Chapter 1

Black Hole Perturbation

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In this chapter, we present analytic calculations of gravitational waves from a particle orbiting a black hole. We first review the Teukolsky formalism for dealing with the gravitational perturbation of a black hole. Then we develop a systematic method to calculate higher order post-Newtonian corrections to the gravitational waves emitted by an orbiting particle. As applications of this method, we consider orbits that are nearly circular, including exactly circular ones, slightly eccentric ones and slightly inclined orbits off the equatorial plane of a Kerr black hole and give the energy flux and angular momentum flux formulas at infinity with higher order post-Newtonian corrections. Using a different method that makes use of an analytic series representation of the solution of the Teukolsky equation, we also give a post-Newtonian expanded formula for the energy flux absorbed by a Kerr black hole for a circular orbit.

§1. Introduction

In this chapter, we review recent progress in the analytic calculations of gravitational waves from a particle orbiting a black hole using a systematic post-Newtonian expansion method. There have been substantial activities in this field recently and there is a diversity of literature. Here we are mostly concerned with the actual calculations of the gravitational waves from an orbiting particle and we intend to make this chapter as self-contained as possible. We do not, however, discuss much about implications of the results to actual astrophysical situations.

In the black hole perturbation approach, one considers gravitational waves from a particle of mass μ orbiting a black hole of mass M assuming $\mu \ll M$. Although this method is restricted to the case when $\mu \ll M$, one can calculate very high order post-Newtonian corrections to gravitational waves using a relatively simple algorithm in contrast with the standard post-Newtonian analysis. This is because the fully relativistic effect of the spacetime curvature is naturally taken into account in the basic perturbation equation. We can also calculate numerically the gravitational waves without assuming the slow motion of its source. Then, we can easily investigate the convergence of the post-Newtonian expansion by comparing the result of the post-Newtonian approximation with the fully relativistic one. In this sense, the black hole perturbation method gives a very important test of the post-Newtonian expansion. Further, since the effect of the spacetime curvature is naturally taken into account, we can easily investigate interesting relativistic effects such as tails of gravitational waves.

We consider the post-Newtonian wave forms and luminosity which are expanded by v/c, where v is the order of the orbital velocity. The lowest order of the gravitational waves are given by the Newtonian quadrupole formula. We call the post-Newtonian formulas for the wave forms and luminosity which contain terms up to $O((v/c)^n)$ beyond the Newtonian quadrupole formula as the $n/2$PN formulas.

Let us first briefly give a historical review. The gravitational perturbation equation of a black hole using the Newman-Penrose formalism [1] was derived by Bardeen and Press [2] for the Schwarzschild black hole, and by Teukolsky [3] for the Kerr black hole. By using these equations, many numerical calculations of gravitational waves induced by the presence of a test particle have been done. We do not list up all such works. Here, we only refer to three articles; Breuer, [4] Chandrasekhar, [5] and Nakamura, Oohara and Kojima. [6]

On the other hand, analytic calculations of gravitational waves produced by the motion of a test particle have not been developed very much until recently. This direction of research was first done by Gal'tsov, Matiukhin and Petukhov [7] in which they considered a case when a particle moves a slightly eccentric orbit around a Schwarzschild black hole, and calculated the gravitational waves up to 1PN order. Then, Poisson [8] considered a case of circular orbit around a Schwarzschild black hole and calculated the wave forms and luminosity to 1.5PN order at which the tail effect appears. Cutler, Finn, Poisson and Sussman [9] also worked on the same problem numerically by using the least square fitting method, and obtained a formula for the luminosity to 2.5PN order. Subsequently, a highly accurate numerical calculation was carried out by Tagoshi and Nakamura. [10] They obtained the formulas for the luminosity to 4PN order numerically by using the least square fitting method. They found the $\log v$ terms in the luminosity formula at 3PN and 4PN orders. They showed that, although the convergence of the post-Newtonian expansion is slow, the luminosity formula which is accurate to 3.5PN order will be good enough to represent the orbital phase evolution of coalescing compact binaries accurately. After that, Sasaki [11] found an analytic method and obtained the formulas which are needed to calculate the gravitational waves to 4PN order. Then, Tagoshi and Sasaki [12] obtained the gravitational wave forms and luminosity to 4PN order analytically, and confirmed the results of Tagoshi and Nakamura. These calculations were extended to 5.5PN order by Tanaka, Tagoshi and Sasaki. [13]

In the case of orbits around a Kerr black hole, Poisson calculated the 1.5PN order corrections to the wave forms and luminosity due to the rotation of the black hole and showed that the result agrees with the standard post-Newtonian effect due to spin-orbit coupling. [14] Then, Shibata, Sasaki, Tagoshi and Tanaka [15] calculated the luminosity to 2.5PN order. They calculated the luminosity from a particle in circular orbit with small inclination from the equatorial plane. They used the Sasaki-Nakamura equation as well as the Teukolsky equation. This analysis was extended to 4PN order by Tagoshi, Shibata, Tanaka and Sasaki [16] in which the orbit of the test particle was restricted to circular ones on the equatorial plane. The analysis in the case of slightly eccentric orbit on the equatorial plane was also done by Tagoshi [17] to 2.5PN order.

Tanaka, Mino, Sasaki and Shibata[18] considered the case when a spinning particle moves a circular orbit near the equatorial plane around a Kerr black hole, and derived the luminosity formula to 2.5PN order including the linear order effect of the particle's spin. They used the equations of motion of Papapetrou's[19] and the energy momentum tensor of the spinning particle given by Dixon.[20]

The absorption of gravitational waves into the black hole horizon, appearing at 4PN order in the Schwarzschild case, was calculated by Poisson and Sasaki in the case when a test particle is in a circular orbit.[21] The black hole absorption in the case of rotating black hole appears at 2.5PN order.[22] Recently a new analytic method to solve the homogeneous Teukolsky equation was found by Mano, Suzuki, and Takasugi.[23] Using this method, the black hole absorption in the case of rotating black hole was calculated by Tagoshi, Mano and Takasugi[24] to 6.5PN order beyond the quadrupole formula.

If gravity is not described by the Einstein theory but by the Brans-Dicke theory, there will appear scalar type gravitational waves as well as transverse-traceless gravitational waves. Such scalar type gravitational waves were calculated by Ohashi, Tagoshi and Sasaki[25] in a case when a compact star is in a circular orbit on the equatorial plane around a Kerr black hole.

The organization of this chapter is as follows. We review the Teukolsky formalism for the black hole perturbation in §2 and formulate a post-Newtonian expansion method of the Teukolsky equation in §3. Then we turn to the evaluation of gravitational waves by an orbiting particle in the rest of sections.

First we consider circular orbits. In §4, we calculate the gravitational wave luminosity from a test particle in circular orbit around a Schwarzschild black hole to 5.5PN order, based on Tanaka, Tagoshi and Sasaki.[13] This is the highest post-Newtonian order achieved so far. Based on this result, we investigate the convergence property of the post-Newtonian expansion in §5. In §6, we consider circular orbits on the equatorial plane around a Kerr black hole and calculate the luminosity to 4PN order, based on Tagoshi, Shibata, Tanaka and Sasaki.[16] We find the luminosity contains the terms which describe the effect of not only spin-orbit coupling but also the effect of higher multipole moments of the Kerr black hole.

Next we consider slightly noncircular orbits. In §7, we calculate the $O(e^2)$ corrections to the 4PN energy and angular momentum flux formulas in the case of a slightly eccentric orbit around a Schwarzschild black hole, where e is the eccentricity. In §8, we consider a slightly eccentric orbit on the equatorial plane around a Kerr black hole and evaluate the $O(e^2)$ corrections to 2.5PN order, based on Tagoshi.[17] Then in §9, we calculate the gravitational waves induced by a test particle in circular orbit with small inclination from the equatorial plane around a Kerr black hole and evaluate the 2.5PN energy and angular momentum fluxes, based on Shibata, Sasaki, Tagoshi and Tanaka.[15] In §10, we discuss the adiabatic orbital evolution around a Kerr black hole under radiation reaction and show that circular orbits will remain circular under adiabatic radiation reaction but the stability of circular orbits can only be examined by an explicit evaluation of the backreaction force.

In §11, we consider the effect of the spin of a particle. We first give a general formalism to treat the gravitational radiation from a spinning particle orbiting a

Kerr black hole. Then we calculate the 2.5PN luminosity formula with the first order corrections of the spin for circular orbits which are slightly inclined due to the spin of the particle.

Finally, in §12, we review a calculation of the black hole absorption based on Tagoshi, Mano and Takasugi.[24] The black hole absorption effect appears at $O(v^5)$ relative to the Newtonian quadrupole luminosity for a Kerr black hole, while at $O(v^8)$ for a Schwarzschild black hole. We show the energy absorption rate to $O(v^8)$ beyond the lowest order for the Kerr case, i.e., $O(v^{13})$ or 6.5PN order beyond the Newtonian quadrupole luminosity.

Since many of the calculations encountered in this black hole perturbation approach are lengthy, various subsidiary equations and formulas are deferred to Appendices A to J. In the rest of this chapter, we use the units of $c = G = 1$.

§2. Teukolsky formalism

In terms of the conventional Boyer-Lindquist coordinates, the metric of a Kerr black hole is expressed as

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\Sigma}(dt - a\sin^2\theta\, d\varphi)^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\varphi - a\, dt\right]^2 \\ &+\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 , \end{aligned} \tag{2·1}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. In the Teukolsky formalism,[3] the gravitational perturbations of a Kerr black hole are described by a Newman-Penrose quantity $\psi_4 = -C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$, where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, $n^\alpha = ((r^2+a^2), -\Delta, 0, a)/(2\Sigma)$ and $m^\alpha = (ia\sin\theta, 0, 1, i/\sin\theta)/(\sqrt{2}(r + ia\cos\theta))$.

We decompose ψ_4 into Fourier-harmonic components according to

$$(r - ia\cos\theta)^4\psi_4 = \sum_{\ell m}\int d\omega e^{-i\omega t + im\varphi}\ {}_{-2}S_{\ell m}(\theta)R_{\ell m\omega}(r). \tag{2·2}$$

The radial function $R_{\ell m\omega}$ and the angular function ${}_sS_{\ell m}(\theta)$ satisfy the Teukolsky equations with $s = -2$ as

$$\Delta^2\frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_{\ell m\omega}}{dr}\right) - V(r)R_{\ell m\omega} = T_{\ell m\omega}, \tag{2·3}$$

$$\begin{aligned} &\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{d}{d\theta}\right\} - a^2\omega^2\sin^2\theta - \frac{(m-2\cos\theta)^2}{\sin^2\theta}\right. \\ &\qquad\left. + 4a\omega\cos\theta - 2 + 2ma\omega + \lambda\right]{}_{-2}S_{\ell m} = 0. \end{aligned} \tag{2·4}$$

The potential $V(r)$ is given by

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda, \tag{2·5}$$

where $K = (r^2 + a^2)\omega - ma$, and λ is the eigenvalue of ${}_{-2}S^{a\omega}_{\ell m}$. The angular function ${}_sS_{\ell m}(\theta)$ is the spin-weighted spheroidal harmonic which may be normalized as

$$\int_0^\pi |{}_{-2}S_{\ell m}|^2 \sin\theta d\theta = 1. \tag{2·6}$$

The source term $T_{\ell m\omega}$ is specified later. Here we only mention that for orbits of our interest, which are bounded, $T_{\ell m\omega}$ has support in a compact range of r.

We define two kinds of homogeneous solutions of the radial Teukolsky equation:

$$R^{\rm in}_{\ell m\omega} \to \begin{cases} B^{\rm trans}_{\ell m\omega}\Delta^2 e^{-ikr^*} & \text{for } r \to r_+, \\ r^3 B^{\rm ref}_{\ell m\omega} e^{i\omega r^*} + r^{-1} B^{\rm inc}_{\ell m\omega} e^{-i\omega r^*} & \text{for } r \to +\infty, \end{cases} \tag{2·7}$$

$$R^{\rm up}_{\ell m\omega} \to \begin{cases} C^{\rm up}_{\ell m\omega} e^{ikr^*} + \Delta^2 C^{\rm ref}_{\ell m\omega} e^{-ikr^*} & \text{for } r \to r_+, \\ C^{\rm trans}_{\ell m\omega} r^3 e^{i\omega r^*} & \text{for } r \to +\infty, \end{cases} \tag{2·8}$$

where $k = \omega - ma/2Mr_+$, and r^* is the tortoise coordinate defined by

$$\begin{aligned} r^* &= \int \frac{dr^*}{dr} dr \\ &= r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M}, \end{aligned} \tag{2·9}$$

where $r_\pm = M \pm \sqrt{M^2 - a^2}$, and for definiteness, we have fixed the integration constant.

