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Ashtekar's Formulation for $N=1, 2$ Supergravities as "Constrained" BF Theories

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It is known that Ashtekar's formulation for pure Einstein gravity can be cast into the form of a topological field theory, namely, the $SU(2)$ BF theory, with the two-form fields subject to an algebraic constraint. We extend this relation between Ashtekar's formalism and BF theories to $N=1$ and $N=2$ supergravities. The relevant gauge groups in these cases become graded Lie groups of $SU(2)$, which are generated by left-handed local Lorentz transformations and left-supersymmetry transformations. As a corollary of these relations, we provide topological solutions for $N=2$ supergravity with a vanishing cosmological constant. It is also shown that, due to the algebraic constraints, the Kalb-Ramond symmetry which is characteristic of BF theories breaks down to the symmetry under diffeomorphisms and right-supersymmetry transformations.

§ 1. Introduction

Since its birth in the mid eighties, Ashtekar's formulation for canonical gravity[1] has been vigorously investigated by many researchers as a promising approach to the nonperturbative quantum gravity.[2] A merit of Ashtekar's formalism is that the Hamiltonian constraint, or the Wheeler-DeWitt equation,[3] takes a polynomial form in terms of new canonical variables. Thus we expect that, using Ashtekar's formalism, we can solve the constraint equations, which have not been solved in the conventional metric formulation.[4]

In fact, several types of solutions are found. These are roughly classified into two types, "loop solutions", which consist of Wilson loops[5)~7)] and "topological solutions", which are also solutions for a topological field theory.[8] The latter type includes the Chern-Simons solution in the case with a nonvanishing cosmological constant Λ.[9] The existence of these topological solutions suggests a relationship between Ashtekar's formalism and a topological field theory, namely, the $SU(2)$ BF theory.[10] It was indeed shown that Ashtekar's formalism can be obtained from the $SU(2)$ (strictly speaking the chiral $SL(2, C)$) BF theory with the two-form field subject to an algebraic constraint.[11)~13)]

Ashtekar's formalism is also applied to supergravities with $N=1$[14),11)] and with $N=2$.[15] In the case $N=1$ supergravity also, topological solutions[16] including Chern-Simons solutions[17),18)] were found. As for the $N=2$ case, only the Chern-Simons solution was found.[17] This fact implies a relation between $N=1, 2$ supergravities and BF theories with appropriate gauge groups. We expect that, if these relations can be made transparent, we can make further progress in the investigation of quantum gravity both technically and conceptually.

In this paper we show explicitly that Ashtekar's formulation for $N=1$ and $N=2$ supergravities can indeed be cast into the form of BF theories with two-form fields

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subject to algebraic constraints. The relevant gauge groups in these cases are provided by graded versions of $SU(2)$, which are generated by left-handed local Lorentz transformations and local left-supersymmetry transformations (plus $U(1)$ gauge transformations in the $N=2$ case). These relations not only elegantly explain the existence of the above-mentioned topological solutions but also predict the existence of topological solutions in the case $N=2$ and $\Lambda=0$. We also show how the algebraic constraints for two-form fields break the Kalb-Ramond symmetry of the BF theories[19),10)] down to symmetry under diffeomorphisms and right-supersymmetry transformations.

The presentation of this paper is as follows. Once the relations to the BF theories are established, the arguments are almost parallel for the cases of pure gravity and $N=1, 2$ supergravities. So only the $N=1$ case is dealt with in detail. After briefly reviewing the relation between pure gravity and the $SU(2)$ BF theory in § 2, we derive the action of $N=1$ chiral supergravity from that of the $GSU(2)$ BF theory in § 3. Canonical quantization of the $GSU(2)$ BF theory is also discussed in § 3. The $\Lambda=0$ case is a little peculiar and the reduced phase space in this case is shown to be the cotangent bundle over the moduli space of flat $GSU(2)$ connections on the spatial manifold $M^{(3)}$ modulo gauge transformations. In § 4 we show that $N=2$ supergravity is obtained from the BF theory whose gauge group is an appropriate graded version of $SU(2)$, which we will henceforth call $G^2SU(2)$. Unlike in the pure gravity and $N=1$ cases, we cannot find the relation easily because the $N=2$ chiral action involves a quadratic term in auxiliary fields. Replacing this quadratic term by linear terms in auxiliary fields, we make the relation to the $G^2SU(2)$ BF theory manifest. For $N=2$ supergravity the $\Lambda=0$ case is somewhat different from those in the $N=0, 1$ cases because the reduced phase space does not possess cotangent bundle structure. We also provide the formal "topological" solutions for the $N=2$, $\Lambda=0$ case. In § 5 we discuss possibilities for future developments.

Let us now define the notation used in this paper: i) $\mu, \nu, \cdots$ stand for spacetime indices; ii) $a, b, \cdots$ are used for spatial indices; iii) $A, B, \cdots$ represent left-handed $SL(2, \boldsymbol{C})$ spinor indices; iv) $i, j, \cdots$ denote indices for the adjoint representation of (the left-handed part of) $SL(2, \boldsymbol{C})$; v) $\tilde{\epsilon}^{abc}(\underset{\sim}{\epsilon}_{abc})$ is the Levi-Civita alternating tensor density of weight $+1$ (-1) with $\tilde{\epsilon}^{123}=\underset{\sim}{\epsilon}_{123}=1$; vi) ϵ^{ijk} is the antisymmetric (pseudo-) tensor with $\epsilon^{123}=1$; vii) $\epsilon^{AB}(\epsilon_{AB})$ is the antisymmetric spinor with $\epsilon^{12}=\epsilon_{12}=1$;*) viii) the relation between a symmetric rank-2 spinor ϕ^{AB} and its equivalent vector ϕ^i in the adjoint representation is given by $\phi^{AB}=\phi^i(\sigma^i/2i)^{AB}$, where $(\sigma^i)^A{}_B$ are the Pauli matrices with $(\sigma^i)^A{}_C(\sigma^j)^C{}_B=\delta^{ij}\delta^A_B+i\epsilon^{ijk}(\sigma^k)^A{}_B$; ix) $D=dx^\mu D_\mu$ denotes the covariant exterior derivative with respect to the $SU(2)$ connection $A=A^iJ_i$; x) indices located between (and) ([and]) are regarded as symmetrized (antisymmetrized).

For simplicity we will restrict our analysis to the case where the spacetime has the topology $\boldsymbol{R}\times M^{(3)}$, with $M^{(3)}$ being a compact, oriented, 3-dimensional manifold without boundary.

*) These antisymmetric spinors are used to raise and lower the spinor index: $\varphi^A=\epsilon^{AB}\varphi_B$, $\varphi_A=\varphi^B\epsilon_{BA}$.

§ 2. $SU(2)$ BF theory and Ashtekar's formalism

In this section we provide a brief review of the relationship between $SU(2)$ (or, strictly speaking, chiral $SL(2, \boldsymbol{C})$) BF theory and Ashtekar's formulation for pure gravity.[11)~13)] We start with the action of the BF theory:

$$-iI_{\mathrm{BF}}=\int \mathrm{Tr}\left(\Sigma\wedge F-\frac{\Lambda}{6}\Sigma\wedge\Sigma\right), \tag{2·1}$$

where $\Sigma=\Sigma^i J_i=(1/2)\Sigma^i_{\mu\nu}dx^\mu\wedge dx^\nu J_i$ is an $SU(2)$ Lie algebra-valued two-form and $F=dA+A\wedge A$ is the curvature two-form of an $SU(2)$ connection $A=A^iJ_i=A_\mu{}^i dx^\mu J_i$.*)

This action is invariant under $SU(2)$ gauge (or left-handed local Lorentz) transformations:

$$\begin{aligned}&\delta_\theta\Sigma=[\theta,\Sigma],\\ &\delta_\theta A=-D\theta\equiv-d\theta-[A,\theta],\end{aligned} \tag{2·2}$$

where $\theta=\theta^iJ_i$ is an $SU(2)$-valued scalar. The action (2·1) has an additional symmetry, the (generalized) Kalb-Ramond symmetry:[19),10)]

$$\begin{aligned}&\delta_\phi\Sigma=-D\phi\equiv-d\phi-A\wedge\phi-\phi\wedge A,\\ &\delta_\phi A=-\frac{\Lambda}{3}\phi,\end{aligned} \tag{2·3}$$

where $\phi=\phi^iJ_i$ is an $SU(2)$-valued one form. This Kalb-Ramond symmetry includes symmetry under the diffeomorphisms. We use as ϕ the field-dependent parameter $\phi_\mu=v^\nu\Sigma_{\mu\nu}$. Then by using equations of motion

$$F-\frac{\Lambda}{3}\Sigma=D\Sigma=0,$$

we obtain the infinitesimal diffeomorphism generated by the vector $v=v^\mu(\partial/\partial x^\mu)$ plus the $SU(2)$ gauge transformation generated by $\theta=v^\mu A_\mu$:

$$\begin{aligned}&\delta_\phi\Sigma|_{\phi_\mu=v^\nu\Sigma_{\mu\nu}}=\mathcal{L}_v\Sigma+\delta_\theta\Sigma|_{\theta=v^\mu A_\mu},\\ &\delta_\phi A|_{\phi_\mu=v^\nu\Sigma_{\mu\nu}}=\mathcal{L}_vA+\delta_\theta A|_{\theta=v^\mu A_\mu},\end{aligned} \tag{2·4}$$

where $\mathcal{L}_v$ denotes the Lie derivative with respect to the vector v. The derivation of these equations requires the equations of motion. However, the equations of motion for the BF theory are either first class constraints or equations which yield conditions for the temporal components of the fields. As is seen shortly, in the canonical formalism, temporal components are considered to be Lagrange multipliers which play the role of the gauge parameters. The diffeomorphism invariance is thus considered to be a particular form of the Kalb-Ramond symmetry as far as the physical content of the BF theory is concerned.

