mu\nu}(\rho) \equiv \rho^4 g_{\mu\nu},$$

$$R \rightarrow R(\rho) \equiv \rho^{-2} R.$$

The stationary points of $S_{\text{eff}}$ for this variation are given by

$$
\frac{1}{4} R - \Lambda = 2\pi G T(R). \quad (T(R) \equiv 4V_m - 2RV_m')
$$

This is nothing but the trace of Eq. (2) for a de Sitter metric, for which $(1)H_{\mu\nu} = (2)H_{\mu\nu} = 0$. For the massless free scalar theory,

$$T = kR^2,$$

$$k = \left\{ \frac{1}{16\pi^2} \left( \frac{1}{2160} \frac{1}{2} \left( \xi - \frac{1}{6} \right)^2 \right), \quad \xi > 0 \right\}$$

which means, from Eq. (A2),

$$V_m = -\frac{1}{2}kR^2 \log R/R_0 \quad (R_0; \text{an integration constant}).$$

Now we rewrite $S_{\text{eff}}(g_{\mu\nu}(\rho))$ in terms of $r$. Under the scale transformation Eq. (A1)

$$r \rightarrow r_\rho \equiv \rho^{-1} r.$$

Fig. 9. Two other typical relations between the cosmological constant and the constant scalar curvature of de Sitter metric.

Fig. 10. (a) The potential $-S_{\text{eff}}(r)(A4)$ or $V$ in Eq. (8) with $l<0$ and $\Lambda>0$. (b) The potential with $l<0$ and $\Lambda<0$. (c) A possible behaviour of $G_{\text{sub}}(x,x;m^2)$. (d) A possible behaviour of the potential in the massive scalar theory.
Then using $\rho = r_\rho^{-1} \cdot r, R = 12 r^2$ and Eq. (A3), we get

$$S_{\text{eff}}(r_\rho) = N_0 \{-2A r_\rho^{-4} + 12 r_\rho^{-2} + 12 \log r_\rho + \text{const}\}, \quad (A4)$$

where $N_0$ is a $\rho$-independent constant and

$$l = 96 \pi G \kappa,$$

which is defined consistently with Eq. (5).

This effective potential $(-S_{\text{eff}})$ and $V$ in Eq. (8) are equivalent in a sense that both have the same extrema at the same positions. The apparent difference comes from the fact that the 'kinetic term' in Eq. (7) has an unconventional form. Anyway, $S_{\text{eff}}(r)$ looks like one of Fig. 2 as long as $l > 0$. In the nonconformal case, however, $l$ can be negative, and the potential may look like Fig. 10(a) ($\Lambda > 0$) or Fig. 10(b) ($\Lambda < 0$). There is only one extremum in Fig. 10(a) and none in Fig. 10(b), which explains Fig. 9(a).

In the massive case we cannot get a compact expression for $V_m$ in $S_{\text{eff}}(R)$. However, it is numerically calculable from the known subtracted Green function $G_{\text{sub}}(x, x; R, m^2)$ by

$$V_m = \frac{1}{2} \int_{m^2}^\infty dm^2 G_{\text{sub}}(x, x; R, m^2) + \text{const}.$$  

For some values of $m$ and $\xi$, $G_{\text{sub}}$ looks like Fig. 10(c) (curve C is Fig. 2 of Ref. 4)). Adding this behaviour to $2A/r^4 - 12/r^2$, the whole potential may look like Fig. 10(d) for some $\Lambda$. This has three de Sitter solutions and explains the region $\Lambda_1 < \Lambda < \Lambda_2$ in Fig. 9(b).

Even for the nonconformal case the higher derivative terms in the action gives rise to the 'kinetic part' $m(-\cdots)$ in Eq. (4). However, $\langle T_{\mu\nu} \rangle$ may have additional contribution which we do not know, and we cannot discuss general solutions in the nonconformal case at present.

References

1) For reviews see, for example, N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

2) The following is a very incomplete list of literature on this subject:
   - V. Ts. Garovich and A. A. Starobinski, Sov. Phys.-JETP 50 (1979), 844.