We solve the radial Teukolsky equation by using the Green function method. A solution of the Teukolsky equation which has purely outgoing property at infinity and has purely ingoing property at the horizon is given by

$$R_{\ell m\omega} = \frac{1}{W_{\ell m\omega}} \left\{ R^{\rm up}_{\ell m\omega} \int_{r_+}^{r} dr' R^{\rm in}_{\ell m\omega} T_{\ell m\omega} \Delta^{-2} + R^{\rm in}_{\ell m\omega} \int_r^\infty dr' R^{\rm up}_{\ell m\omega} T_{\ell m\omega} \Delta^{-2} \right\}, \tag{2·10}$$

where the Wronskian $W_{\ell m\omega}$ is given by

$$W_{\ell m\omega} = 2i\omega C^{\rm trans}_{\ell m\omega} B^{\rm inc}_{\ell m\omega}. \tag{2·11}$$

Then, the solution has an asymptotic property at the horizon as

$$R_{\ell m\omega}(r \to r_+) \to \frac{B^{\rm trans}_{\ell m\omega}\Delta^2 e^{-ikr^*}}{2i\omega C^{\rm trans}_{\ell m\omega} B^{\rm inc}_{\ell m\omega}} \int_{r_+}^\infty dr' R^{\rm up}_{\ell m\omega} T_{\ell m\omega} \Delta^{-2} \equiv \tilde{Z}^{\rm H}_{\ell m\omega} \Delta^2 e^{-ikr^*}. \tag{2·12}$$

The solution at infinity is also expressed as

$$R_{\ell m\omega}(r \to \infty) \to \frac{r^3 e^{i\omega r^*}}{2i\omega B^{\rm inc}_{\ell m\omega}} \int_{r_+}^\infty dr' \frac{T_{\ell m\omega}(r') R^{\rm in}_{\ell m\omega}(r')}{\Delta^2(r')} \equiv \tilde{Z}^\infty_{\ell m\omega} r^3 e^{i\omega r^*}. \tag{2·13}$$

Here and in the following sections except for §12, we focus on the gravitational waves emitted to infinity. Hence $\tilde{Z}^\infty_{\ell m\omega}$ will be simply denoted as $\tilde{Z}_{\ell m\omega}$. The gravitational waves absorbed into the black hole horizon will be treated separately in §12.

Now let us discuss the general form of the source term $T_{\ell m\omega}$. It is given by

$$T_{\ell m\omega} = 4\int d\Omega dt \rho^{-5}\overline{\rho}^{-1}(B_2' + B_2'^{*})e^{-im\varphi+i\omega t}\frac{{}_{-2}S^{a\omega}_{\ell m}}{\sqrt{2\pi}}, \tag{2·14}$$

where

$$\begin{aligned} B_2' &= -\frac{1}{2}\rho^8\overline{\rho}L_{-1}[\rho^{-4}L_0(\rho^{-2}\overline{\rho}^{-1}T_{nn})] \\ &\quad - \frac{1}{2\sqrt{2}}\rho^8\overline{\rho}\Delta^2 L_{-1}[\rho^{-4}\overline{\rho}^2 J_+(\rho^{-2}\overline{\rho}^{-2}\Delta^{-1}T_{\overline{m}n})], \\ B_2'^{*} &= -\frac{1}{4}\rho^8\overline{\rho}\Delta^2 J_+[\rho^{-4}J_+(\rho^{-2}\overline{\rho}T_{\overline{m}\overline{m}})] \\ &\quad - \frac{1}{2\sqrt{2}}\rho^8\overline{\rho}\Delta^2 J_+[\rho^{-4}\overline{\rho}^2\Delta^{-1}L_{-1}(\rho^{-2}\overline{\rho}^{-2}T_{\overline{m}n})], \end{aligned} \tag{2·15}$$

with

$$\begin{aligned} \rho &= (r - ia\cos\theta)^{-1}, \\ L_s &= \partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + s\cot\theta, \\ J_+ &= \partial_r + iK/\Delta\,. \end{aligned} \tag{2·16}$$

In the above, T_{nn}, $T_{\overline{m}n}$ and $T_{\overline{m}\overline{m}}$ are the tetrad components of the energy momentum tensor ($T_{nn} = T_{\mu\nu}n^\mu n^\nu$, etc.), and the bar denotes the complex conjugation.

We consider $T_{\mu\nu}$ of a monopole particle of mass μ. The case of a spinning particle will be discussed in §11 separately. The energy momentum tensor takes the form

$$T^{\mu\nu} = \frac{\mu}{\Sigma\sin\theta dt/d\tau}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}\delta(r-r(t))\delta(\theta-\theta(t))\delta(\varphi-\varphi(t)), \tag{2·17}$$

where $z^\mu = (t, r(t), \theta(t), \varphi(t))$ is a geodesic trajectory and $\tau = \tau(t)$ is the proper time along the geodesic. The geodesic equations in Kerr geometry are given by

$$\begin{aligned} \Sigma\frac{d\theta}{d\tau} &= \pm\left[C - \cos^2\theta\left\{a^2(1-E^2) + \frac{l_z^2}{\sin^2\theta}\right\}\right]^{1/2} \equiv \Theta(\theta), \\ \Sigma\frac{d\varphi}{d\tau} &= -\left(aE - \frac{l_z}{\sin^2\theta}\right) + \frac{a}{\Delta}\left(E(r^2+a^2) - al_z\right) \equiv \Phi, \\ \Sigma\frac{dt}{d\tau} &= -\left(aE - \frac{l_z}{\sin^2\theta}\right)a\sin^2\theta + \frac{r^2+a^2}{\Delta}\left(E(r^2+a^2) - al_z\right) \equiv T, \\ \Sigma\frac{dr}{d\tau} &= \pm\sqrt{R}, \end{aligned} \tag{2·18}$$

where E, l_z and C are the energy, the z-component of the angular momentum and the Carter constant of a test particle, respectively.*) $\Sigma = r^2 + a^2\cos^2\theta$ and

$$R = [E(r^2+a^2) - al_z]^2 - \Delta[(Ea - l_z)^2 + r^2 + C]. \tag{2·19}$$

*) These constants of motion are those measured in units of μ. That is, if expressed in the standard units, E, l_z and C in Eq. (2·18) are to be replaced with E/μ, l_z/μ and C/μ^2, respectively.

Using Eq. (2·18), the tetrad components of the energy momentum tensor are expressed as

$$T_{nn} = \mu\frac{C_{nn}}{\sin\theta}\delta(r-r(t))\delta(\theta-\theta(t))\delta(\varphi-\varphi(t)),$$
$$T_{\overline{m}n} = \mu\frac{C_{\overline{m}n}}{\sin\theta}\delta(r-r(t))\delta(\theta-\theta(t))\delta(\varphi-\varphi(t)),$$
$$T_{\overline{m}\,\overline{m}} = \mu\frac{C_{\overline{m}\,\overline{m}}}{\sin\theta}\delta(r-r(t))\delta(\theta-\theta(t))\delta(\varphi-\varphi(t)), \tag{2·20}$$

where

$$C_{nn} = \frac{1}{4\Sigma^3\dot{t}}\left[E(r^2+a^2)-al_z+\Sigma\frac{dr}{d\tau}\right]^2,$$
$$C_{\overline{m}n} = -\frac{\rho}{2\sqrt{2}\Sigma^2\dot{t}}\left[E(r^2+a^2)-al_z+\Sigma\frac{dr}{d\tau}\right]\left[i\sin\theta\left(aE-\frac{l_z}{\sin^2\theta}\right)\right],$$
$$C_{\overline{m}\,\overline{m}} = \frac{\rho^2}{2\Sigma\dot{t}}\left[i\sin\theta\left(aE-\frac{l_z}{\sin^2\theta}\right)\right]^2, \tag{2·21}$$

and $\dot{t} = dt/d\tau$. Substituting Eq. (2·15) into Eq. (2·14) and performing integration by part, we obtain

$$\begin{aligned}
T_{\ell m\omega}
&= \frac{4\mu}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\int d\theta e^{i\omega t-im\varphi(t)}\\
&\times\Big[-\frac{1}{2}L_1^\dagger\Big\{\rho^{-4}L_2^\dagger(\rho^3S)\Big\}C_{nn}\rho^{-2}\overline{\rho}^{-1}\delta(r-r(t))\delta(\theta-\theta(t))\\
&+\frac{\Delta^2\overline{\rho}^2}{\sqrt{2}\rho}\Big(L_2^\dagger S+ia(\overline{\rho}-\rho)\sin\theta S\Big)J_+\Big\{C_{\overline{m}n}\rho^{-2}\overline{\rho}^{-2}\Delta^{-1}\delta(r-r(t))\delta(\theta-\theta(t))\Big\}\\
&+\frac{1}{2\sqrt{2}}L_2^\dagger\Big\{\rho^3S(\overline{\rho}^2\rho^{-4})_{,r}\Big\}C_{\overline{m}n}\Delta\rho^{-2}\overline{\rho}^{-2}\delta(r-r(t))\delta(\theta-\theta(t))\\
&-\frac{1}{4}\rho^3\Delta^2SJ_+\Big\{\rho^{-4}J_+\Big(\overline{\rho}\rho^{-2}C_{\overline{m}\,\overline{m}}\delta(r-r(t))\delta(\theta-\theta(t))\Big)\Big\}\Big],
\end{aligned} \tag{2·22}$$

where

$$L_s^\dagger = \partial_\theta - \frac{m}{\sin\theta}+a\omega\sin\theta+s\cot\theta, \tag{2·23}$$

and S denotes ${}_{-2}S_{\ell m}^{a\omega}(\theta)$ for simplicity.

We further rewrite Eq. (2·22) as

$$\begin{aligned}
T_{\ell m\omega} = \mu\int_{-\infty}^{\infty}dte^{i\omega t-im\varphi(t)}\Delta^2\Big[(A_{nn0}+A_{\overline{m}n0}+A_{\overline{m}\,\overline{m}0})\delta(r-r(t))\\
+\Big\{(A_{\overline{m}n1}+A_{\overline{m}\,\overline{m}1})\delta(r-r(t))\Big\}_{,r}+\Big\{A_{\overline{m}\,\overline{m}2}\delta(r-r(t))\Big\}_{,rr}\Big],
\end{aligned} \tag{2·24}$$

where

$$
\begin{aligned}
A_{n\,n\,0} &= \frac{-2}{\sqrt{2\pi}\Delta^2} C_{n\,n} \rho^{-2}\overline{\rho}^{-1} L_1^\dagger \{\rho^{-4} L_2^\dagger(\rho^3 S)\}, \\
A_{\overline{m}\,n\,0} &= \frac{2}{\sqrt{\pi}\Delta} C_{\overline{m}\,n}\rho^{-3}\left[\left(L_2^\dagger S\right)\left(\frac{iK}{\Delta}+\rho+\overline{\rho}\right) - a\sin\theta S\frac{K}{\Delta}(\overline{\rho}-\rho)\right], \\
A_{\overline{m}\,\overline{m}\,0} &= -\frac{1}{\sqrt{2\pi}}\rho^{-3}\overline{\rho} C_{\overline{m}\,\overline{m}} S\left[-i\left(\frac{K}{\Delta}\right)_{,r} - \frac{K^2}{\Delta^2} + 2i\rho\frac{K}{\Delta}\right], \\
A_{\overline{m}\,n\,1} &= \frac{2}{\sqrt{\pi}\Delta}\rho^{-3} C_{\overline{m}\,n}[L_2^\dagger S + ia\sin\theta(\overline{\rho}-\rho)S], \\
A_{\overline{m}\,\overline{m}\,1} &= -\frac{2}{\sqrt{2\pi}}\rho^{-3}\overline{\rho} C_{\overline{m}\,\overline{m}} S\left(i\frac{K}{\Delta}+\rho\right), \\
A_{\overline{m}\,\overline{m}\,2} &= -\frac{1}{\sqrt{2\pi}}\rho^{-3}\overline{\rho} C_{\overline{m}\,\overline{m}} S.
\end{aligned} \tag{2·25}
$$

Inserting Eq. (2·24) into Eq. (2·13), we obtain $\tilde{Z}_{\ell m\omega}$ as

$$
\tilde{Z}_{\ell m\omega} = \frac{\mu}{2i\omega B^{\rm inc}_{\ell m\omega}} \int_{-\infty}^{\infty} dt e^{i\omega t - im\varphi(t)} W_{\ell m\omega}\,, \tag{2·26}
$$

where

$$
\begin{aligned}
W_{\ell m\omega} = \Bigg[& R^{\rm in}_{\ell m\omega}\{A_{n\,n\,0} + A_{\overline{m}\,n\,0} + A_{\overline{m}\,\overline{m}\,0}\} \\
& - \frac{dR^{\rm in}_{\ell m\omega}}{dr}\{A_{\overline{m}\,n\,1} + A_{\overline{m}\,\overline{m}\,1}\} + \frac{d^2 R^{\rm in}_{\ell m\omega}}{dr^2} A_{\overline{m}\,\overline{m}\,2}\Bigg]_{r=r(t)}
\end{aligned} \tag{2·27}
$$