*) J_i denote the $SU(2)$ generators subject to the commutation relations $[J_i, J_j]=\epsilon_{ijk}J_k$. Tr is the $SU(2)$-invariant bilinear form: $\mathrm{Tr}(J_iJ_j)=\delta_{ij}$.

In order to rewrite the action (2·1) in canonical form, we simply identify the zeroth coordinate x^0 with time t. The result is

$$-iI_{\mathrm{BF}}=\int dt\int_{M^{(3)}}d^3x\mathrm{Tr}[\tilde{\pi}^a\dot{A}_a+A_tG+\Sigma_{ta}\Phi^a]$$
$$=\int dt\int_{M^{(3)}}d^3x(\tilde{\pi}^{ai}\dot{A}_a{}^i+A_t{}^iG^i+\Sigma^i_{ta}\Phi^{ai}), \qquad (2\cdot5)$$

where we have set $\tilde{\pi}^a=\tilde{\pi}^{ai}J_i\equiv(1/2)\tilde{\epsilon}^{abc}\Sigma_{bc}$ and $\dot{A}=(\partial/\partial t)A$. This system involves two types of first class constraints. Gauss' law constraint

$$G=G^iJ_i\equiv D_a\tilde{\pi}^a \qquad (2\cdot6)$$

generates the $SU(2)$ gauge transformations, and the remaining constraint

$$\Phi^a=\Phi_i{}^aJ_i\equiv\frac{1}{2}\tilde{\epsilon}^{abc}F_{bc}-\frac{\Lambda}{3}\tilde{\pi}^a \qquad (2\cdot7)$$

generates the Kalb-Ramond transformations.

Let us now quantize this system following Dirac's quantization procedure.[20] We first read off canonical commutation relations from the symplectic structure. The result is

$$[\hat{A}_a{}^i(x),\ \hat{\tilde{\pi}}^{bj}(y)]=\delta_a{}^b\delta^{ij}\delta^3(x,y)\,.$$

If we use as wavefunctions the functionals of the connection $A_a{}^i$, the conjugate momenta are represented by functional differentiations:

$$\hat{\tilde{\pi}}_i{}^a(x)\cdot\Psi[A_a{}^i]=-\frac{\delta}{\delta A_a{}^i(x)}\Psi[A_a{}^i]\,. \qquad (2\cdot8)$$

Next we impose the first class constraints as conditions to which the physical wavefunctions are subject. Gauss' law constraint simply tells us that the wavefunctions are $SU(2)$ gauge invariant. The other constraint can also be solved easily.

For $\Lambda=0$, this is solved by the wavefunctions with support only on the flat connections. The formal solutions for the $\Lambda=0$ case are given by

$$\Psi[A]=\psi[A_a{}^i]\prod_{x\in M^{(3)}}\prod_{i,a}\delta(\tilde{\epsilon}^{abc}F^i_{bc}(x))\,, \qquad (2\cdot9)$$

where ψ is an arbitrary $SU(2)$ gauge invariant functional of the connection. This solution coincides with that obtained in Ref. 8). This is effectively equivalent to dealing with the functions on the moduli space of flat $SU(2)$ connections modulo (the identity-connected component of) gauge transformations.*)

For $\Lambda\neq0$ this remaining constraint has the unique solution

$$\Psi[A]=\exp\left[-\frac{3}{2\Lambda}\int_{M^{(3)}}\mathrm{Tr}\left(AdA+\frac{2}{3}A\wedge A\wedge A\right)\right], \qquad (2\cdot10)$$

*) Because the constraints are at most linear in the conjugate momenta $\tilde{\pi}^a$, we expect that Dirac's quantization yields the same result as the reduced phase space quantization, up to minor subtleties.[21] Particularly in the $\Lambda=0$ case, the result of these two quantizations should be identical, because the reduced phase space turns out to be the cotangent bundle on the moduli space of flat $SU(2)$ connections.

which coincides with the Chern-Simons solution found in Ref. 9).

Ashtekar's formulation for pure gravity is obtained from the action (2·1), accompanied by the following algebraic constraint on the two-form field (we set $\Sigma^{AB} = \Sigma^i(\sigma^i/2i)^{AB}$):

$$\Sigma^{(AB} \wedge \Sigma^{CD)} = 0 . \tag{2·11}$$

Solving this algebraic constraint for Σ^i_{ta} and substituting the result into the action (2·5), we find

$$-iI_{\mathrm{Ash}} = \int dt \int_{M^{(3)}} d^3x \mathrm{Tr}\Big(\tilde{\pi}^a \dot{A}_a + A_t D_a \tilde{\pi}^a - i \underset{\sim}{N} \frac{1}{2} \tilde{\pi}^b \tilde{\pi}^c \Big(F_{bc} - \frac{\Lambda}{3} \underset{\sim}{\epsilon}_{bca} \tilde{\pi}^a \Big)$$
$$+ N^b \tilde{\pi}^c \Big(F_{bc} - \frac{\Lambda}{3} \underset{\sim}{\epsilon}_{bca} \tilde{\pi}^a \Big) \Big) . \tag{2·12}$$

This is nothing but the action for Ashtekar's formalism. Thus we easily see that the solutions to the $SU(2)$ BF theory are necessarily included in the solution space of Ashtekar's constraints provided that we take the ordering with the momenta $\tilde{\pi}^{ai}$ to the left. This seems to be natural, because we know that the constraint algebra formally closes under such ordering.

§ 3. $GSU(2)$ BF theory and Ashtekar's formulation for $N=1$ supergravity

In this section we show explicitly that $N=1$ supergravity in Ashtekar's form can be cast into the form of the $GSU(2)$ BF theory with the two-form field subject to algebraic constraints.

3.1. *$GSU(2)$ BF theory*

We start with the BF action

$$-iI^{N=1}_{\mathrm{BF}} = \int \mathrm{STr}\Big(\mathcal{B} \wedge \mathcal{F} - \frac{g^2}{6} \mathcal{B} \wedge \mathcal{B} \Big) , \tag{3·1}$$

where $\mathcal{B} = \Sigma^i J_i - (1/\lambda g)\chi^A J_A$ is a $GSU(2)$-valued two-form, and $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the curvature two-form of the $GSU(2)$ connection $\mathcal{A} = A^i J_i + \psi^A J_A$.*) (J_i, J_A) are the generators of the graded Lie albebra $GSU(2)$[22]:

$$[J_i, J_j] = \epsilon_{ijk} J_k , \quad [J_i, J_A] = \Big(\frac{\sigma^i}{2i} \Big)_A{}^B J_B , \quad \{J_A, J_B\} = -2\lambda g \Big(\frac{\sigma^i}{2i} \Big)_{AB} J_i , \tag{3·2}$$

where $\{ , \}$ denotes the anti-commutation relation. STr stands for the $GSU(2)$ invariant bilinear form which is unique up to an overall constant factor

$$\mathrm{STr}(J_i J_j) = \delta_{ij} , \quad \mathrm{STr}(J_A J_B) = -2\lambda g \epsilon_{AB} , \quad \mathrm{STr}(J_A J_i) = \mathrm{STr}(J_i J_A) = 0 . \tag{3·3}$$

*) Note that χ^A and ψ^A are Grassmann odd fields. Whether an object is Grassmann even or odd can be determined by whether the number of its Lorentz spinor indices is even or odd.

If we use Eqs. (3·2) and (3·3) and rewrite the action (3·1) in terms of component fields, the result is as follows:

$$-iI_{\rm BF}^{N=1}=\int\left(\Sigma^i\wedge\left(F^i+\lambda g\left(\frac{\sigma^i}{2i}\right)_{AB}\phi^A\wedge\phi^B\right)+2\chi^A\wedge D\phi_A-\frac{g^2}{6}\Sigma^i\wedge\Sigma^i-\frac{g}{3\lambda}\chi_A\wedge\chi^A\right). \quad (3\cdot4)$$

This action (3·1), or equivalently the action (3·4), necessarily possesses the symmetry under $GSU(2)$ gauge transformations

$$\delta_\rho\mathcal{A}=-\mathcal{D}\rho\equiv-d\rho-[\mathcal{A},\rho]\,,$$
$$\delta_\rho\mathcal{B}=[\rho,\mathcal{B}]\,, \quad (3\cdot5)$$

where $\rho=\theta^iJ_i+\epsilon^AJ_A$ is a $GSU(2)$-valued scalar, and the Kalb-Ramond symmetry

$$\delta_\xi\mathcal{A}=-\frac{g^2}{3}\xi\,,$$
$$\delta_\xi\mathcal{B}=-\mathcal{D}\xi\equiv-d\xi-\mathcal{A}\wedge\xi-\xi\wedge\mathcal{A}\,, \quad (3\cdot6)$$

where $\xi=\phi^iJ_i-(1/\lambda g)\eta^AJ_A$ is a $GSU(2)$-valued one-form. Of course these transformations can be translated in terms of component fields. The $GSU(2)$ gauge transformation (3·7) is

$$\delta_\rho A^i=-D\theta^i+2\lambda g\left(\frac{\sigma^i}{2i}\right)_{AB}\epsilon^A\phi^B\,,$$
$$\delta_\rho\phi^A=\theta^i\left(\frac{\sigma^i}{2i}\right)^A{}_B\phi^B-D\epsilon^A\,,$$
$$\delta_\rho\Sigma^i=\epsilon^{ijk}\theta^j\Sigma^k-2\left(\frac{\sigma^i}{2i}\right)_{AB}\epsilon^A\chi^B\,,$$
$$\delta_\rho\chi^A=\theta^i\left(\frac{\sigma^i}{2i}\right)^A{}_B\chi^B+\lambda g\epsilon^B\left(\frac{\sigma^i}{2i}\right)_B{}^A\Sigma^i\,. \quad (3\cdot7)$$

And the Kalb-Ramond symmetry is decomposed as

$$\delta_\xi A^i=-\frac{g^2}{3}\phi^i\,,$$
$$\delta_\xi\phi^A=\frac{g}{3\lambda}\eta^A\,,$$
$$\delta_\xi\Sigma^i=-D\phi^i+2\left(\frac{\sigma^i}{2i}\right)_{AB}\eta^A\wedge\phi^B\,,$$
$$\delta_\xi\chi^A=\lambda g\phi^i\left(\frac{\sigma^i}{2i}\right)^A{}_B\wedge\phi^B-D\eta^A\,. \quad (3\cdot8)$$

In almost the same way as in the $SU(2)$ case, we can show that a diffeomorphism is generated by the Kalb-Ramond transformation (3·6) (or the transformation (3·8)) with $\xi_\mu=v^\nu\mathcal{B}_{\mu\nu}$, up to a $GSU(2)$ gauge transformation generated by $\rho=v^\mu\mathcal{A}_\mu$.

Let us now investigate the canonical formalism. In a manner similar to the $SU(2)$ case, we perform a (3+1)-decomposition of the action. The result is

$$-iI_{\mathrm{BF}}^{N=1}=\int dt\int_{M^{(3)}}\mathrm{STr}(\tilde{\Pi}^a\dot{\mathcal{A}}_a+\mathcal{A}_t\boldsymbol{G}+\mathcal{B}_{ta}\boldsymbol{\Phi}^a)$$

$$=\int dt\int_{M^{(3)}}(\tilde{\pi}^{ai}\dot{A}_a{}^i+2\tilde{\pi}^A\dot{\psi}_{aA}+A_t{}^iG^i-2\psi_{tA}L^A+\Sigma_{ta}^i\Phi^{ai}-2\chi_{taA}\Phi^{aA})\,,\quad(3\cdot9)$$

where we have set $\tilde{\Pi}^a=(1/2)\tilde{\epsilon}^{abc}\mathcal{B}_{bc}=\tilde{\pi}^{ai}J_i-(1/\lambda g)\tilde{\pi}^{aA}J_A$. From the symplectic potential

$$\Theta=i\int_{M^{(3)}}d^3x\mathrm{STr}(\tilde{\Pi}^a\delta\mathcal{A}_a)=i\int_{M^{(3)}}d^3x(\tilde{\pi}^{ai}\delta A_a{}^i+2\tilde{\pi}^{aA}\delta\psi_{aA})\,,\quad(3\cdot10)$$

we can read off Poisson brackets between the canonical variables:

$$\{A_a{}^i(x),\,\tilde{\pi}^{bj}(y)\}_{PB}=-i\delta_a{}^b\delta^{ij}\delta^3(x,y)\,,\quad\{\psi_{aA}(x),\,\tilde{\pi}^{bB}(y)\}_{PB}=\frac{-i}{2}\delta_a{}^b\delta_A{}^B\delta^3(x,y)\quad(3\cdot11)$$

with the rest being zero.

There are two types of first class constraints. One is the Gauss' law constraint which generates $GSU(2)$ gauge transformations of the canonical variables under the Poisson bracket

$$\boldsymbol{G}=\mathcal{D}_a\tilde{\Pi}^a=G^iJ_i-\frac{1}{\lambda g}L^AJ_A\,,$$

$$G^i=D_a\tilde{\pi}^{ai}-2\left(\frac{\sigma^i}{2i}\right)_{AB}\psi_a{}^A\tilde{\pi}^{aB}\,,$$

$$L^A=D_a\tilde{\pi}^{aA}+\lambda g\tilde{\pi}^{ai}\left(\frac{\sigma^i}{2i}\right)^A{}_B\psi_a{}^B\,.\quad(3\cdot12)$$

The other type of constraint is that which generates the Kalb-Ramond transformations

$$\boldsymbol{\Phi}^a=\frac{1}{2}\tilde{\epsilon}^{abc}\mathcal{F}_{bc}-\frac{g^2}{3}\tilde{\Pi}^a=\Phi^{ai}J_i+\Phi^{aA}J_A\,,$$

$$\Phi^{ai}=\frac{1}{2}\tilde{\epsilon}^{abc}\left(F_{bc}^i+2\lambda g\left(\frac{\sigma^i}{2i}\right)_{AB}\psi_b{}^A\psi_c{}^B\right)-\frac{g^2}{3}\tilde{\pi}^{ai}\,,$$

$$\Phi^{aA}=\tilde{\epsilon}^{abc}D_b\psi_c{}^A+\frac{g}{3\lambda}\tilde{\pi}^{aA}\,.\quad(3\cdot13)$$

These constraints indeed form a closed algebra under the Poisson bracket. To see this it is convenient to use smeared constraints:

$$\boldsymbol{G}(\rho)\equiv i\int_{M^{(3)}}d^3x\mathrm{STr}(\rho\boldsymbol{G})=i\int_{M^{(3)}}d^3x(\theta^iG^i-2\epsilon_AL^A)$$

$$\equiv G^i(\theta^i)+L^A(\epsilon_A)\,,\quad(3\cdot14)$$

$$\boldsymbol{\Phi}(\xi)\equiv i\int_{M^{(3)}}d^3x\mathrm{STr}(\xi_a\boldsymbol{\Phi}^a)=i\int_{M^{(3)}}d^3x(\phi_a{}^i\Phi^{ai}-2\eta_{aA}\Phi^{aA})$$

$$\equiv\Phi^{ai}(\phi_a{}^i)+\Phi^{aA}(\eta_{aA})\,.\quad(3\cdot15)$$

The calculation of Poisson brackets is not so difficult if we recall that these smeared constraints generate the $GSU(2)$ gauge transformation (3·5) and the Kalb-Ramond transformation (3·6) on the canonical variables. The result is neatly written as

$$\{\boldsymbol{G}(\rho),\ \boldsymbol{G}(\rho')\}_{PB}=\boldsymbol{G}([\rho,\rho']),$$
$$\{\boldsymbol{\Phi}(\xi),\ \boldsymbol{G}(\rho)\}_{PB}=\boldsymbol{\Phi}([\xi,\rho]),$$
$$\{\boldsymbol{\Phi}(\xi),\ \boldsymbol{\Phi}(\xi')\}_{PB}=0\,. \tag{3·16}$$

Of course these involve all the information on the constraint algebra written in terms of component fields. For instance, the Poisson algebra between the components of Gauss' law constraint reads

$$\{iG^i(x),\ iG^j(y)\}_{PB}=\delta^3(x,y)\epsilon^{ijk}(iG^k(x))\,,$$
$$\{iG^i(x),\ -2iL_A(y)\}_{PB}=\delta^3(x,y)\left(\frac{\sigma^i}{2i}\right)_A{}^B(-2iL_B(x))\,,$$
$$\{-2iL_A(x),\ -2iL_B(y)\}_{PB}=\delta^3(x,y)(-2\lambda g)\left(\frac{\sigma^i}{2i}\right)_{AB}(iG^i(x))\,. \tag{3·17}$$

This precisely coincides with the $GSU(2)$ algebra.

One may suspect that the case with $g=0$ requires careful consideration because the definition of $\mathscr{B}$ is singular at $g=0$. However, we do not have to be so careful since no negative power of g appears either in the action or in the smeared constraints, provided that these are expressed in terms of component fields. Indeed if we start with the action for the component fields

$$-iI_{\rm BF}^{N=1}=\int(\Sigma^i\wedge F^i+2\chi^A\wedge D\phi_A)\,, \tag{3·18}$$

and consider linear combinations of the constraints appearing in the last expressions of Eqs. (3·14) and (3·15), the result of the constraint algebra reproduces the $g\to 0$ limit of Eqs. (3·16) and (3·17).