In this paper, we focus on orbits which are either circular (with or without inclination) or eccentric but confined on the equatorial plane. In either case, the frequency spectrum of $T_{\ell m\omega}$ becomes discrete. Accordingly $\tilde{Z}_{\ell m\omega}$ in Eq. (2·12) or (2·13) takes the form

$$
\tilde{Z}_{\ell m\omega} = \sum_n \delta(\omega - \omega_n) Z_{\ell m\omega}\,. \tag{2·28}
$$

Then, in particular, ψ_4 at $r \to \infty$ is obtained from Eq. (2·2) as

$$
\psi_4 = \frac{1}{r}\sum_{\ell m n} Z_{\ell m\omega_n} \frac{{}_{-2}S^{a\omega_n}_{\ell m}}{\sqrt{2\pi}} e^{i\omega_n(r^*-t)+im\varphi}. \tag{2·29}
$$

At infinity, ψ_4 is related to the two independent modes of gravitational waves h_+ and $h_\times$ as

$$
\psi_4 = \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times). \tag{2·30}
$$

From Eqs. (2·29) and (2·30), the luminosity averaged over $t \gg \Delta t$, where Δt is the characteristic time scale of the orbital motion (e.g., a period between the two consecutive apastrons), is given by

$$
\left\langle \frac{dE}{dt} \right\rangle = \sum_{\ell,m,n} \frac{|Z_{\ell m\omega_n}|^2}{4\pi\omega_n^2} \equiv \sum_{\ell,m,n}\left(\frac{dE}{dt}\right)_{\ell mn}. \tag{2·31}
$$

In the same way, the time-averaged angular momentum flux is given by

$$\left\langle \frac{dJ_z}{dt} \right\rangle = \sum_{\ell,m,n} \frac{m\,|Z_{\ell m\omega_n}|^2}{4\pi\omega_n^3} \equiv \sum_{\ell,m,n} \left(\frac{dJ_z}{dt}\right)_{\ell mn} = \sum_{\ell,m,n} \frac{m}{\omega_n}\left(\frac{dE}{dt}\right)_{\ell mn}. \qquad (2\cdot 32)$$

§3. Post-Newtonian expansion of the ingoing wave solutions

We consider the case when a test particle of mass μ is in an orbit which is nearly circular around a Kerr black hole of mass $M \gg \mu$ and describe a method to calculate the ingoing wave Teukolsky functions $R^{\rm in}_{\ell m\omega}$ which are necessary to evaluate the 4PN formulas for the gravitational waves energy and angular momentum fluxes emitted to infinity. In the Schwarzschild case, we shall derive the 5.5PN luminosity formula in §4. A method to calculate the ingoing wave solutions in this case is separately discussed in Appendix D because it is considerably more complicated than the method explained in this section.

Using non-dimensional variables in the Teukolsky equation, we can see that the Teukolsky equation is expressed in terms of three basic variables, $z \equiv \omega r$, $\epsilon \equiv 2M\omega$ and $a\omega \equiv q\epsilon/2$ where $q \equiv a/M$. In order to calculate the gravitational waves induced by a particle, we need to know the explicit form of the source terms $T_{\ell m\omega}(r)$. They will be given in the proceeding sections for specified orbits. Here it is sufficient to note that they have support only around $r = r_0$ where r_0 is the orbital radius for a circular orbit or the mean radius in the case of an eccentric orbit (with small eccentricity). Hence from Eq. (2·13), what we need to know are the ingoing wave functions $R^{\rm in}(r)$ around $r = r_0$, and their incident amplitudes $B^{\rm inc}$. Note that we do not need the transmission amplitudes $B^{\rm trans}$ to evaluate the gravitational waves at infinity. This fact considerably simplifies the calculations. Since we treat a test particle in a bound orbit which is nearly circular, the contribution of ω to the Teukolsky functions comes from $\omega \sim m\Omega_\varphi$, where $\Omega_\varphi \sim (M/r_0^3)^{1/2}$ is the orbital angular frequency. We will evaluate $R^{\rm in}$ by setting three basic variables to be $z = \omega r_0 \sim m(M/r_0)^{1/2} \sim v$, $\epsilon \sim 2m(M/r_0)^{3/2} \sim v^3$ and $a\omega \sim qm(M/r_0)^{3/2} \sim v^3$. Here, we have introduced a parameter $v \equiv (M/r_0)^{1/2}$ which represents the magnitude of the orbital velocity. We assume that v is much smaller than the velocity of light; $v \ll 1$. Consequently, we also assume that $\epsilon \ll v \ll 1$. This relation is the basic assumption in obtaining the homogeneous solutions below.

Now we calculate the ingoing wave solutions which are necessary to calculate the luminosity to $O(v^8)$ beyond the lowest order for the Kerr case. The method is mainly based on Shibata et al.[15] and Tagoshi et al..[16] An extension to $O(v^{11})$ calculations done by Tanaka, Tagoshi and Sasaki[13] for the Schwarzschild case is given in Appendix D.

First, we discuss the angular solutions. The angular solutions are the spin-weighted spheroidal harmonics. The angular equation (2·4) contains only one small parameter $a\omega$. It is straightforward to calculate the spin-weighted spheroidal harmonic ${}_{-2}S_{\ell m}$ and its eigenvalue λ by expanding the solution in power of $a\omega$. It can be done by the usual perturbation method.[26),16),15)] It is also possible to obtain them by using an expansion by means of Jacobi functions.[27] The method and the results are given explicitly in Appendix A. Here we only show the eigenvalue λ which is used to calculate the radial functions. The eigenvalue λ is given by

$$\lambda = \lambda_0 + a\omega\lambda_1 + a^2\omega^2\lambda_2 + O((a\omega)^3), \tag{3·1}$$

where

$$\begin{aligned} \lambda_0 &= \ell(\ell+1) - 2 = (\ell-1)(\ell+2), \\ \lambda_1 &= -2m\frac{\ell(\ell+1)+4}{\ell(\ell+1)}, \\ \lambda_2 &= -2(\ell+1)(c_{\ell m}^{\ell+1})^2 + 2\ell(c_{\ell m}^{\ell-1})^2 + \frac{2}{3} - \frac{2}{3}\frac{(\ell+4)(\ell-3)(\ell^2+\ell-3m^2)}{\ell(\ell+1)(2\ell+3)(2\ell-1)}, \end{aligned} \tag{3·2}$$

with

$$\begin{aligned} c_{\ell m}^{\ell+1} &= \frac{2}{(\ell+1)^2}\left[\frac{(\ell+3)(\ell-1)(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right]^{1/2}, \\ c_{\ell m}^{\ell-1} &= -\frac{2}{\ell^2}\left[\frac{(\ell+2)(\ell-2)(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2}. \end{aligned} \tag{3·3}$$

Next we consider the homogeneous solution $R^{\rm in}$. We assume $\omega > 0$ below. The solution for $\omega < 0$ can be obtained from the one for $\omega > 0$ by using the symmetry of the homogeneous Teukolsky equation which implies $\overline{R_{\ell,m,\omega}} = R_{\ell,-m,-\omega}$. Here, we do not treat the Teukolsky equation directly. Instead, we transform the homogeneous Teukolsky equation to the Sasaki-Nakamura equation,[28] which is given by

$$\left[\frac{d^2}{dr^{*2}} - F(r)\frac{d}{dr^*} - U(r)\right]X_{\ell m\omega} = 0. \tag{3·4}$$

The function $F(r)$ is given by

$$F(r) = \frac{\eta_{,r}}{\eta}\frac{\Delta}{r^2+a^2}, \tag{3·5}$$

where

$$\eta = c_0 + c_1/r + c_2/r^2 + c_3/r^3 + c_4/r^4 \tag{3·6}$$

with

$$\begin{aligned} c_0 &= -12i\omega M + \lambda(\lambda+2) - 12a\omega(a\omega - m), \\ c_1 &= 8ia[3a\omega - \lambda(a\omega - m)], \\ c_2 &= -24iaM(a\omega - m) + 12a^2[1 - 2(a\omega - m)^2], \\ c_3 &= 24ia^3(a\omega - m) - 24Ma^2, \\ c_4 &= 12a^4. \end{aligned} \tag{3·7}$$

The function $U(r)$ is given by

$$U(r) = \frac{\Delta U_1}{(r^2+a^2)^2} + G^2 + \frac{\Delta G_{,r}}{r^2+a^2} - FG, \tag{3·8}$$

where

$$\begin{aligned}
G &= -\frac{2(r-M)}{r^2+a^2} + \frac{r\Delta}{(r^2+a^2)^2},\\
U_1 &= V + \frac{\Delta^2}{\beta}\left[\left(2\alpha + \frac{\beta_{,r}}{\Delta}\right)_{,r} - \frac{\eta_{,r}}{\eta}\left(\alpha + \frac{\beta_{,r}}{\Delta}\right)\right],\\
\alpha &= -i\frac{K\beta}{\Delta^2} + 3iK_{,r} + \lambda + \frac{6\Delta}{r^2},\\
\beta &= 2\Delta\left(-iK + r - M - \frac{2\Delta}{r}\right).
\end{aligned} \tag{3·9}$$

When we set $a = 0$, this transformation becomes the Chandrasekhar transformation[29] for the Schwarzschild black hole. The Sasaki-Nakamura equation was originally introduced, for the inhomogeneous case, to make the potential term short-ranged and to make the source term well-behaved at infinity. It is not necessary to perform this transformation in this case, since we are interested only in bound orbits. Nevertheless we choose to do this because the lowest order solution becomes the spherical Bessel function and we can apply the post-Newtonian expansion techniques developed for the Schwarzschild case by Poisson[8] and Sasaki.[11]

The relation between $R_{\ell m\omega}$ and $X_{\ell m\omega}$ is

$$R_{\ell m\omega} = \frac{1}{\eta}\left\{\left(\alpha + \frac{\beta_{,r}}{\Delta}\right)\chi_{\ell m\omega} - \frac{\beta}{\Delta}\chi_{\ell m\omega,r}\right\}, \tag{3·10}$$

where $\chi_{\ell m\omega} = X_{\ell m\omega}\Delta/(r^2+a^2)^{1/2}$. Conversely, we can express $X_{\ell m\omega}$ in terms of $R_{\ell m\omega}$ as

$$X_{\ell m\omega} = (r^2+a^2)^{1/2}\, r^2\, J_- J_- \left[\frac{1}{r^2}R_{\ell m\omega}\right], \tag{3·11}$$

where $J_- = (d/dr) - i(K/\Delta)$. Then the asymptotic behavior of the ingoing-wave solution $X^{\rm in}$ which corresponds to Eq. (2·7) is

$$X^{\rm in}_{\ell m\omega} \to \begin{cases} A^{\rm ref}_{\ell m\omega} e^{i\omega r^*} + A^{\rm inc}_{\ell m\omega} e^{-i\omega r^*} & \text{for } r^* \to \infty, \\ A^{\rm trans}_{\ell m\omega} e^{-ikr^*} & \text{for } r^* \to -\infty. \end{cases} \tag{3·12}$$

The coefficients $A^{\rm inc}$, $A^{\rm ref}$ and $A^{\rm trans}$ are respectively related to $B^{\rm inc}$, $B^{\rm ref}$ and $B^{\rm trans}$, defined in Eq. (2·7), by

$$B^{\rm inc}_{\ell m\omega} = -\frac{1}{4\omega^2}A^{\rm inc}_{\ell m\omega}, \tag{3·13}$$

$$B^{\rm ref}_{\ell m\omega} = -\frac{4\omega^2}{c_0}A^{\rm ref}_{\ell m\omega}, \tag{3·14}$$

$$B^{\rm trans}_{\ell m\omega} = \frac{1}{d_{\ell m\omega}}A^{\rm trans}_{\ell m\omega}, \tag{3·15}$$

where c_0 is given in Eq. (3·7) and

$$d_{\ell m\omega} = \sqrt{2Mr_+}[(8 - 24iM\omega - 16M^2\omega^2)r_+^2 + (12iam - 16M + 16amM\omega + 24iM^2\omega)r_+ - 4a^2m^2 - 12iamM + 8M^2].$$

Now we introduce the variable z^* as

$$\begin{aligned} z^* &= z + \epsilon\left[\frac{z_+}{z_+ - z_-}\ln(z - z_+) - \frac{z_-}{z_+ - z_-}\ln(z - z_-)\right] \\ &= \omega r^* + \epsilon\ln\epsilon, \end{aligned} \qquad (3\cdot16)$$

where $z = \omega r$ and $z_\pm = \omega r_\pm$. To solve for $X^{\rm in}$, we set

$$X^{\rm in}_{\ell m\omega} = \sqrt{z^2 + a^2\omega^2}\,\xi_{\ell m}(z)\exp\left(-i\phi(z)\right), \qquad (3\cdot17)$$

where

$$\begin{aligned} \phi(z) &= \int dr\left(\frac{K}{\Delta} - \omega\right) \\ &= z^* - z - \frac{\epsilon}{2}mq\frac{1}{z_+ - z_-}\ln\frac{z - z_+}{z - z_-}. \end{aligned} \qquad (3\cdot18)$$

With this choice of the phase function, the ingoing wave boundary condition at horizon reduces to that $\xi_{\ell m}$ is regular and finite at $z = z_+$.