One of the properties characteristic of the $g=0$ case is that the symplectic potential (3·10) is inherited by the reduced phase space.*) To see this explicitly we compute the transformation property of the symplectic potential under the $GSU(2)$ gauge transformations and the Kalb-Ramond transformations. We find

$$\delta_\theta\Theta=i\int_{M^{(3)}}d^3xG^i\delta\theta^i\,,$$
$$\delta_\epsilon\Theta=2i\int_{M^{(3)}}d^3xL^A\delta\epsilon_A\,,$$
$$\delta_\phi\Theta=i\int_{M3}d^3x\phi_a{}^i\delta\Phi^{ai}\,,$$

*) Reduced phase space is the quotient space of the constraint surface modulo gauge transformations in a broader sense. The constraint surface is the subspace of the phase space on which the first class constraints vanish. Gauge transformations in a broader sense are the transformations generated by the first class constraints.

$$\delta_\eta \Theta = 2i \int_{M^{(3)}} d^3x \eta_a{}^A \delta \Phi_A{}^a . \tag{3·19}$$

These expressions vanish on the constraint surface $G^i = L^A = \Phi^{ai} = \Phi_A{}^a = 0$. This implies that the reduced phase space has a well-defined cotangent bundle structure. The base space of this cotangent bundle is provided by the reduced configuration space, which in this case turns out to be the moduli space $\mathcal{N}_0$ of flat $GSU(2)$ connections $\mathcal{A}_a = A_a{}^i J_i + \psi_a{}^A J_A$ on $M^{(3)}$ modulo $GSU(2)$ gauge transformations. The reduced phase space in the $g=0$ case is therefore the cotangent bundle $T^*\mathcal{N}_0$ over the moduli space $\mathcal{N}_0$ of flat $GSU(2)$ connections.

To quantize this system canonically, we have only to replace (i-times) the basic poisson brackets (3·11) by the commutation relations. If we use as wave functions the functionals $\Psi[\mathcal{A}_a]$ of the connection $\mathcal{A}_a = A_a{}^i J_i + \psi_a{}^A J_A$, the conjugate momenta become the functional derivatives:

$$\hat{\tilde{\pi}}^{ai}(x) \cdot \Psi[\mathcal{A}] = -\frac{\delta}{\delta A_a{}^i(x)} \Psi[\mathcal{A}] , \quad \hat{\tilde{\pi}}^{aA}(x) \cdot \Psi[\mathcal{A}] = \frac{1}{2} \frac{\delta}{\delta \psi_{aA}(x)} \Psi[\mathcal{A}] . \tag{3·20}$$

Next we solve the constraint equations. Gauss' law constraint

$$\hat{G}^i \cdot \Psi[\mathcal{A}] = \hat{L}^A \cdot \Psi[\mathcal{A}] = 0 \tag{3·21}$$

requires the wavefunctions to be invariant under the (identity-connected component of the) $GSU(2)$ gauge transformations. The remaining constraint

$$\hat{\Phi}^{ai} \cdot \Psi[\mathcal{A}] = \hat{\Phi}^{aA} \cdot \Psi[\mathcal{A}] = 0 \tag{3·22}$$

can easily be solved (at least formally).

For $g=0$, this constraint requires the wavefunctions to have support only on the flat $GSU(2)$ connections. The solutions to all the constraints are therefore provided formally by

$$\Psi[\mathcal{A}] = F[\mathcal{A}_a] \prod_{x \in M^{(3)}} \Big(\prod_{a,i} \delta(\tilde{\epsilon}^{abc} F^i_{bc}(x)) \prod_{a,A} \delta(\tilde{\epsilon}^{abc} D_b \psi_c{}^A(x)) \Big) , \tag{3·23}$$

where $F[\mathcal{A}_a]$ is an arbitrary $GSU(2)$-invariant functional of the connection $\mathcal{A}_a$. Due to the delta functions, $F[\mathcal{A}]$ reduces to the function on the moduli space $\mathcal{N}_0$ of flat $GSU(2)$ connections. Thus, naively, these solutions are considered to be "Fourier transforms" of the topological solutions found in Ref. 16).

For $g \neq 0$, we can rewrite Eq. (3·22) as

$$\Big(\frac{\delta}{\delta A_a{}^i} + \frac{3}{2g^2} \mathrm{STr}(J_i \tilde{\epsilon}^{abc} \mathcal{F}_{bc}) \Big) \cdot \Psi[\mathcal{A}_a] = 0 ,$$

$$\Big(\frac{\delta}{\delta \psi_{aA}} - \frac{3}{2g^2} \mathrm{STr}(J^A \tilde{\epsilon}^{abc} \mathcal{F}_{bc}) \Big) \cdot \Psi[\mathcal{A}_a] = 0 . \tag{3·24}$$

These equations have a unique solution,

$$\Psi[\mathcal{A}] = e^{-(3/2g^2) W_{CS}^{N=1}} , \tag{3·25}$$

where $W_{CS}^{N=1}$ is the Chern-Simons functional for the $GSU(2)$ connection $\mathcal{A}$:

$$W_{\rm CS}^{N=1}=\int_{M^{(3)}}{\rm STr}\Big(\mathcal{A}d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big)$$

$$=\int_{M^{(3)}}\Big(A^i dA^i+\frac{1}{3}\epsilon^{ijk}A^i\wedge A^j\wedge A^k-2\lambda g\psi^A\wedge D\psi_A\Big). \qquad (3\cdot26)$$

The solution (3·25) coincides with the $N=1$ Chern-Simons solution found in Refs. 17) and 18).

3.2. *Ashtekar's formalism for $N=1$ supergravity*

We are now in a position to discuss the relation of the $N=1$ Ashtekar formalism to the $GSU(2)$ BF theory. First we note that the action (3·4) is identical to the chiral action for $N=1$ supergravity[11] with the cosmological constant $\Lambda=g^2$ if we identify A^i, ψ^A, $\Sigma^{AB}=\Sigma^i(\sigma^i/2i)^{AB}$ and χ^A with the anti-self-dual part of the spin connection, the left-handed gravitino, the chiral two-form $e^A_{A'}\wedge e^{BA'}$ constructed from the vierbein $e^{AA'}=e_\mu{}^{AA'}dx^\mu$, and with the chiral two-form $e^A_{A'}\wedge\psi^{A'}$ constructed from the right-handed gravitino $\psi^{A'}$, respectively.*) As a consequence of this identification the components (Σ^i,χ^A) of the $\mathcal{B}$ field are subject to the algebraic constraints

$$\Sigma^{ABCD}\equiv\Sigma^{(AB}\wedge\Sigma^{CD)}=0\,,$$

$$\Xi^{ABC}\equiv\Sigma^{(AB}\wedge\chi^{C)}\,. \qquad (3\cdot27)$$

In order to obtain the action for Ashtekar's formalism we first solve the algebraic constraints (3·27) for the time components $(\Sigma^i_{ta},\chi^A_{ta})$ and then substitute the result into the canonical action (3·9). General solutions to Eq. (3·27) are given by

$$\Sigma^i_{ta}=-\frac{1}{2}\underset{\sim}{\epsilon}_{abc}\Big(-i\underset{\sim}{N}\frac{\epsilon^{ijk}}{2}\tilde{\pi}^{bj}\tilde{\pi}^{ck}+2N^b\tilde{\pi}^{ci}\Big),$$

$$\chi^A_{ta}=-\underset{\sim}{\epsilon}_{abc}\Big(-i\underset{\sim}{N}\tilde{\pi}_B{}^{bA}\tilde{\pi}^{cB}+N^b\tilde{\pi}^{cA}\Big)+\underset{\sim}{\epsilon}_{abc}\tilde{\pi}_B{}^{bA}\tilde{\pi}^{cBC}\underset{\sim}{M}_C\,. \qquad (3\cdot28)$$

By substituting this expression into Eq. (3·9), we find

$$-iI_{\rm Ash}^{N=1}=\int dt\int_{M^{(3)}}d^3x(\tilde{\pi}^{ai}\dot{A}_a{}^i+2\tilde{\pi}^{aA}\dot{\psi}_{aA}$$

$$+A_t^iG^i-2\psi_{tA}L^A+2\underset{\sim}{M}_AR^A+i\underset{\sim}{N}\mathcal{H}-2N^a\mathcal{H}_a) \qquad (3\cdot29)$$

with the new constraints

$$R^A=\frac{1}{2}\underset{\sim}{\epsilon}_{abc}\epsilon^{ijk}\tilde{\pi}^{bj}\tilde{\pi}^{ck}\Big(\frac{\sigma^i}{2i}\Big)^{AB}\Phi_B{}^a\,,$$

$$\mathcal{H}=\frac{1}{4}\underset{\sim}{\epsilon}_{abc}\epsilon^{ijk}\tilde{\pi}^{bj}\tilde{\pi}^{ck}\Phi^{ai}-2\underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{bi}\Big(\frac{\sigma^i}{2i}\Big)_{AB}\tilde{\pi}^{cB}\Phi^{aA}\,,$$

$$\mathcal{H}_a=\frac{1}{2}\underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{bi}\Phi^{ci}+\underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{bA}\Phi_A{}^c\,. \qquad (3\cdot30)$$

Physically, R^A generates right-supersymmetry transformations, $\mathcal{H}$ generates bubble-

*) Our action is in fact twice the action used in Refs. 17) and 18).

time evolutions, and $\mathcal{H}_a$ generates spatial diffeomorphisms. In passing, we note that among the $GSU(2)$ gauge transformations, the transformations generated by the G^i are reinterpreted as local Lorentz transformations for left-handed fields, and the transformations generated by L^A are regarded as left-supersymmetry transformations.

Let us now briefly consider the canonical quantization. Here we also use $\Psi[\mathcal{A}_a]$ as wavefunctions. Because the Gauss' law constraint remains intact, the wavefunctions must be invariant under the $GSU(2)$ gauge transformations. When we solve the new constraints $(R^A, \mathcal{H}, \mathcal{H}_a)$, we should note that these constraints are linear combinations of the constraints (Φ^{ai}, Φ^{aA}) in the BF theory, with the coefficients being polynomials in the momenta $(\tilde{\pi}^{ai}, \tilde{\pi}^{aA})$. As a consequence, if we take the ordering with the momenta to the left, the solutions (3·23) and (3·25) for the $GSU(2)$ BF theory are involved into the solution space for quantum $N=1$ supergravity in the Ashtekar form. These solutions are the topological solutions found in Refs. 16)~18).

Before ending this section we investigate how the symmetry of the theory is influenced by the algebraic constraints (3·27). For this purpose we look into the variation of the constraints $(\Sigma^{ABCD}, \Xi^{ABC})$ under the gauge transformations (in a broader sense). Because these constraints transform covariantly under the $GSU(2)$ gauge transformations (see the Appendix), the $GSU(2)$ gauge symmetry is preserved even after imposing the algebraic constraints. However, the Kalb-Ramond symmetry (3·6) in general breaks down because the variation of $(\Sigma^{ABCD}, \Xi^{ABC})$ does not vanish even after imposing all the constraints. More precisely, by computing the variation using Eq. (3·8) and equations of motion which are derived from the variation of the action (3·1) w.r.t. the connection $\mathcal{A}$, we find

$$\delta_\xi \Sigma^{ABCD} = -2D(\phi^{(AB} \wedge \Sigma^{CD)}) + 2\{\phi^{(AB} \wedge \chi^C + \Sigma^{(AB} \wedge \eta^C\} \wedge \phi^{D)},$$

$$\delta_\xi \Xi^{ABC} = -D(\phi^{(AB} \wedge \chi^{C)} + \Sigma^{(AB} \wedge \eta^{C)}) - 2\lambda g \phi^{(AB} \wedge \Sigma^{CD)} \phi_D . \tag{3·31}$$

In order words, the Kalb-Ramond symmetry survives if the parameter ξ is such that the variation (3·31) vanishes. A sufficient condition for not violating the Kalb-Ramond symmetry is provided by

$$\phi^{(AB} \wedge \Sigma^{CD)} = \phi^{(AB} \wedge \chi^{C)} + \Sigma^{(AB} \wedge \eta^{C)} = 0 . \tag{3·32}$$

If we assume the vierbein $e^{AA'}$ to be nondegenerate, this equation is completely solved by the superposition of the diffeomorphisms

$$\phi_\mu{}^i = v^\nu \Sigma^i_{\mu\nu} , \quad \eta_\mu{}^A = v^\nu \chi^A_{\mu\nu} , \tag{3·33}$$

and the right-supersymmetry transformations

$$\phi^i = 0 , \quad \Sigma^{(AB} \wedge \eta^{C)} = 0 . \tag{3·34}$$

Thus we have seen explicitly that the imposition of the algebraic constraints (3·27) breaks the Kalb-Ramond symmetry down to a symmetry under the diffeomorphisms and the right-supersymmetry transformations.

In the Lagrangian formalism, we impose the algebraic constraints by introducing the linear terms in the auxiliary fields ($\Psi_{ABCD} = \Psi_{(ABCD)}$, $\kappa_{ABC} = \kappa_{(ABC)}$):

$$-iI_{\text{aux}}^{N=1}=\int(-\Psi_{ABCD}\Sigma^{AB}\wedge\Sigma^{CD}-2\kappa_{ABC}\Sigma^{AB}\wedge\chi^{C})\,. \tag{3·35}$$

The transformation properties of the fields are somewhat modified, while the essential features remain valid. This is explained in the Appendix.

§ 4. $G^2SU(2)$ BF theory and Ashtekar's formulation for $N=2$ supergravity

In this section we demonstrate that $N=2$ supergravity can be cast into the form of the "constrained" BF theory with the gauge group being an appropriate graded version of $SU(2)$. Except for a few subtleties, the argument proceeds in almost the same manner as in the previous two cases. Thus we briefly explain the overview, focusing on the subtleties.

The relevant graded Lie algebra is provided by

$$\begin{aligned}
&[J_i, J_j]=\epsilon_{ijk}J_k\,, \quad [J_i, J_A{}^{(\alpha)}]=\left(\frac{\sigma^i}{2i}\right)_A{}^B J_B{}^{(\alpha)}\,, \quad [J_i, J]=0\,,\\
&[J, J_A{}^{(\alpha)}]=g(\tau^3)^\alpha{}_\beta J_A{}^{(\beta)}\,, \quad [J, J]=0\,,\\
&\{J_A{}^{(\alpha)}, J_B{}^{(\beta)}\}=-\epsilon^{\alpha\beta}\epsilon_{AB}J+4g(\tau^3)^{\alpha\beta}\left(\frac{\sigma^i}{2i}\right)_{AB}J_i\,,
\end{aligned} \tag{4·1}$$

where $\alpha, \beta, \cdots$ denotes the spinor indices for the internal $SU(2)$ symmetry existing in the $N=2$ supergravity with a vanishing cosmological constant $\Lambda\equiv-6g^2=0$. τ^3 is the third component of the Pauli matrices:

$$(\tau^3)^\alpha{}_\beta=(\tau^3)_\beta{}^\alpha\equiv(\tau^3)^\alpha_\beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{4·2}$$

In this paper we tentatively refer to this graded algebra (4·1) as $G^2SU(2)$.

Let us investigate the $G^2SU(2)$ BF theory. The action is

$$-iI_{\text{BF}}^{N=2}=\int \text{STr}(\mathcal{B}\wedge\mathcal{F}+g^2\mathcal{B}\wedge\mathcal{B})\,, \tag{4·3}$$

where $\mathcal{B}=\Sigma^iJ_i-(1/2g)(\tau^3)^\beta_\alpha\chi_\beta{}^A J_A{}^{(\alpha)}+(1/4g^2)BJ$ is a $G^2SU(2)$-valued two-form, and $\mathcal{F}=d\mathcal{A}+\mathcal{A}\wedge\mathcal{A}$ is the curvature two-form of a $G^2SU(2)$ connection $\mathcal{A}=A^iJ_i+\psi_\alpha{}^A J_A{}^{(\alpha)}+AJ$. STr used here is the unique $G^2SU(2)$-invariant bilinear form:

$$\begin{aligned}
&\text{STr}(J_iJ_j)=\delta_{ij}\,, \quad \text{STr}(J_A{}^{(\alpha)}J_B{}^{(\beta)})=4g\epsilon_{AB}(\tau^3)^{\alpha\beta}\,, \quad \text{STr}(JJ)=4g^2\,,\\
&\text{STr}(J_iJ_A{}^{(\alpha)})=\text{STr}(J_iJ)=\text{STr}(J_A{}^{(\alpha)}J)=0\,.
\end{aligned} \tag{4·4}$$

We can now rewrite the action (4·3) in terms of the component fields

$$\begin{aligned}
-iI_{\text{BF}}^{N=2}=\int\bigg(&\Sigma^i\wedge\left(F^i+2g\psi^{A\alpha}\wedge\psi_\beta{}^B(\tau^3)^\beta_\alpha\left(\frac{\sigma^i}{2i}\right)_{AB}\right)\\
&+2\chi_\alpha{}^A\wedge(D\psi_A{}^\alpha-g(\tau^3)^\alpha_\beta A\wedge\psi_A{}^\beta)+B\wedge\hat{F}
\end{aligned}$$

$$+g^2\Sigma^i\wedge\Sigma^i-g(\tau^3)^\beta_\alpha\chi_A{}^\alpha\wedge\chi_\beta{}^A+\frac{1}{4}B\wedge B\Big)\,, \tag{4·5}$$

where we have set $\hat{F}=dA-(1/2)\psi_\alpha{}^A\wedge\psi_A{}^\alpha$. This action obviously possesses the symmetry under the $G^2SU(2)$ gauge transformations

$$\delta_\rho\mathcal{A}=-\mathcal{D}\rho\equiv-d\rho-[\mathcal{A},\rho]\,,$$
$$\delta_\rho\mathcal{B}=[\rho,\mathcal{B}] \tag{4·6}$$

with $\rho=\theta^iJ_i+\epsilon_\alpha{}^AJ_A{}^{(\alpha)}+\lambda J$ being a $G^2SU(2)$-valued scalar, and the Kalb-Ramond symmetry

$$\delta_\xi\mathcal{A}=2g^2\xi\,,$$
$$\delta_\xi\mathcal{B}=-\mathcal{D}\xi\equiv-d\xi-\mathcal{A}\wedge\xi-\xi\wedge\mathcal{A} \tag{4·7}$$

with $\xi=\phi^iJ^i-(1/2g)(\tau^3)^\beta_\alpha\eta_\beta{}^AJ_A{}^{(\alpha)}+(1/4g^2)\kappa J$ being a $G^2SU(2)$-valued one-form. These transformations written in terms of component fields are as follows. The $G^2SU(2)$ gauge transformations are