Inserting Eq. (3·17) into Eq. (3·4) and expanding it in powers of $\epsilon = 2M\omega$, we obtain

$$L^{(0)}[\xi_{\ell m}] = \epsilon L^{(1)}[\xi_{\ell m}] + \epsilon Q^{(1)}[\xi_{\ell m}] + \epsilon^2 Q^{(2)}[\xi_{\ell m}] + \epsilon^3 Q^{(3)}[\xi_{\ell m}] + \epsilon^4 Q^{(4)}[\xi_{\ell m}] + O(\epsilon^5), \qquad (3\cdot19)$$

where $L^{(0)}$, $L^{(1)}$, $Q^{(1)}$ and $Q^{(2)}$ are differential operators given by

$$L^{(0)} = \frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} + \left(1 - \frac{\ell(\ell+1)}{z^2}\right), \qquad (3\cdot20)$$

$$L^{(1)} = \frac{1}{z}\frac{d^2}{dz^2} + \left(\frac{1}{z^2} + \frac{2i}{z}\right)\frac{d}{dz} - \left(\frac{4}{z^3} - \frac{i}{z^2} + \frac{1}{z}\right), \qquad (3\cdot21)$$

$$Q^{(1)} = \frac{iq\lambda_1}{2z^2}\frac{d}{dz} - \frac{4imq}{l(l+1)z^3}. \qquad (3\cdot22)$$

The formulas for $Q^{(2)}$, $Q^{(3)}$ and $Q^{(4)}$ are very complicated, and they are given explicitly in Appendix B. Note that, when we set $a = 0$, all $Q^{(n)}$ vanish.

By expanding $\xi_{\ell m}$ in terms of ϵ as

$$\xi_{\ell m} = \sum_{n=0}^{\infty}\epsilon^n\xi^{(n)}_{\ell m}(z), \qquad (3\cdot23)$$

we obtain from Eq. (3·19) the iterative equations,

$$L^{(0)}[\xi^{(n)}_{\ell m}] = z^{-2}W^{(n)}_{\ell m}, \qquad (3\cdot24)$$

where

$$W_{\ell m}^{(0)} = 0, \tag{3·25}$$

$$W_{\ell m}^{(1)} = z^2 \left(L^{(1)}[\xi_{\ell m}^{(0)}] + Q^{(1)}[\xi_{\ell m}^{(0)}]\right), \tag{3·26}$$

$$W_{\ell m}^{(2)} = z^2 \left(L^{(1)}[\xi_{\ell m}^{(1)}] + Q^{(1)}[\xi_{\ell m}^{(1)}] + Q^{(2)}[\xi_{\ell m}^{(0)}]\right), \tag{3·27}$$

$$W_{\ell m}^{(3)} = z^2 \left(L^{(1)}[\xi_{\ell m}^{(2)}] + Q^{(1)}[\xi_{\ell m}^{(2)}] + Q^{(2)}[\xi_{\ell m}^{(1)}] + Q^{(3)}[\xi_{\ell m}^{(0)}]\right), \tag{3·28}$$

$$W_{\ell m}^{(4)} = z^2 \left(L^{(1)}[\xi_{\ell m}^{(3)}] + Q^{(1)}[\xi_{\ell m}^{(3)}] + Q^{(2)}[\xi_{\ell m}^{(2)}] + Q^{(3)}[\xi_{\ell m}^{(1)}] + Q^{(4)}[\xi_{\ell m}^{(0)}]\right). \tag{3·29}$$

As $\epsilon = 2GM\omega$ if we recover G, the above expansion corresponds to the post-Minkowski expansion of the vacuum Einstein equations.

The iterative equations (3·24) have been obtained by expanding Eq. (3·4) in powers of ϵ by regarding $z = \omega r$ as the independent variable. Since the horizon is at $z = z_+ = O(\epsilon)$, this procedure implicitly assumes that $\epsilon \ll z$. Consequently, we cannot apply the above expansion near the horizon where the ingoing wave boundary condition is to be imposed. To implement the boundary condition correctly, we have to consider a series solution of $\xi_{\ell m}$ which is valid near the horizon as well as in the range $\epsilon \ll z$ and match it to the series solution of the form (3·23). Recently this matching problem has been rigorously solved by Mano, Suzuki and Takasugi [23] for the original Teukolsky equation. However, for our present purpose, it is sufficient to resort to a simple power-counting argument, by which it is possible to implement the boundary condition of $\xi_{\ell m}$ at the horizon to the behavior of $\xi_{\ell m}^{(n)}$ at $z \gg \epsilon$ for $n \le 2\ell$ (for $n \le 2\ell + 1$ in the Schwarzschild case; see Appendix D).

Since the ingoing wave boundary condition is that $\xi_{\ell m}$ is regular at horizon, if we introduce an independent variable $x := (z - z_+)/\epsilon$ we can expand $\xi_{\ell m}$ near the horizon as

$$\xi_{\ell m} = \sum_{n=0}^{\infty} \epsilon^n \xi_{\ell m}^{\{n\}}(x). \tag{3·30}$$

This means we have

$$\xi_{\ell m}(0) = \xi_{\ell m}^{\{0\}}(0) \sum_{n=0}^{\infty} \epsilon^n C_n, \tag{3·31}$$

where $C_n = \xi_{\ell m}^{\{n\}}(0)/\xi_{\ell m}^{\{0\}}(0)$. In other words, $\xi_{\ell m}(0)$ should have a well-defined limit for $\epsilon \to 0$ except for the overall normalization factor $\xi_{\ell m}^{\{0\}}(0)$. Keeping this property in mind, let us consider the boundary conditions for Eqs. (3·25)∼(3·29).

The general solution to Eq. (3·25) is immediately obtained as

$$\xi_{\ell m}^{(0)} = \alpha_{\ell m}^{(0)} j_\ell + \beta_{\ell m}^{(0)} n_\ell, \tag{3·32}$$

where j_ℓ and n_ℓ are the spherical Bessel functions. The coefficients $\alpha_{\ell m}^{(0)}$ and $\beta_{\ell m}^{(0)}$ are to be determined by the boundary condition. For convenience, we normalize the solution $X^{\rm in}$ such that the incident amplitude $A^{\rm inc}$ is of order unity. Then both $\alpha_{\ell m}^{(0)}$

and $\beta_{\ell m}^{(0)}$ must be of order unity. Since $j_\ell(z) \sim z^\ell$ and $n_\ell(z) \sim z^{-\ell-1}$ for $z \ll 1$, the latter is $O(\epsilon^{2\ell+1})$ larger than the former near the horizon where $z = O(\epsilon)$. Hence Eq. (3·31) implies we should set $\beta_{\ell m}^{(0)} = 0$. As for the value of $\alpha_{\ell m}^{(0)}$, since it only contributes to the overall normalization of $X^{\rm in}$, we set $\alpha_{\ell m}^{(0)} = 1$ for convenience.

Inspection of Eqs. (3·26)∼(3·29) reveals that the solution $\xi_{\ell m}^{(n)}$ behaves as $z^{\ell-n}$ plus the homogeneous solution $\alpha_{\ell m}^{(n)} j_\ell + \beta_{\ell m}^{(n)} n_\ell$ for $z \to 0$. As for $\alpha_{\ell m}^{(n)}$ $(n \geq 1)$, they simply contribute to renormalizations of $\alpha_{\ell m}^{(0)}$. Hence we put them to zero. As for $\beta_{\ell m}^{(n)}$, from the same argument as given above, we find they may become non-zero only for $n \geq 2\ell + 1$. Since $\ell \geq 2$ and $\epsilon = O(v^3)$, this implies that the near zone contribution of $n_\ell \sim z^{-\ell-1}$, which is $O(v^{-(2\ell+1)})$ greater than the lowest order term j_ℓ, to the gravitational waves emitted to infinity may arise only at $O(v^{10})$ beyond the quadrupole order. Since the post-Newtonian corrections we shall consider for the Kerr case are those up to $O(v^8)$, we set $\beta_{\ell m}^{(n)} = 0$ and solve the iterative equations (3·24) to $O(\epsilon^4)$ with the boundary conditions that $\xi_{\ell m}^{(n)} \sim z^{\ell-n}$ at $z \to 0$. We note that, in the Schwarzschild case which is discussed separately in Appendix D, these boundary conditions turn out to be appropriate for $n \leq 2\ell + 1$; i.e., up to one power of ϵ higher than the Kerr case.

To calculate $\xi_{\ell m}^{(n)}$ for $n \geq 1$, we rewrite Eqs. (3·26)∼(3·29) in the indefinite integral form by using the spherical Bessel functions as

$$\xi_{\ell m}^{(n)} = n_\ell \int^z dz j_\ell W_{\ell m}^{(n)} - j_\ell \int^z dz n_\ell W_{\ell m}^{(n)} . \tag{3·33}$$

The calculation is straightforward but tedious. All the formulas which are needed to calculate the above integration to obtain $\xi_{\ell m}^{(n)}$ for $n \leq 2$ are shown in Appendix of Ref. 11). They are recapitulated in an alternative way in Appendix D. Using those formulas, we have for $n = 1$,

$$\begin{aligned}\xi_{\ell m}^{(1)} = & \frac{(\ell-1)(\ell+3)}{2(\ell+1)(2\ell+1)} j_{\ell+1} - \left(\frac{\ell^2-4}{2\ell(2\ell+1)} + \frac{2\ell-1}{\ell(\ell-1)}\right) j_{\ell-1} \\ & + R_{\ell,0} j_0 + \sum_{m=1}^{\ell-2} \left(\frac{1}{m} + \frac{1}{m+1}\right) R_{\ell,m} j_m - 2D_\ell^{nj} + i j_\ell \ln z \\ & + \frac{imq}{2}\left(\frac{\ell^2+4}{\ell^2(2\ell+1)}\right) j_{\ell-1} + \frac{imq}{2}\left(\frac{(\ell+1)^2+4}{(\ell+1)^2(2\ell+1)}\right) j_{\ell+1},\end{aligned} \tag{3·34}$$

where D_ℓ^{nj} is an extension of the spherical Bessel function defined in §D.2.1 of Appendix D

$$D_\ell^{nj} = \frac{1}{2}\left[j_\ell {\rm Si}(2z) - n_\ell \left({\rm Ci}(2z) - \gamma - \ln 2z\right)\right], \tag{3·35}$$

where $\gamma = 0.5772\cdots$ is the Euler constant, ${\rm Ci}(x) = -\int_x^\infty dt \cos t/t$ and ${\rm Si}(x) = \int_0^x dt \sin t/t$, and $R_{m,k}$ is a polynomial of the inverse power of z defined by

$$R_{m,k} = z^2(n_m j_k - j_m n_k)$$

$$= -\sum_{r=0}^{[(m-k-1)/2]} (-1)^r \frac{(m-k-1-r)!\, \Gamma\left(m+\frac{1}{2}-r\right)}{r!\,(m-k-1-2r)!\,\Gamma\left(k+\frac{3}{2}+r\right)} \left(\frac{2}{z}\right)^{m-k-1-2r} \tag{3·36}$$

for $m > k$, and

$$R_{m,k} = -R_{k,m} \tag{3·37}$$

for $m < k$.