$$\delta_\rho A^i=-D\theta^i+4g\left(\frac{\sigma^i}{2i}\right)_{AB}(\tau^3)^{\alpha\beta}\psi_\alpha{}^A\epsilon_\beta{}^B\,,$$
$$\delta_\rho\psi_\alpha{}^A=\theta^i\left(\frac{\sigma^i}{2i}\right)^A{}_B\psi_\alpha{}^B-D\epsilon_\alpha{}^A-g(\tau^3)^\beta_\alpha\epsilon_\beta{}^AA+g\lambda(\tau^3)^\beta_\alpha\psi_\beta{}^A\,,$$
$$\delta_\rho A=-d\lambda-\epsilon_\alpha{}^A\psi_A{}^\alpha\,,$$
$$\delta_\rho\Sigma^i=\epsilon^{ijk}\theta^j\Sigma^k+2\epsilon_B{}^\alpha\left(\frac{\sigma^i}{2i}\right)^B{}_C\chi_\alpha{}^C\,,$$
$$\delta_\rho\chi_\alpha{}^A=\theta^i\left(\frac{\sigma^i}{2i}\right)^A{}_B\chi_\alpha{}^B+2g(\tau^3)^\beta_\alpha\epsilon_\beta{}^B\left(\frac{\sigma^i}{2i}\right)^A{}_B\Sigma^i+\frac{1}{2}\epsilon_\alpha{}^AB+g\lambda(\tau^3)^\beta_\alpha\chi_\beta{}^A\,,$$
$$\delta_\rho B=2g\epsilon_A{}^\alpha(\tau^3)^\beta_\alpha\chi_\beta{}^A\,. \tag{4·8}$$

In the $N=2$ supergravity, transformations generated by θ^i, by $\epsilon_\alpha{}^A$ and by λ are respectively interpreted as local Lorentz transformations, left-SUSY transformations, and $U(1)$ gauge transformations. The Kalb-Ramons transformations for the component fields are given by

$$\delta_\xi A^i=2g^2\phi^i\,,$$
$$\delta_\xi\psi_\alpha{}^A=-g(\tau^3)^\beta_\alpha\eta_\beta{}^A\,,$$
$$\delta_\xi A=\frac{1}{2}\kappa\,,$$
$$\delta_\xi\Sigma^i=-D\phi^i-2\left(\frac{\sigma^i}{2i}\right)_{AB}\psi_\alpha{}^A\wedge\eta^{C\alpha}\,,$$
$$\delta_\xi\chi_\alpha{}^A=2g(\tau^3)^\beta_\alpha\left(\frac{\sigma^i}{2i}\right)^A{}_B\phi^i\wedge\psi_\beta{}^B-D\eta_\alpha{}^A-g(\tau^3)^\beta_\alpha A\wedge\eta_\beta{}^A-\frac{1}{2}\psi_\alpha{}^A\wedge\kappa\,,$$
$$\delta_\xi B=-2g(\tau^3)^\beta_\alpha\psi_A{}^\alpha\wedge\eta_\beta{}^A-d\kappa\,. \tag{4·9}$$

As we will see shortly, these transformations are closely related to the diffeomorphisms and the right-SUSY transformations in the $N=2$ supergravity.

Let us now briefly look into the canonical quantization. In the canonical formalism, the action (4·3) is rewritten as follows:

$$-iI_{\mathrm{BF}}^{N=2}=\int dt\int_{M^{(3)}} d^3x\,\mathrm{STr}[\tilde{\Pi}^a\mathcal{A}_a+\mathcal{A}_t\boldsymbol{G}+\mathcal{B}_{ta}\boldsymbol{\Phi}^a]\,, \tag{4·10}$$

where we have set $\tilde{\Pi}^a=(1/2)\tilde{\epsilon}^{abc}\mathcal{B}_{bc}=\tilde{\pi}^{ai}J_i-(1/2g)(\tau^3)_\alpha^\beta\tilde{\pi}_\beta{}^{A\alpha}J_A{}^{(\alpha)}+(1/4g^2)\tilde{\pi}^aJ$. In terms of the component fields, this canonical action becomes

$$\begin{aligned}-iI_{\mathrm{BF}}^{N=2}=\int dt\int_{M^{(3)}} d^3x\Big(&\tilde{\pi}^{ai}\dot{A}_a{}^i+2\tilde{\pi}_\alpha{}^{aA}\dot{\psi}_{Aa}^\alpha+\tilde{\pi}^a\dot{A}_a\\ &+A_t{}^iG^i-2\psi_{At}^\alpha L_\alpha{}^A+A_tG\\ &+\Sigma_{ta}^i\Phi^{ai}+2\chi_{ata}^A\Phi_A{}^{\alpha a}+B_{ta}\Phi^a\Big)\,.\end{aligned} \tag{4·11}$$

As in the previous cases, this system has two types of first class constraints. Gauss' law constraint

$$\begin{aligned}\boldsymbol{G}&=\mathcal{D}_a\tilde{\Pi}^a\\ &=G^iJ_i-\frac{1}{2g}(\tau^3)_\alpha^\beta L_\beta{}^AJ_A{}^{(\alpha)}+\frac{1}{4g^2}GJ\end{aligned} \tag{4·12}$$

generates the $G^2SU(2)$ transformations (4·8). And the remaining constraint

$$\begin{aligned}\boldsymbol{\Phi}^a&=\frac{1}{2}\tilde{\epsilon}^{abc}\mathcal{F}_{bc}+2g^2\tilde{\Pi}^a\\ &=\Phi^{ai}J_i+\Phi_\alpha{}^{aA}J_A{}^\alpha+\Phi^aJ\end{aligned} \tag{4·13}$$

generates the Kalb-Ramond symmetry (4·9). The explicit form of Gauss' law constraint can be seen in Refs. 15) and 17). Expressions for the remaining constraints are also seen implicitly in these references.

Canonical quantization of this theory can be handled in a manner analogous to the $GSU(2)$ case (except for $g=0$). We will use as wavefunctions the functionals $\Psi[\mathcal{A}_a]$ of the $G^2SU(2)$ connection $\mathcal{A}_adx^a$. Gauss' law constraint tells us that $\Psi[\mathcal{A}_a]$ should be invariant under (the identity-connected component of) the $G^2SU(2)$ gauge transformations. For $g\neq0$, the remaining constraint can be solved similarly to the way it was solved in the $GSU(2)$ case. In this case we have the unique solution

$$\begin{aligned}&\Psi[\mathcal{A}_a]=e^{(1/4g^2)W_{\mathrm{CS}}^{N=2}}\,,\\ &W_{\mathrm{CS}}^{N=2}=\int_{M^{(3)}}\mathrm{STr}\Big(\mathcal{A}d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big)\\ &\quad=\int_{M^{(3)}}\Big[A^idA^i+\frac{1}{3}\epsilon^{ijk}A^i\wedge A^j\wedge A^k\\ &\qquad-4g(\tau^3)^{\alpha\beta}\psi_{\alpha A}\wedge(D\psi_\beta{}^A+g(\tau^3)_\beta^\gamma A\wedge\psi_\gamma{}^A)+4g^2AdA\Big]\,.\end{aligned} \tag{4·14}$$

This coincides with the $N=2$ super-extended version of the Chern-Simons solution found in Ref. 17).

For $g=0$, a special consideration is needed, because in the present case a remnant of the "cosmological term" $g^2\mathcal{B}\wedge\mathcal{B}$ exists even in the limit $g\to 0$. Particularly, the reduced phase space loses the cotangent bundle structure, unlike in the case of $SU(2)$ or $GSU(2)$. We can nevertheless construct solutions to the quantum constraints, at least formally. As in the $g\neq 0$ case, Gauss' law constraint merely requires the wavefunctions to be $G^2SU(2)$ gauge invariant. The remaining constraints in the $g=0$ case are written as

$$\hat{\Phi}^{ai}\cdot\Psi[\mathcal{A}]=\frac{1}{2}\tilde{\epsilon}^{abc}F^i_{bc}\cdot\Psi[\mathcal{A}]=0\,,$$

$$\hat{\Phi}_A{}^{a\alpha}\cdot\Psi[\mathcal{A}]=\tilde{\epsilon}^{abc}D_b\psi^\alpha_{Ac}\cdot\Psi[\mathcal{A}]=0\,,$$

$$\hat{\Phi}^a\cdot\Psi[\mathcal{A}]=\left[\frac{1}{2}\tilde{\epsilon}^{abc}(F_{bc}-\psi^A_{\alpha b}\psi^\alpha_{Ac})-\frac{1}{2}\frac{\delta}{\delta A_a}\right]\Psi[\mathcal{A}]=0\,, \tag{4·15}$$

where $F_{bc}=2\partial_{[b}A_{c]}$ is the field strength of the $U(1)$ connection A. Formal solutions to these equations are given by

$$\Psi[\mathcal{A}]=e^{W_{U(1)}}F[A^i,\psi_A{}^\alpha]\prod_{x\in M^{(3)}}(\prod_{a,i}\delta(\tilde{\epsilon}^{abc}F^i_{bc}(x))\prod_{a,A,\alpha}\delta(\tilde{\epsilon}^{abc}D_b\psi^\alpha_{Ac}(x)))\,, \tag{4·16}$$

where $F[A^i,\psi_A{}^\alpha]$ is a $G^2SU(2)$ gauge invariant function of $(A_a{}^i,\psi^\alpha_{Aa})$, and

$$W_{U(1)}\equiv\int_{M^{(3)}}(AdA-A\wedge\psi_\alpha{}^A\wedge\psi_A{}^\alpha)\,.$$