Next we consider $\xi^{(2)}_{\ell m}$. From Eqs. (3·33) and (3·34), we obtain $\xi^{(2)}_{\ell m}$ as

$$\xi^{(2)}_{\ell m} = f^{(2)}_\ell + i g^{(2)}_\ell + k^{(2)}_{\ell m}(q)\,, \tag{3·38}$$

where $f^{(2)}_\ell$ and $g^{(2)}_\ell$ are the real and imaginary parts of $\xi^{(2)}_{\ell m}$ in the Schwarzschild limit, respectively, and $k^{(2)}_{\ell m}(q)$ is the correction term due to non-vanishing $q = a/M$. For $\ell = 2$ and $\ell = 3$, $f^{(2)}_\ell$ are given by

$$\begin{aligned}
f^{(2)}_2 &= -\frac{389}{70z^2} j_0 - \frac{113}{420z} j_1 + \frac{1}{7z} j_3 + 4D^{nnj}_2 \\
&\quad -\frac{5}{3z} D^{nj}_2 + \frac{10}{3} D^{nj}_1 + \frac{6}{z} D^{nj}_0 + \frac{107}{105} D^{nj}_{-3} \\
&\quad -\frac{107}{210} j_2 \ln z - \frac{1}{2} j_2 (\ln z)^2\,, \\
f^{(2)}_3 &= \frac{1}{4z} j_4 + \frac{323}{49z} j_2 - \frac{5065}{294z^2} j_1 - \left(\frac{1031}{588z} + \frac{445}{14z^3}\right) j_0 + \frac{65}{6z^2} n_0 - \frac{65}{6z} n_1 \\
&\quad -\frac{3}{z} D^{nj}_3 + \frac{13}{3} D^{nj}_2 + \frac{9}{z} D^{nj}_1 + \frac{30}{z^2} D^{nj}_0 - \frac{13}{21} D^{nj}_{-4} \\
&\quad +4D^{nnj}_3 - \frac{13}{42} j_3 \ln z - \frac{1}{2} j_3 (\ln z)^2\,,
\end{aligned} \tag{3·39}$$

where D^{nnj}_ℓ is defined in Appendix D, Eq. (D·20). As explained in §D.1 of Appendix D, the term $g^{(2)}_\ell$ is given for any ℓ as

$$g^{(2)}_\ell = -\frac{1}{z} j_\ell + f^{(1)}_\ell \ln z. \tag{3·40}$$

The term $k^{(2)}_{\ell m}(q)$ is given for $\ell = 2$ and $\ell = 3$ by

$$\begin{aligned}
k^{(2)}_{2m} &= \frac{191\, i}{180} m\, q\, j_0 - \frac{m^2 q^2 j_0}{30} - \frac{m\, q\, j_1}{10} \\
&\quad -\frac{68\, i}{63} m\, q j_2 - \frac{q^2}{392} j_2 - \frac{73\, m^2 q^2}{1764} j_2 + \frac{7\, m\, q\, j_3}{180} - \frac{i}{72} q^2 j_3 + \frac{i}{324} m^2 q^2 j_3 \\
&\quad +\frac{11\, i}{420} m\, q\, j_4 - \frac{q^2 j_4}{392} - \frac{71\, m^2 q^2 j_4}{8820} + \frac{13\, i}{6} m\, q\, n_1 \\
&\quad +mq\left(-\frac{j_1}{5} - \frac{13\, j_3}{90}\right) \ln z + imq\left(\frac{-2}{5} D^{nj}_1 - \frac{13}{45} D^{nj}_3\right),
\end{aligned} \tag{3·41}$$

$$k_{3m}^{(2)} = \frac{3527\,i}{840}\,m\,q\,j_1 - \frac{2\,m^2\,q^2\,j_1}{315} - \frac{m\,q\,j_2}{36} - \frac{5\,i}{504}\,q^2\,j_2 + \frac{5\,i}{2268}\,m^2\,q^2\,j_2$$
$$-\frac{379\,i}{360}\,m\,q\,j_3 - \frac{q^2\,j_3}{360} - \frac{7\,m^2\,q^2\,j_3}{720} + \frac{3\,m\,q\,j_4}{160} - \frac{i}{140}\,q^2\,j_4 + \frac{i}{1120}\,m^2\,q^2\,j_4$$
$$+\frac{97\,i}{5040}\,m\,q\,j_5 - \frac{q^2\,j_5}{360} - \frac{17\,m^2\,q^2\,j_5}{5040} - \frac{103\,i}{48}\,m\,q\,n_0 + \frac{25\,i}{8}\,m\,q\,n_2$$
$$-\left(\frac{13\,m\,q\,j_2}{126} + \frac{5\,m\,q\,j_4}{56}\right)\ln z + imq\left(\frac{-13}{63}\,D_2^{nj} - \frac{5}{28}\,D_4^{nj}\right). \qquad (3{\cdot}42)$$

As noted previously, the source term $T_{\ell m\omega}$ has support only around $r = r_0$, hence around $z = \omega r_0 = O(v)$. Therefore, to evaluate the source integral, we only need $X^{\rm in}$ at $z = O(v) \ll 1$, apart from the value of the incident amplitude $A^{\rm inc}$. Hence the post-Newtonian expansion of $X^{\rm in}$ corresponds to the expansion not only in terms of $\epsilon = O(v^3)$, but also of z by assuming $\epsilon \ll z \ll 1$. In order to evaluate the gravitational wave luminosity to $O(v^8)$ beyond the leading order, we must calculate the series expansion of $\xi_{\ell m}^{(n)}$ in powers of z for $n \le 6 - \ell$ for each $2 \le \ell \le 6$. This follows from a simple power counting. The leading order contribution of the ℓ-th pole is $O(v^{2(\ell-2)})$ smaller than that of the quadrupole, while the n-th post-Minkowski terms are $O(\epsilon^n z^{-n}) = O(v^{2n})$ relative to the lowest order terms in the near-zone. Hence the leading term of $\xi_{\ell m}^{(n)}$ contributes at $O(v^{2(\ell-2)+2n})$ and $O(v^8)$ is attained for $\ell + n = 6$ (see Appendix C of Ref. 15)).

To evaluate $A^{\rm inc}$, we need to know the asymptotic behavior of $\xi_{\ell m}^{(n)}$ at infinity. Since the accuracy of $A^{\rm inc}$ we need is $O(\epsilon^2)$, we do not have to calculate $\xi_{\ell m}^{(3)}$ and $\xi_{\ell m}^{(4)}$ in closed analytic form. We need only the series expansion formulas for $\xi_{\ell m}^{(3)}$ and $\xi_{\ell m}^{(4)}$ around $z = 0$, which are easily obtained from Eq. (3·33). This is also true for $\xi_{4m}^{(2)}$ for $\ell = 4$. Inserting $\xi_{\ell m}^{(n)}$ into Eq. (3·17) and expanding it by z and ϵ assuming $\epsilon \ll z \ll 1$, we obtain

$$\xi_{2m}^{(3)} = \frac{-q^2}{30z} - \frac{i}{30z}\,m\,q^3 + \frac{-i}{30} + \frac{7\,m\,q}{180} - \frac{i}{60}\,m^2\,q^2 + \frac{m\,q^3}{36} - \frac{m^3\,q^3}{90}$$
$$-\frac{m\,q\,\ln z}{30} - \frac{i}{30}\,m^2\,q^2\ln z$$
$$+z\left(\frac{319}{6300} + \frac{100637\,i}{441000}\,m\,q - \frac{q^2}{180} + \frac{17\,m^2\,q^2}{1134} + \frac{83\,i}{5880}\,m\,q^3\right.$$
$$\left.-\frac{61\,i}{13230}\,m^3\,q^3 + \frac{\ln z}{15} - \frac{106\,i}{1575}\,m\,q\ln z - \frac{i}{30}\,m\,q\,(\ln z)^2\right)$$
$$+O(z^2), \qquad (3{\cdot}43)$$

$$\xi_{2m}^{(4)} = \frac{q^4}{80\,z^2} + O(z^{-1}), \qquad (3{\cdot}44)$$

$$\xi_{3m}^{(3)} = \frac{-i}{1260}\,m\,q + \frac{m^2\,q^2}{1890} - \frac{i}{1260}\,m\,q^3 - \frac{i}{3780}\,m^3\,q^3 + O(z), \qquad (3{\cdot}45)$$

$$\xi_{4m}^{(2)} = \left(\frac{1}{1764} - \frac{11\,i}{15120}\,m\,q + \frac{q^2}{10584} - \frac{19\,m^2\,q^2}{105840}\right) z^2 + O(z^3). \tag{3·46}$$

Inserting these $\xi_{\ell m}^{(n)}$ into Eq. (3·17) and expanding the result in terms of $\epsilon = 2M\omega$, we obtain the post-Newtonian expansion of $X^{\rm in}$. The transformation from $X^{\rm in}$ to $R^{\rm in}$ is done by using Eq. (3·10).

Next, we consider $A^{\rm inc}$ to $O(\epsilon^2)$. Using the relation $j_{\ell+1} \sim -j_{\ell-1} \sim (-1)^{\ell+n} n_{2n-\ell}$ at $z \sim \infty$, etc., we obtain the asymptotic behavior of $\xi_{\ell m}^{(1)}$ and $\xi_{\ell m}^{(2)}$ at $z \sim \infty$ as

$$\xi_{\ell m}^{(1)} \sim p_{\ell m}^{(1)} j_\ell + (q_{\ell m}^{(1)} - \ln z) n_\ell + i j_\ell \ln z, \tag{3·47}$$

$$\begin{aligned}\xi_{\ell m}^{(2)} \sim &\left(p_{\ell m}^{(2)} + q_{\ell m}^{(1)} \ln z - (\ln z)^2\right) j_\ell + (q_{\ell m}^{(2)} - p_{\ell m}^{(1)} \ln z) n_\ell \\ &+ i p_{\ell m}^{(1)} j_\ell \ln z + i(q_{\ell m}^{(1)} - \ln z) n_\ell \ln z,\end{aligned} \tag{3·48}$$

where

$$p_{\ell m}^{(1)} = -\frac{\pi}{2}, \tag{3·49}$$

$$q_{\ell m}^{(1)} = \frac{1}{2}\left[\psi(\ell) + \psi(\ell+1) + \frac{(\ell-1)(\ell+3)}{\ell(\ell+1)}\right] - \ln 2 - \frac{2imq}{\ell^2(\ell+1)^2}, \tag{3·50}$$

$$\psi(\ell) = \sum_{k=1}^{\ell-1} \frac{1}{k} - \gamma \tag{3·51}$$

for any ℓ, and

$$\begin{aligned}p_{2m}^{(2)} = &\frac{457\,\gamma}{210} - \frac{\gamma^2}{2} + \frac{5\,\pi^2}{24} - \frac{i}{18}\gamma\,m\,q + \frac{457\,\ln 2}{210} \\ &-\gamma\,\ln 2 - \frac{i}{18}\,m\,q\,\ln 2 - \frac{(\ln 2)^2}{2},\end{aligned} \tag{3·52}$$

$$q_{2m}^{(2)} = \frac{-457\,\pi}{420} + \frac{\gamma\,\pi}{2} + \frac{5\,m\,q}{36} + \frac{i}{36}\,m\,\pi\,q - \frac{i}{72}\,q^2 + \frac{i}{324}\,m^2\,q^2 + \frac{\pi\,\ln 2}{2}, \tag{3·53}$$

$$p_{3m}^{(2)} = \frac{52\,\gamma}{21} - \frac{\gamma^2}{2} + \frac{5\,\pi^2}{24} - \frac{i}{72}\gamma\,m\,q + \frac{52\,\ln 2}{21} - \gamma\,\ln 2 - \frac{i}{72}\,m\,q\,\ln 2 - \frac{(\ln 2)^2}{2}, \tag{3·54}$$

$$q_{3m}^{(2)} = \frac{-26\,\pi}{21} + \frac{\gamma\,\pi}{2} + \frac{67\,m\,q}{1440} + \frac{i}{144}\,m\,\pi\,q + \frac{i}{360}\,q^2 - \frac{17\,i}{12960}\,m^2\,q^2 + \frac{\pi\,\ln 2}{2}. \tag{3·55}$$

Then noting that $\exp(-i\phi) \sim \exp(-i(z^* - z))$ at $z \sim \infty$, the asymptotic form of $X^{\rm in}$ is expressed as

$$\begin{aligned}X^{\rm in} &= \sqrt{z^2 + a^2\omega^2}\,\exp(-i\phi)\left\{f_{\ell m}^{(0)} + \epsilon\xi_{\ell m}^{(1)} + \epsilon^2\xi_{\ell m}^{(2)} + \cdots\right\} \\ &\sim e^{-iz^*}(z h_\ell^{(2)} e^{iz})\left[1 + \epsilon(p_{\ell m}^{(1)} + i q_{\ell m}^{(1)}) + \epsilon^2(p_{\ell m}^{(2)} + i q_{\ell m}^{(2)})\right] \\ &\quad + e^{iz^*}(z h_\ell^{(1)} e^{-iz})\left[1 + \epsilon(p_{\ell m}^{(1)} - i q_{\ell m}^{(1)}) + \epsilon^2(p_{\ell m}^{(2)} - i q_{\ell m}^{(2)})\right],\end{aligned} \tag{3·56}$$

where $h_\ell^{(1)}$ and $h_\ell^{(2)}$ are the spherical Hankel functions of the first and second kinds, respectively, which are given by

$$h_\ell^{(1)} = j_\ell + i n_\ell \to (-1)^{\ell+1}\frac{e^{iz}}{z},\ h_\ell^{(2)} = j_\ell - i n_\ell \to (-1)^{\ell+1}\frac{e^{-iz}}{z}. \tag{3·57}$$

From these equations, noting $\omega r^* = z^* - \epsilon \ln \epsilon$, we obtain

$$A^{\rm inc} = \frac{1}{2} i^{\ell+1} e^{-i\epsilon \ln\epsilon}\left[1 + \epsilon(p_{\ell m}^{(1)} + i q_{\ell m}^{(1)}) + \epsilon^2(p_{\ell m}^{(2)} + i q_{\ell m}^{(2)}) + \cdots\right]. \tag{3·58}$$

The corresponding incident amplitude $B^{\rm inc}$ for the Teukolsky function is obtained from Eq. (3·13).