As it is, however, Eq. (4·16) is not $G^2SU(2)$ gauge invariant. There is no problem in the delta function part, because the curvatures $(F^i, D\psi_A{}^\alpha)$ transform covariantly under the $G^2SU(2)$ gauge transformations, and because their gauge transformations do not involve the $U(1)$ part $\hat{F}=dA-(1/2)\psi_\alpha{}^A\wedge\psi_A{}^\alpha$. The functional $W_{U(1)}$ is, however, not invariant under the left-SUSY transformations. After a somewhat lengthy calculation, we see that $W_{U(1)}$ transforms as*)

$$e^{-i\hat{L}(\epsilon)}W_{U(1)}e^{i\hat{L}(\epsilon)}-W_{U(1)}$$
$$=\int_{M^{(3)}}\Big[\epsilon_\alpha{}^A\psi_A{}^\alpha\wedge\psi_\beta{}^B\wedge\psi_B{}^\beta-\frac{1}{2}\epsilon_A{}^\alpha D\epsilon_\alpha{}^A\wedge\psi_\beta{}^B\wedge\psi_B{}^\beta-\epsilon_\alpha{}^A\psi_A{}^\alpha\wedge d(\epsilon_\beta{}^B\psi_B{}^\beta)$$
$$+D\epsilon_\alpha{}^A\wedge D\epsilon_A{}^\alpha\wedge\epsilon_\beta{}^B\psi_B{}^\beta-\frac{1}{4}\epsilon_A{}^\alpha D\epsilon_\alpha{}^A\wedge d(\epsilon_B{}^\beta D\epsilon_\beta{}^B)\Big]\,, \tag{4·17}$$

where $\hat{L}(\epsilon)\equiv -2i\int_{M^{(3)}}d^3x\epsilon_\alpha{}^A\hat{L}_A{}^\alpha$ is the generator of the left-SUSY transformations. We should note that the $U(1)$ connection A does not appear anywhere on the r.h.s of the above expression. The wavefunction (4·16) with $W_{U(1)}$ replaced by $e^{-i\hat{L}(\epsilon)}W_{U(1)}e^{i\hat{L}(\epsilon)}$ therefore remains as the solution of Eq. (4·15). Now we can give formal solutions to all the constaint equations in the $g=0$ case:

$$\Psi[\mathcal{A}]=F[A^i,\psi_A{}^\alpha]\prod_{x\in M^{(3)}}(\prod_{a,i}\delta(\tilde{\epsilon}^{abc}F^i_{bc}(x))\prod_{a,A,\alpha}\delta(\tilde{\epsilon}^{abc}D_b\psi^\alpha_{Ac}(x)))$$

*) We have assumed that $F^i_{bc}=D_{[b}\psi^\alpha_{c]A}=0$ hold.

$$\times \int [d\epsilon_\alpha{}^A] \exp(e^{-i\hat{L}(\epsilon)} W_{U(1)} e^{i\hat{L}(\epsilon)}) , \tag{4·18}$$

where $[d\epsilon_\alpha{}^A]$ denotes an $SU(2)$ invariant measure.

In passing, let us note that $F[A^i, \psi_A{}^\alpha]$ can be interpreted as the gauge invariant functional of the "truncated" connection $\hat{\mathcal{A}} \equiv A^i J_i + \psi_\alpha{}^A \hat{J}_A{}^{(\alpha)}$, where $(J_i, \hat{J}_A{}^{(\alpha)})$ are the generators of the following truncated algebra:

$$[J_i, J_j] = \epsilon_{ijk} J_k , \quad [J_i, \hat{J}_A{}^{(\alpha)}] = \left(\frac{\sigma^i}{2i}\right)_A{}^B \hat{J}_B{}^{(\alpha)} , \quad \{\hat{J}_A{}^{(\alpha)}, \hat{J}_B{}^{(\beta)}\} = 0 . \tag{4·19}$$

This is possible because the $U(1)$ part J in the $G^2SU(2)$ algebra (4·1) with $g=0$ almost decouples from the remaining generators $(J_i, J_A{}^{(\alpha)})$.

The relation to $N=2$ supergravity is not so simple. This is because the chiral action of $N=2$ supergravity[15),17)]

$$\begin{aligned} -iI_{\rm Ash}^{N=2} = \int \Big(& \Sigma^i \wedge \Big(F^i + 2g(\tau^3)_\alpha^\beta \Big(\frac{\sigma^i}{2i}\Big)_{AB} \psi^{\alpha A} \wedge \psi_\beta{}^B \Big) + 2\chi_\alpha{}^A \wedge (D\psi_A{}^\alpha - g(\tau^3)_\beta^\alpha A \wedge \psi_A{}^\beta) \\ & + g^2 \Sigma^i \wedge \Sigma^i - g(\tau^3)_\alpha^\beta \chi_A{}^\alpha \wedge \chi_\beta{}^A - \Psi_{ABCD} \Sigma^{AB} \wedge \Sigma^{CD} - 2\kappa_{ABC}^\alpha \Sigma^{AB} \wedge \chi_\alpha{}^C \\ & - \hat{F} \wedge \hat{F} + \varphi^i \hat{F} \wedge \Sigma^i - \frac{1}{4} \varphi^i \varphi^j \Sigma^i \wedge \Sigma^j + \varphi^i \Big(\frac{\sigma^i}{2i}\Big)_{AB} \chi_\alpha{}^A \wedge \chi^{B\alpha} \Big) , \end{aligned} \tag{4·20}$$

involves the terms which are (at most) quadratic in the auxiliary field φ^i:

$$-iL_{U(1)} \equiv -\hat{F} \wedge \hat{F} + \varphi^i \hat{F} \wedge \Sigma^i - \frac{1}{4} \varphi^i \varphi^j \Sigma^i \wedge \Sigma^j + \varphi^i \left(\frac{\sigma^i}{2i}\right)_{AB} \chi_\alpha{}^A \wedge \chi^{B\alpha} . \tag{4·21}$$

First we translate this quadratic part $-iL_{U(1)}$ into terms which are at most linear in the auxiliary fields as follows. We know how to deal with auxiliary fields which appear in the action at most quadratically: we have only to solve equations of motion obtained from the variation w.r.t the auxiliary fields and to substitute the result into the action. In the present case the desired equations of motion are

$$\Sigma^i \wedge B = 2\left(\frac{\sigma^i}{2i}\right)_{AB} \chi_\alpha{}^A \wedge \chi^{B\alpha} , \tag{4·22}$$

where we have set $B = \varphi^i \Sigma^i - 2\hat{F}$. Using this, Eq. (4·21) is rewritten as

$$-iL_{U(1)} = -\frac{1}{4} B \wedge B + (B + 2\hat{F}) \wedge \frac{1}{2} B .$$

Arranging this expression neatly and taking account of the algebraic constraint (4·22), the quadratic part (4·21) turns out to be equivalent to the following expression, which is at most linear in the new auxiliary field φ'_{AB}:*)

$$-iL_{U(1)} = B \wedge \hat{F} + \frac{1}{4} B \wedge B - \varphi'_{AB} (\Sigma^{AB} \wedge B - \chi_\alpha{}^A \wedge \chi^{B\alpha}) . \tag{4·23}$$

Substituting this into the chiral $N=2$ action (4·20) and comparing the result with the $G^2SU(2)$ BF action (4·5), we find

*) φ'_{AB} can be identified with $\varphi_{AB} = \varphi^i (\sigma^i/2i)_{AB}$ if B is integrated out.

$$-iI_{\rm Ash}^{N=2}=-iI_{\rm BF}^{N=2}-iI_{\rm aux}^{N=2}$$

$$-iI_{\rm aux}^{N=2}=\int[-\Psi_{ABCD}\Sigma^{AB}\wedge\Sigma^{CD}-2\kappa^{\alpha}_{ABC}\Sigma^{AB}\wedge\chi_{\alpha}{}^{C}-\varphi'_{AB}(\Sigma^{AB}\wedge B-\chi_{\alpha}{}^{A}\wedge\chi^{B\alpha})]\,. \tag{4·24}$$

Thus we have established the relation between $N=2$ supergravity and the $G^2SU(2)$ BF theory. Namely, $N=2$ supergravity in Ashtekar's form is regarded as the $G^2SU(2)$ BF theory (4·3), with the $\mathcal{B}$ fields being subject to the algebraic constraints

$$\Sigma^{ABCD}\equiv\Sigma^{(AB}\wedge\Sigma^{CD)}=0\,,$$

$$\Xi_{\alpha}{}^{ABC}\equiv\Sigma^{(AB}\wedge\chi_{\alpha}{}^{C)}=0\,,$$

$$\boldsymbol{B}^{AB}\equiv\Sigma^{AB}\wedge B-\chi_{\alpha}{}^{A}\wedge\chi^{B\alpha}=0\,. \tag{4·25}$$