§4. Gravitational waves to $O(v^{11})$ in Schwarzschild case

In this section we consider a circular orbit around a Schwarzschild black hole and derive the 5.5PN formula for the energy flux emitted to infinity. In this case, we can take the orbit to lie on the equatorial plane ($\theta = \pi/2$) without loss of generality. Then E and l_z are given by setting $R(r_0) = \partial R/\partial r(r_0) = 0$ where R is given by Eq. (2·19). This gives

$$E = (r_0 - 2M)/\sqrt{r_0(r_0 - 3M)}, \quad l_z = \sqrt{M r_0}/\sqrt{1 - 3M/r_0}, \tag{4·1}$$

where r_0 is the orbital radius. The angular frequency is given by $\Omega_\varphi = (M/r_0^3)^{1/2}$.

Defining ${}_s b_{\ell m}$ by

$$\begin{aligned}
{}_0 b_{\ell m} &= \frac{1}{2}\left[(\ell-1)\ell(\ell+1)(\ell+2)\right]^{1/2} {}_0 Y_{\ell m}\left(\frac{\pi}{2}, 0\right)\frac{E r_0}{r_0 - 2M}, \\
{}_{-1} b_{\ell m} &= \left[(\ell-1)(\ell+2)\right]^{1/2} {}_{-1} Y_{\ell m}\left(\frac{\pi}{2}, 0\right)\frac{l_z}{r_0}, \\
{}_{-2} b_{\ell m} &= {}_{-2} Y_{\ell m}\left(\frac{\pi}{2}, 0\right) l_z \Omega_\varphi,
\end{aligned} \tag{4·2}$$

where ${}_s Y_{\ell m}(\theta, \varphi)$ are the spin-weighted spherical harmonics,[30)] $\tilde{Z}_{\ell m\omega}$ is found to take the form

$$\tilde{Z}_{\ell m\omega} = Z_{\ell m}\delta(\omega - m\Omega_\varphi), \tag{4·3}$$

where

$$\begin{aligned}
Z_{\ell m} = \frac{\pi}{i\omega r_0^2 B_{\ell\omega}^{\rm inc}} \Bigg\{ \Bigg[& -{}_0 b_{\ell m} - 2i\,{}_{-1} b_{\ell m}\left(1 + \frac{i}{2}\omega r_0^2/(r_0 - 2M)\right) \\
& + i\,{}_{-2} b_{\ell m}\omega r_0 (1 - 2M/r_0)^{-2}\left(1 - M/r_0 + \frac{1}{2} i\omega r_0\right)\Bigg] R_{\ell m}^{\rm in} \\
& + \left[i\,{}_{-1} b_{\ell m} - {}_{-2} b_{\ell m}\left(1 + i\omega r_0^2/(r_0 - 2M)\right)\right] r_0 R_{\ell\omega}^{\rm in\,\prime}(r_0) \\
& + \frac{1}{2}\,{}_{-2} b_{\ell m} r_0^2 R_{\ell\omega}^{\rm in\,\prime\prime}(r_0) \Bigg\},
\end{aligned} \tag{4·4}$$

where prime denotes the derivative with respect to the radial coordinate r. In terms of the amplitudes $Z_{\ell m}$, the gravitational wave luminosity at infinity is given by

$$\frac{dE}{dt} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{|Z_{\ell m}|^2}{4\pi\omega^2}, \tag{4·5}$$

where $\omega = m\Omega_\varphi$. Since the dominant frequency of the gravitational waves at infinity is $2\Omega_\varphi$, an observationally relevant post-Newtonian parameter is $x \equiv (M\Omega_\varphi)^{1/3}$. We mention that our post-Newtonian expansion parameter is defined by $v := (M/r_0)^{1/2}$. In the case of a circular orbit around a Schwarzschild black hole, however, we have $v = x$. Hence the parameter v is directly related to the observable frequency in the present case.

Following the method given in §3, instead of directly calculating $R^{\rm in}_{\ell\omega}$ from the homogeneous Teukolsky equation, we calculate the corresponding Regge-Wheeler function $X^{\rm in}_{\ell\omega}$ first and then transform it to $R^{\rm in}_{\ell\omega}$. The homogeneous Regge-Wheeler equation, which is given by setting $a = 0$ in Eq. (3·4), takes the form,[31]

$$\left[\frac{d^2}{dr^{*2}} + \omega^2 - V(r)\right] X_{\ell\omega}(r) = 0, \tag{4·6}$$

where

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right). \tag{4·7}$$

The transformation (3·10) reduces to[32]

$$R_{\ell m\omega} = \frac{\Delta}{c_0}\left(\frac{d}{dr^*} + i\omega\right)\frac{r^2}{\Delta}\left(\frac{d}{dr^*} + i\omega\right) r X_{\ell\omega}, \tag{4·8}$$

where c_0, defined in Eq. (3·7), reduces to $c_0 = (\ell-1)\ell(\ell+1)(\ell+2) - 12iM\omega$. The inverse transformation (3·11) reduces to

$$X_{\ell\omega} = \frac{r^5}{\Delta}\left(\frac{d}{dr^*} - i\omega\right)\frac{r^2}{\Delta}\left(\frac{d}{dr^*} - i\omega\right)\frac{R_{\ell\omega}}{r^2}. \tag{4·9}$$

The asymptotic forms of $X^{\rm in}$ are the same as given in Eq. (3·12) except that now we have $k = \omega$. The coefficients $A^{\rm inc}$, $A^{\rm ref}$ and $A^{\rm trans}$ are also respectively related to $B^{\rm inc}$, $B^{\rm ref}$ and $B^{\rm trans}$ as before. (See Eqs. (3·13), (3·14) and (3·15).) Note that the coefficient that appears in Eq. (3·15) now reduces to

$$d_{\ell m\omega} = 16M^3(1 - 2iM\omega)(1 - 4iM\omega). \tag{4·10}$$

Corresponding to Eq. (3·16), we introduce the variable $z^* = z + \epsilon\ln(z-\epsilon)$. Then Eq. (3·17) reduces to

$$X^{\rm in}_\ell = e^{-i(z^*-z)} z\xi_\ell(z) = e^{-i\epsilon\ln(z-\epsilon)} z\xi_\ell(z), \tag{4·11}$$

and Eq. (3·19) becomes $L^{(0)}[\xi_\ell] = \epsilon L^{(1)}[\xi_\ell]$. Thus Eq. (3·24) simplifies considerably to become

$$L^{(0)}[\xi^{(n)}_\ell] = L^{(1)}[\xi^{(n-1)}_\ell]. \tag{4·12}$$

It may be worthwhile to note that the left-hand side can be expressed concisely as

$$L^{(1)}[\xi_\ell^{(n-1)}] = e^{-iz}\frac{d}{dz}\left[\frac{1}{z^3}\frac{d}{dz}\left(e^{iz}z^2\xi_\ell^{(n-1)}\right)\right]. \tag{4·13}$$

The calculations to $O(\epsilon^2)$ are already done in §3. When we consider the gravitational wave luminosity to $O(v^{11})$, we need to calculate $A^{\rm inc}$ to $O(\epsilon^3)$ for $\ell = 2$ and 3 and to $O(\epsilon^2)$ and for $\ell = 4$. Thus we need the closed analytic forms of $\xi_\ell^{(3)}$ for $\ell = 2$ and 3 and $\xi_4^{(2)}$. The latter can be obtained in the same way as in the previous section. The procedure to obtain $\xi_\ell^{(3)}$ is explained in detail in Appendix D.

The real parts of $\xi_\ell^{(3)}$, $f_\ell^{(3)}$, for $\ell = 2$ and 3 are given as

$$\begin{aligned} f_2^{(3)} =& \frac{214\,F_{2,0}[z(\ln z)j_0]}{105} - \frac{107\,D_{-4}^{nj}}{630} - \frac{457\,D_{-2}^{nj}}{70} - \frac{2629\,D_0^{nj}}{630} + \frac{16949\,D_2^{nj}}{4410} \\ & -\frac{107\,(\ln z)\,D_2^{nj}}{105} + (\ln z)^2\,D_2^{nj} - \frac{2\,D_4^{nj}}{49} - 12\,D_{-1}^{nnj} - 18\,D_1^{nnj} + \frac{2\,D_3^{nnj}}{3} \\ & -8\,D_2^{nnnj} - \frac{197\,j_{-3}}{126} + \frac{2539\,j_{-1}}{3780} + \frac{107\,(\ln z)\,j_{-1}}{70} + \frac{3\,(\ln z)^2\,j_{-1}}{2} + \frac{21\,j_1}{100} \\ & +\frac{349\,(\ln z)\,j_1}{140} + \frac{9\,(\ln z)^2\,j_1}{4} - \frac{457\,j_3}{1050} + \frac{29\,(\ln z)\,j_3}{252} - \frac{(\ln z)^2\,j_3}{12} + \frac{j_5}{504}, \end{aligned} \tag{4·14}$$

$$\begin{aligned} f_3^{(3)} =& \frac{26\,F_{3,0}[z(\log z)j_0]}{21} + \frac{13\,D_{-5}^{nj}}{98} + \frac{14075\,D_{-3}^{nj}}{882} \\ & -\frac{1424\,D_{-1}^{nj}}{63} - \frac{2511\,D_1^{nj}}{70} + \frac{269\,D_3^{nj}}{70} - \frac{13\,(\log z)\,D_3^{nj}}{21} \\ & +(\log z)^2\,D_3^{nj} - \frac{D_5^{nj}}{18} + 60\,D_{-2}^{nnj} + 34\,D_0^{nnj} - \frac{710\,D_2^{nnj}}{21} + \frac{6\,D_4^{nnj}}{7} \\ & -8\,D_3^{nnnj} + \frac{75\,j_{-4}}{28} - \frac{19\,j_{-2}}{56} - \frac{65\,(\log z)\,j_{-2}}{14} - \frac{15\,(\log z)^2\,j_{-2}}{2} - \frac{2789\,j_0}{3780} \\ & -\frac{221\,(\log z)\,j_0}{84} - \frac{17\,(\log z)^2\,j_0}{4} - \frac{7495\,j_2}{5292} + \frac{4867\,(\log z)\,j_2}{1764} \\ & +\frac{355\,(\log z)^2\,j_2}{84} - \frac{10963\,j_4}{32340} + \frac{15\,(\log z)\,j_4}{196} - \frac{3\,(\log z)^2\,j_4}{28} + \frac{4\,j_6}{1485}, \end{aligned} \tag{4·15}$$

where the definitions of the functions D_ℓ^{nnj}, etc., are given in Eqs. (D·20) and (D·24) of Appendix D. The imaginary parts $g_\ell^{(3)}$ are expressed in terms of $f_\ell^{(1)}$ and $f_\ell^{(2)}$ as given in Eq. (D·9). As for the real part of $\xi_4^{(2)}$, $f_4^{(2)}$, it is calculated to be

$$\begin{aligned} f_4^{(2)} =& \frac{56}{165\,z}j_5 + \left(-\frac{5036}{33\,z^4} + \frac{30334}{1155\,z^2}\right)j_4 + \left(\frac{35252}{33\,z^5} - \frac{30334}{165\,z^3} + \frac{14401}{3465\,z}\right)j_3 \\ & -\left(\frac{5036}{11\,z^5} + \frac{45461}{693\,z^3} + \frac{36287}{9240\,z}\right)j_1 + \left(\frac{140}{z^3} - \frac{5}{18\,z}\right)n_0 - \frac{49}{6\,z}n_2 \end{aligned}$$

$$
-\frac{21}{5\,z}D_4^{nj}+\frac{149}{30}D_3^{nj}-\frac{10}{3\,z}D_2^{nj}+\frac{105}{z^2}D_1^{nj}+\frac{210}{z^3}D_0^{nj}-\frac{20}{z}D_0^{nj}+\frac{1571}{3465}D_{-5}^{nj}
+4D_4^{nnj}-\frac{1571}{6930}j_4\ln z-\frac{1}{2}j_4(\ln z)^2\,. \qquad (4\cdot 16)
$$

Using the analysis given in §D.4.2 of Appendix D, the above results readily give us the asymptotic forms of $\xi_\ell^{(3)}$ ($\ell=2,3$) and $\xi_4^{(2)}$ at $z\to\infty$, from which the amplitudes $A^{\rm inc}$ to the required order are calculated. The results are

$$
\begin{aligned}
A_2^{\rm inc} &= -\frac{1}{2}ie^{-i\epsilon(\ln 2\epsilon+\gamma)}\exp\left[i\left\{\epsilon\frac{5}{3}-\epsilon^2\frac{107}{420}\pi+\epsilon^3\left(\frac{29}{648}-\frac{107}{1260}\pi^2+\frac{\zeta(3)}{3}\right)+\cdots\right\}\right]\\
&\quad\times\left[1-\epsilon\frac{\pi}{2}+\epsilon^2\left(\frac{25}{18}+\frac{5}{24}\pi^2+\frac{107}{210}(\gamma+\ln 2)\right)\right.\\
&\qquad\left.+\epsilon^3\left(-\frac{25}{36}\pi-\frac{107}{420}(\gamma+\ln 2)\pi-\frac{\pi^3}{16}\right)+\cdots\right],\\
A_3^{\rm inc} &= \frac{1}{2}e^{-i\epsilon(\ln 2\epsilon+\gamma)}\exp\left[i\left\{\epsilon\frac{13}{6}-\epsilon^2\frac{13}{84}\pi+\epsilon^3\left(-\frac{29}{810}-\frac{13}{252}\pi^2+\frac{\zeta(3)}{3}\right)+\cdots\right\}\right]\\
&\quad\times\left[1-\epsilon\frac{\pi}{2}+\epsilon^2\left(\frac{169}{72}+\frac{5}{24}\pi^2+\frac{13}{42}(\gamma+\ln 2)\right)\right.\\
&\qquad\left.+\epsilon^3\left(-\frac{169}{144}\pi-\frac{13}{84}(\gamma+\ln 2)\pi-\frac{\pi^3}{16}\right)+\cdots\right],\\
A_4^{\rm inc} &= \frac{1}{2}ie^{-i\epsilon(\ln 2\epsilon+\gamma)}\exp\left[i\left\{\epsilon\frac{149}{60}-i\epsilon^2\frac{1571}{13860}\pi+\cdots\right\}\right]\\
&\quad\times\left[1-\epsilon\frac{\pi}{2}+\epsilon^2\left(\frac{22201}{7200}+\frac{5}{24}\pi^2+\frac{1571}{6930}(\gamma+\ln 2)\right)+\cdots\right]. \qquad (4\cdot 17)
\end{aligned}
$$

The corresponding amplitudes $B^{\rm inc}$ are readily obtained from Eq. (3·13).

As in the previous section, from Eqs. (4·11) and (4·8), it is also straightforward to obtain the near-zone post-Newtonian expansion of $X^{\rm in}$ and hence of $R^{\rm in}$, assuming $z\ll 1$. As discussed there, we need the series expansion formulas for $R^{\rm in}$ for $2(n+\ell-2)\le 11$, hence for $n\le 7-\ell$ for each $2\le\ell\le 7$. The resulting $R^{\rm in}$ for $2\le\ell\le 7$ which are necessary to calculate the luminosity to $O(v^{11})$ are given in Appendix E.

Finally, from Eq. (4·5), we obtain the luminosity to $O(v^{11})$ as

$$
\begin{aligned}
\left\langle\frac{dE}{dt}\right\rangle = \left(\frac{dE}{dt}\right)_{\rm N}\Biggl[1 &- \frac{1247}{336}v^2+4\pi v^3-\frac{44711}{9072}v^4-\frac{8191\,\pi}{672}v^5\\
&+\left(\frac{6643739519}{69854400}-\frac{1712\,\gamma}{105}+\frac{16\,\pi^2}{3}-\frac{3424\ln 2}{105}-\frac{1712\ln v}{105}\right)v^6\\
&-\frac{16285\,\pi}{504}v^7\\
&+\left(-\frac{323105549467}{3178375200}+\frac{232597\,\gamma}{4410}-\frac{1369\,\pi^2}{126}\right.
\end{aligned}
$$

$$+\frac{39931\ln 2}{294}-\frac{47385\ln 3}{1568}+\frac{232597\ln v}{4410}\bigg)v^8$$
$$+\left(\frac{265978667519\pi}{745113600}-\frac{6848\gamma\pi}{105}-\frac{13696\pi\ln 2}{105}-\frac{6848\pi\ln v}{105}\right)v^9$$
$$+\bigg(-\frac{2500861660823683}{2831932303200}+\frac{916628467\gamma}{7858620}-\frac{424223\pi^2}{6804}$$
$$-\frac{83217611\ln 2}{1122660}+\frac{47385\ln 3}{196}+\frac{916628467\ln v}{7858620}\bigg)v^{10}$$
$$+\bigg(\frac{8399309750401\pi}{101708006400}+\frac{177293\gamma\pi}{1176}$$
$$+\frac{8521283\pi\ln 2}{17640}-\frac{142155\pi\ln 3}{784}+\frac{177293\pi\ln v}{1176}\bigg)v^{11}\bigg], \tag{4·18}$$

where $(dE/dt)_{\rm N}$ is the Newtonian quadrupole luminosity given by

$$\left(\frac{dE}{dt}\right)_{\rm N}=\frac{32\mu^2M^3}{5r_0^5}=\frac{32}{5}\left(\frac{\mu}{M}\right)^2v^{10}. \tag{4·19}$$

To compare the above result with those obtained previously by the standard post-Newtonian method, we note that $v=x\equiv(M\Omega_\varphi)^{1/3}$ in the present case. Then we find our result agrees with the standard post-Newtonian results up to $O(x^5)$ [33)-38)] in the limit $\mu/M\ll 1$. The contributions to the luminosity from individual ℓ modes are given in Appendix E.

§5. Convergence of the post-Newtonian expansion

Using the results obtained in the previous section, we compare the formula for the gravitational wave flux with the corresponding numerical results and investigate the accuracy of the post-Newtonian expansion.

A high precision numerical calculation of gravitational waves from a particle in a circular orbit around a Schwarzschild black hole has been performed by Tagoshi and Nakamura. [10)] Since no assumption was made about the velocity of the test particle, their results are correct relativistically in the limit $\mu\ll M$. In that work, dE/dt was calculated only for $\ell=2\sim 6$. Here, for the orbital radius $r_0\le 100M$, we calculate dE/dt again for all modes of $\ell=2\sim 6$ and for $\ell=7$ with odd m. The estimated accuracy of the calculation is about 10^{-11}, which turns out to be accurate enough for the present purpose. As for the radius $r_0>100M$, we use the data calculated by Tagoshi and Nakamura [10)] which contain modes from $\ell=2$ to 6.

In Figs. 1 and 2, we show the error in the post-Newtonian formulas as a function of the orbital radius r. The error of the post-Newtonian formula is defined as

$$\text{error}=\left|1-\left(\frac{dE}{dt}\right)_n\bigg/\left(\frac{dE}{dt}\right)\right|, \tag{5·1}$$

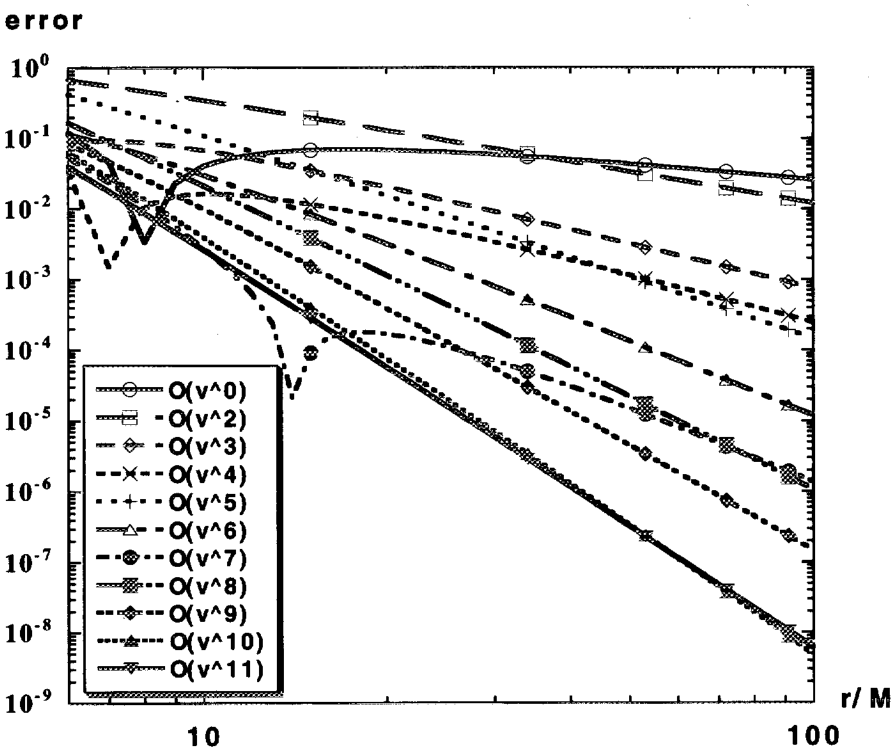

Fig. 1. The error of the post-Newtonian formulas as functions of the Schwarzschild radial coordinate r for $6 \leq r/M \leq 100$. Contributions from $\ell = 2$ to 7 modes are included.

where $(dE/dt)_n$ and (dE/dt) denote the $(n/2)$PN formula and the numerical result, respectively. In the plot of Fig. 2, only the contributions from $\ell = 2$ to 6 are included in both the post-Newtonian formulas and the numerical data. We can see that, at small radius less than $r \sim 10M$, the error of the 1PN and 2.5PN formulas are larger than the other formulas. On the other hand, the Newtonian and the 2PN formulas are very accurate within this radius. This is because those formulas coincide with the exact one accidentally at a radius between $6M$ and $10M$. The error of each post-Newtonian formula at the inner most stable circular orbit, $r = 6M$, becomes as follows: 12% (Newtonian), 66% (1PN), 8.6% (1.5PN), 3.4% (2PN), 42% (2.5PN), 11% (3PN), 5.4% (3.5PN), 17% (4PN), 8.4% (4.5PN), 6.5% (5PN), 4.1% (5.5PN). As is expected, the errors of the post-Newtonian formulas decrease almost linearly up to $r \sim 5000M$ in a log-log plot. This fact also suggests that the numerical data have accuracy of at least $\sim 10^{-18}$ at $r \sim 5000M$.

In order to examine exactly to what order the post-Newtonian formulas are needed to do accurate estimation of the parameters of a binary, using data from laser interferometers, we must evaluate the systematic error produced by incorrect templates. However, here we simply calculate the total cycle of gravitational waves from a coalescing binary in a laser interferometer band and evaluate the error produced by the post-Newtonian formulas. It has been suggested that whether the error

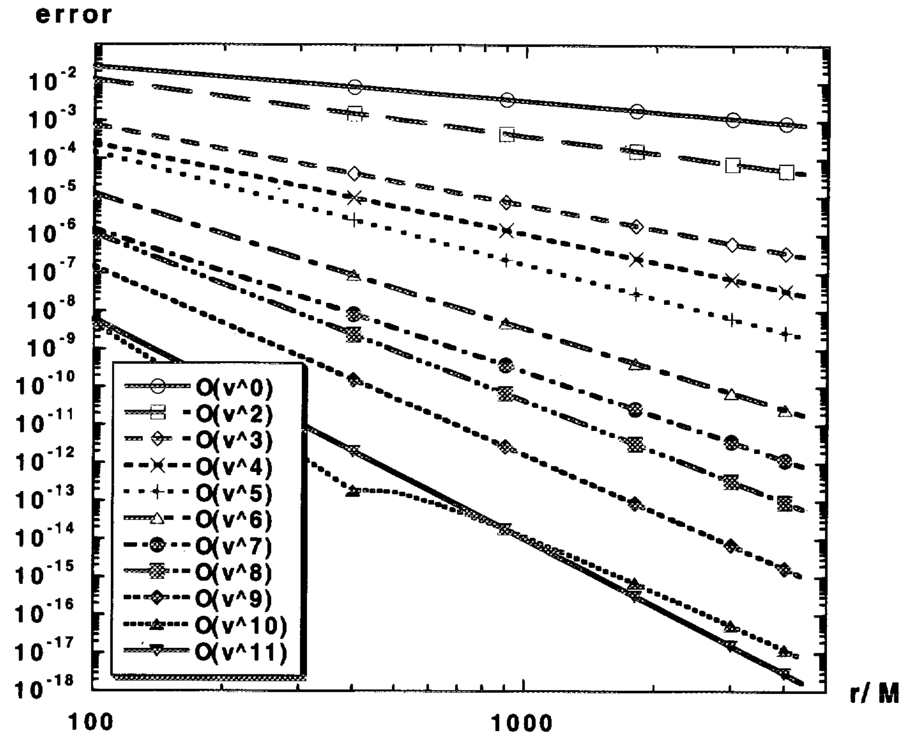

Fig. 2. The same figure for $100 \leq r/M < 5000$. Contributions only from $\ell = 2$ to 6 are included in both the post-Newtonian formulas and in the numerical data.

in the total cycle is less than unity or not gives a useful guideline to examine the accuracy of the post-Newtonian formulas as templates[39] (see also Ref. 40)).