As in the $N=1$ case, Ashtekar's formalism for $N=2$ supergravity is derived by solving these algebraic constraints for the time components $(\Sigma^{i}_{ta},\ \chi^{A}_{\alpha ta},\ B_{ta})$ and by substituting the solution into the canonical BF action (4·11). The Gauss' law constraint is inherited as it is from the BF theory. In addition we have three types of constraints: the Hamiltonian constraint $\mathcal{H}$, the diffeomorphism constraint $\mathcal{H}_a$, and the constraint $R_{\alpha}{}^{A}$ which generates right-SUSY transformations. These are given by linear combinations of the constraints $(\Phi^{ai},\ \Phi_{A}{}^{a\alpha},\ \Phi^{a})$ in the $G^2SU(2)$ BF theory:

$$\mathcal{H}^{\mathcal{J}}=C^{\mathcal{J}}_{ai}(\tilde{\Pi}^{a})\Phi^{ai}+C^{\mathcal{J}A}_{a\alpha}(\tilde{\Pi}^{a})\Phi_{A}{}^{a\alpha}+C_{a}{}^{\mathcal{J}}(\tilde{\Pi}^{a})\Phi^{a}\,, \tag{4·26}$$

where we have set $(\mathcal{H}^{\mathcal{J}})\equiv(\mathcal{H},\ \mathcal{H}_a,\ R_{\alpha}{}^{A})$. The crucial point here is that the coefficients depend only on the momenta $\tilde{\Pi}^{a}$ and not on the connections $\mathcal{A}_a$. The solutions (4·14) and (4·18) to the quantum $G^2SU(2)$ BF theory are thus included in the solution space of canonically quantized $N=2$ supergravity, provided that we take the ordering with the momenta to the left.

Similar to the $N=1$ case, the Kalb-Ramond symmetry (4·9) in general breaks down owing to the algebraic constraint (4·25). By an argument parallel to that in the previous section we can find a sufficient condition for the Kalb-Ramond symmetry to preserve the algebraic constraints

$$\phi^{(AB}\wedge\Sigma^{BC)}=0\,,$$

$$\phi^{(AB}\wedge\chi_{\alpha}{}^{C)}+\Sigma^{(AB}\wedge\eta_{\alpha}{}^{C)}=0\,,$$

$$\phi^{AB}\wedge B+\Sigma^{AB}\wedge\kappa=2\eta_{\alpha}{}^{(A}\wedge\chi^{B)\alpha}\,. \tag{4·27}$$

Assuming that the vierbein $e^{AA'}$ to be nondegenerate, these equations are completely solved by the superposition of the diffeomorphisms

$$\phi_{\mu}{}^{i}=v^{\nu}\Sigma^{i}_{\mu\nu}\,,\quad \eta^{A}_{\alpha\mu}=v^{\nu}\chi^{A}_{\alpha\mu\nu}\,,\quad \kappa_{\mu}=v^{\nu}B_{\mu\nu}\,, \tag{4·28}$$

and the right-SUSY transformations

$$\phi^{i}=0\,,\quad \Sigma^{(AB}\wedge\eta_{\alpha}{}^{C)}=0\,,\quad \Sigma^{AB}\wedge\kappa=2\eta_{\alpha}{}^{(A}\wedge\chi^{B)\alpha}\,. \tag{4·29}$$

§ 5. Discussion

In this paper we have shown explicitly that $N=1$ and $N=2$ supergravities in Ashtekar's form can be cast into the form of BF theories with the two-form fields subject to the algebraic constraints. Once we have established these relations, it is expected that considerable progress will be made on the canonical quantum gravity both technically and conceptually.

For example, we may use the technique developed in the BF theory[10] at least when we investigate the topological sector of the canonical quantum gravity. With regard to pure gravity, some works of this kind can be seen in Refs. 13), 23) and 24). The results in this paper suggest that we can exploit similar methods also for studying $N=1, 2$ supergravities. Because the BF theory resembles the Chern-Simons gauge theory,[25] the methods for studying $(2+1)$-dimensional Einstein gravity in the Chern-Simons form[26),27] may be applied. It is of particular interest to investigate the physical significance of the topological solutions. While geometrical interpretation of the Chern-Simons solutions has been studied in considerably detail,[9),18),28] we do not know any works on the geometrical interpretation of the topological solutions in the case where the cosmological constant vanishes. Naively, we can expect that a topological solution in pure gravity corresponds semiclassically to a family of flat spacetimes. This is because the topological solution has support only on the flat anti-self-dual connections and because classical imposition of the reality conditions indicates that the self-dual connection should also be flat in the semiclassical region.[29] In supergravities, however, topological solutions do not always correspond to flat spacetimes even semiclassically because of the presence of nontrivial gravitino modes. It is interesting to investigate how these gravitino modes influence the spacetime geometry.[30]

Recently an attempt appeared to extend the loop representation[6] to $N=1$ supergravity.[31] As we have shown that $N=2$ supergravity is described by the $G^2SU(2)$ connection, the loop representation may be extended also to $N=2$ supergravity.

There have been several attempts to interpret Einstein gravity as an "unbroken phase" of some topological field theories.[32),33] We may extend these ideas to supergravities. Probably this deserves study because the existence of the supersymmetry is believed by many researchers, and thus supergravities seem to be more realistic than pure gravity.

$N=2$ supergravity is of interest in its own right because the twisted version of $N=2$ supergravity gives rise to a topological gravity.[34] Ashtekar's formalism can be applied also to this twisted $N=2$ supergravity.[35] To see whether twisted $N=2$ supergravity is related to a BF theory or not is left to future investigation.

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Appendix A

In this appendix we look into the symmetry of the $N=1$ Ashtekar's formalism in the Lagrangian form. The relevant action is

$$-iI_{\mathrm{Ash}}^{N=1}=-iI_{\mathrm{BF}}^{N=1}-iI_{\mathrm{aux}}^{N=1}\,, \tag{A·1}$$

where $-iI_{\mathrm{BF}}^{N=1}$ is the $GSU(2)$ BF action (3·1) and $-iI_{\mathrm{aux}}^{N=1}$ is the linear term (3·35) in the auxiliary fields (Ψ_{ABCD}, κ_{ABC}). When we discuss the symmetry of the action, we cannot use equations of motion. This is because the equations of motion are nothing but the condition for the action to be stationary under any variations of the fields. Thus a careful consideration is necessary.

In order to make $I_{\mathrm{Ash}}^{N=1}$ invariant under the $GSU(2)$ gauge transformations, we have only to consider (Ψ_{ABCD}, κ_{ABC}) to be covariant under these transformations. For $SU(2)$ transformations, it is obvious that (Ψ_{ABCD}, κ_{ABC}) should transform as is suggested by their spinor indices. Because the algebraic constraints transform under the left-SUSY transformations as

$$\begin{aligned}&\delta_\epsilon\Sigma^{ABCD}=-2\epsilon^{(A}\Xi^{BCD)}\,,\\&\delta_\epsilon\Xi^{ABC}=-\lambda g\Sigma^{ABCD}\epsilon_D\,,\end{aligned} \tag{A·2}$$

the auxiliary fields are required to transform as follows:

$$\begin{aligned}&\delta_\epsilon\Psi_{ABCD}=2\lambda g\kappa_{(ABC}\epsilon_{D)}\,,\\&\delta_\epsilon\kappa_{ABC}=\Psi_{ABCD}\epsilon^D\,.\end{aligned} \tag{A·3}$$

Next we consider the Kalb-Ramond symmetry. The transformations of the algebraic constraints under off-shell are

$$\begin{aligned}\delta_\phi\Sigma^{ABCD}&=-2\phi^{(AB}\wedge\{D\Sigma^{CD)}-\psi^C\wedge\chi^{D)}\}+(\text{terms appeared in Eq. (3·31)})\,,\\\delta_\phi\Xi^{ABC}&=-\phi^{(AB}\wedge\{D\chi^{C)}-\lambda g\Sigma^{C)D}\wedge\psi_D\}+\{D\Sigma^{(AB}-\psi^{(A}\wedge\chi^{B}\}\wedge\eta^{C)}\\&\qquad+(\text{terms appeared in Eq. (3·31)})\,.\end{aligned} \tag{A·4}$$

We should be aware that the expressions in the braces are the equations of motion obtained from the variation of the action (A·1) w.r.t the connection $\mathcal{A}$. This implies that, under the condition (3·32), we can render the action $-iI_{\mathrm{Ash}}^{N=1}$ invariant by adding some extra terms to the transformation of the connection. Adding these extra terms to the original transformations (3·8), we find the total transformation of the connection

$$\begin{aligned}&\delta_\phi A^i=-\frac{g^2}{3}\phi^i-2\left(\frac{\sigma^i}{2i}\right)^{AB}(\Psi_{ABCD}\phi^{CD}+\kappa_{ABC}\eta^C)\,,\\&\delta_\phi\psi^A=\frac{g}{3\lambda}\eta^A+\kappa^{ABC}\phi_{BC}\,.\end{aligned} \tag{A·5}$$

If we set $\phi^i=0$, this exactly coincides with the right-SUSY transformation in Refs. 17) and 18). We can also show that, similar to the cases of BF theories, the transformation (A·5) with the parameter $\xi_\mu = v^\nu \mathscr{B}_{\mu\nu}$ yields the diffeomorphism generated by $v^\mu(\partial/\partial x^\mu)$ under on-shell.

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