We ignore the finite mass effect in the post-Newtonian formulas and interpret M as the total mass and μ as the reduced mass of the system. The total cycle N of gravitational waves from an inspiraling binary is calculated by using the post-Newtonian energy loss formula $(dE/dt)_n$, and the orbital energy formula $(dE/dv)_n$ which is truncated at $n/2$PN order as

$$N^{(n)} = \int_{v_f}^{v_i} dv \frac{\Omega_\varphi}{\pi} \frac{(dE/dv)_n}{|(dE/dt)_n|}, \tag{5·2}$$

where $v_i = (M/r_i)^{1/2}$, $v_f = (M/r_f)^{1/2}$, and r_i and r_f are the initial and final orbital separations of the binary. We define the relative difference of cycle $\Delta N^{(n)}$ as $\Delta N^{(n)} \equiv |N^{(n)} - N^{(n-1)}|$. We adopt $r_f = 6M$ and r_i is the one at which the frequency of wave is 10Hz and which is given by $r_i/M \sim 347(M_\odot/M)^{2/3}$.

The results for typical binary systems are given in Table I. We only show the results for $q = 0$. The cases for $q \neq 0$ are investigated in Shibata et al.[15] and Tagoshi et al..[16] This table suggests that we need the 3PN$\sim$4PN formula to obtain accurate wave forms for typical binaries whose total mass are less than $20M_\odot$. Although the required post-Newtonian order is very high and it has not been achieved yet in the standard post-Newtonian analysis, this results show that the post-Newtonian

Table I. The relative difference of cycle $\Delta N^{(n)}$ for typical coalescing compact binaries. The last line shows the cycle calculated by Newtonian quadrupole formula.

n	$(1.4M_\odot,1.4M_\odot)$	$(10M_\odot,10M_\odot)$	$(1.4M_\odot,10M_\odot)$	$(1.4M_\odot,70M_\odot)$
2	356	54	216	212
3	228	60	208	296
4	11	5	15	31
5	12	7	20	53
6	11	8	22	75
7	1.2	1.0	2.6	10
8	0.12	0.14	0.3	2.2
9	0.82	0.80	1.9	8.9
10	0.09	0.08	0.20	0.87
11	0.03	0.03	0.07	0.40
$N^{(0)}$	16000	600	3578	898

approximation is applicable to the inspiral phase of coalescing compact binaries. In this sense, we can be optimistic.

On the other hand, the convergence for the case of neutron star-black hole binaries, whose mass is above several ten $M_\odot$, is very slow. This is because r_i/M become smaller for a larger mass black hole, and the higher relativistic correction becomes more important. From Table I, one might think that $N^{(n)}$ converges at $n = 11$ even for $(m_1, m_2) = (1.4M_\odot,70M_\odot)$. However this is not true. Note that Table I shows only the relative difference between the post-Newtonian approximated cycles. If we calculate the difference between the post-Newtonian formula and the fully relativistic one, we find that the 5.5PN formula is not accurate enough for the case $(m_1, m_2) = (1.4M_\odot,70M_\odot)$, as pointed out by Tanaka, Tagoshi and Sasaki.[13]

Finally we comment on the initial frequency. The above results are obtained by setting the initial frequency to 10Hz. However, it may be difficult to observe gravitational wave at this frequency because of the seismic noise. If we set the initial frequency higher than 10Hz, the error ΔN becomes slightly smaller. But since this dependence of ΔN on the initial frequency is very weak, the above results are insensitive to the variation of the initial frequency.

§6. Circular orbit on the equatorial plane around a rotating black hole

In this section, we consider a circular orbit on the equatorial plane of a Kerr black hole and calculate the 4PN luminosity formula.

We define the orbital radius as $r = r_0$. As in §4, we have $C = 0$, and E and l_z are determined by $R(r_0) = 0$ and $\partial R/\partial r\,|_{r=r_0} = 0$ as

$$E = \frac{1 - 2v^2 + qv^3}{(1 - 3v^2 + 2qv^3)^{1/2}}, \quad l_z = \frac{r_0 v(1 - 2qv^3 + q^2v^4)}{(1 - 3v^2 + 2qv^3)^{1/2}}, \tag{6·1}$$

where $v = (M/r_0)^{1/2}$. From these, we can easily obtain $\varphi(t)$ as

$$\varphi(t) = \Omega_\varphi t; \quad \Omega_\varphi = \frac{M^{1/2}}{r_0^{3/2}}\left[1 - qv^3 + q^2v^6 + O(v^9)\right]. \tag{6·2}$$

When $a=0$, this becomes $\Omega_\varphi = (M/r_0^3)^{1/2}$.

The rest of the calculation is almost the same as in §4. The amplitude of the Teukolsky function $\tilde{Z}^\infty_{\ell m\omega}$ at infinity is expressed as

$$\tilde{Z}^\infty_{\ell m\omega} = \mu \frac{2\pi\delta(\omega - m\Omega)}{2i\omega B^{\rm inc}} \Bigg[R^{\rm in}\{A_{n\,n\,0} + A_{\bar{m}\,n\,0} + A_{\bar{m}\,\bar{m}\,0}\} - \frac{dR^{\rm in}}{dr}\{A_{\bar{m}\,n\,1} + A_{\bar{m}\,\bar{m}\,1}\} + \frac{d^2R^{\rm in}}{dr^2} A_{\bar{m}\,\bar{m}\,2}\Bigg]_{r=r_0,\theta=\pi/2} \equiv \delta(\omega - m\Omega) Z^\infty_{\ell m\omega}, \tag{6·3}$$

where A_{nn0}, etc., are given by Eq. (2·25).

The total luminosity up to $O(v^8)$ is expressed as

$$\left\langle \frac{dE}{dt} \right\rangle = \left(\frac{dE}{dt}\right)_{\rm N} \Bigg\{ 1 + (q\text{-independent terms}) - \frac{73\,q}{12} v^3 + + \frac{33\,q^2}{16} v^4 + \frac{3749\,q}{336} v^5 - \left(\frac{169\,\pi\,q}{6} + \frac{3419\,q^2}{168}\right) v^6 + \left(\frac{83819\,q}{1296} + \frac{65\,\pi\,q^2}{8} - \frac{151\,q^3}{12}\right) v^7 + \left(\frac{3389\,\pi\,q}{96} - \frac{124091\,q^2}{9072} \frac{17\,q^4}{16}\right) v^8 \Bigg\}, \tag{6·4}$$

where $(dE/dt)_{\rm N}$ is the Newtonian quadrupole luminosity, Eq. (4·19), and the q-independent terms are identical to those in Eq. (4·18).

From an observational point of view, it is more convenient to express the luminosity in terms of the variable $x \equiv (M\Omega_\varphi)^{1/3}$. Using the relation between $v \equiv (M/r_0)^{1/2}$ and x given by Eq. (6·2), the luminosity is expressed as

$$\left\langle \frac{dE}{dt} \right\rangle = \left(\widetilde{\frac{dE}{dt}}\right)_{\rm N} \Bigg\{ 1 + (q\text{-independent terms}) - \frac{11\,q}{4} x^3 + \frac{33\,q^2}{16} x^4 - \frac{59\,q}{16} x^5 + \left(-\frac{65\,\pi\,q}{6} + \frac{611\,q^2}{504}\right) x^6 + \left(\frac{162035\,q}{3888} + \frac{65\,\pi\,q^2}{8} - \frac{71\,q^3}{24}\right) x^7 + \left(\frac{359\,\pi\,q}{14} + \frac{22667\,q^2}{4536} + \frac{17\,q^4}{16}\right) x^8 \Bigg\}, \tag{6·5}$$

where

$$\left(\widetilde{\frac{dE}{dt}}\right)_{\rm N} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 x^{10}, \tag{6·6}$$

and the q-independent terms are again identical to those in Eq. (4·18) with the replacement $v \to x$. The partial luminosities for individual modes are given in

Appendix F. The spin dependent term at $O(v^3)$ agrees with the standard post-Newtonian result.[41]

Here it is interesting to investigate the origin of some of the spin-dependent terms. As an example, we consider the mode $\ell = |m| = 2$. The luminosity from the $\ell = |m| = 2$ modes is given by

$$\left(\frac{dE}{dt}\right)_{2,2} + \left(\frac{dE}{dt}\right)_{2,-2} = \left(\frac{\widetilde{dE}}{dt}\right)_{\rm N} \Bigg\{ 1 + (q\text{-independent terms}) - \frac{8\,q}{3}\,x^3 + 2\,q^2\,x^4 + \frac{52\,q}{27}\,x^5 + \left(-\frac{32\,\pi\,q}{3} - \frac{1817\,q^2}{567}\right)x^6 + \left(\frac{364856\,q}{11907} + 8\,\pi\,q^2 - \frac{8\,q^3}{3}\right)x^7 + \left(\frac{208\,\pi\,q}{27} + \frac{105022\,q^2}{9261} + q^4\right)x^8 \Bigg\}. \qquad (6\cdot7)$$

We can derive some of the spin-dependent terms in the above formula from the quadrupole formula;[42] $dE/dt = (32/5)\mu^2\hat{r}^4\Omega_\varphi^6$, where $\hat{r}$ is the orbital radius of a test particle in harmonic coordinates. If multipole moments of the black hole exist, the orbital radius changes due to the influence of those moments. The mass and mass current multipole moments of a Kerr black hole is given by $M_l + iS_l = M(ia)^l$. We can express the orbital frequency of the test particle in harmonic coordinates. We find that the dominant effect of the multipole moments of a Kerr black hole to dE/dt can be expressed as

$$\left\langle\frac{dE}{dt}\right\rangle = \left(\frac{\widetilde{dE}}{dt}\right)_{\rm N} \Bigg\{ 1 - \frac{8}{3}\frac{S_1}{M^2}x^3 - 2\frac{M_2}{M^3}x^4 + \left(4\frac{S_3}{M^4} - \frac{4}{3}\frac{M_2S_1}{M^5}\right)x^7 + \left(-\frac{3}{2}\frac{M_2^2}{M^6} + \frac{5}{2}\frac{M_4}{M^5}\right)x^8 \Bigg\}. \qquad (6\cdot8)$$

The terms in this expression agree with the corresponding terms of our result such as $(-8/3)qx^3$, $2q^2x^4$, $(-8/3)q^3x^7$ and q^4x^8. Thus, we may interpret the term $2q^2x^4$ as the effect of the quadrupole moment. The terms $(-8/3)q^3x^7$ and q^4x^8 are not due to a single multipole moment, but to combined effects of the multipole moments.

§7. Slightly eccentric orbit around a Schwarzschild black hole

In this section, we present post-Newtonian formulas of gravitational waves from a particle in slightly eccentric orbits around a Schwarzschild black hole. We derive the 4PN formulas of the energy and angular momentum fluxes to $O(e^2)$ where e is the eccentricity of the orbit.

The solution of the geodesic equations for slightly eccentric orbits has been given by Apostolatos et al..[43] Here we briefly sketch the derivation of it. Since we may consider the orbit to lie on the equatorial plane without loss of generality, we put $\theta = \pi/2$ and $C = 0$ in the geodesic equations (2·18). Then we define a slightly eccentric orbit as follows: First of all, we assume that E and l_z are set to be such that $R(r)/r^4$, which plays the role of an effective potential for the radial motion, has the minimum at $r = r_0$ and that the maximum value of the orbital radius is at $r = r_0(1+e)$, where $e \ll 1$. Thus, the following conditions hold

$$\frac{\partial(R/r^4)}{\partial r}(r = r_0) = 0 \quad \text{and} \quad R(r = r_0(1+e)) = 0. \tag{7·1}$$

From these equations, E and l_z are expressed in terms of r_0 and e as

$$E^2 = \frac{(1-2v^2)^2}{1-3v^2} + \frac{v^2(1-6v^2)}{1-3v^2}e^2 - \frac{2v^2(1-7v^2)}{1-3v^2}e^3 + O(e^4), \tag{7·2}$$

$$l_z = \frac{M}{v\sqrt{1-3v^2}}, \tag{7·3}$$

where $v \equiv \sqrt{M/r_0}$. For convenience, we also define $\Omega \equiv v^3/M$. Then expanding the geodesic equations in powers of e, the solution is found to be [43]

$$\begin{aligned} r(t) = r_0[1 &+ e\cos\Omega_r t \\ &+ e^2\{q_1(v)(1-\cos\Omega_r t) + q_2(v)(1-\cos 2\Omega_r t)\}] + O(e^3), \end{aligned} \tag{7·4}$$

$$\begin{aligned} \varphi(t) = \Omega_\varphi t &- e p_1(v)\sin\Omega_r t \\ &+ e^2\{p_2(v)\sin\Omega_r t + p_3(v)\sin 2\Omega_r t\} + O(e^3), \end{aligned} \tag{7·5}$$

where

$$\Omega_r = \Omega(1-6v^2)^{1/2}, \tag{7·6}$$

$$\Omega_\varphi = \Omega[1 - f(v)e^2]\ ; \quad f(v) = \frac{3(1-3v^2)(1-8v^2)}{2(1-2v^2)(1-6v^2)}, \tag{7·7}$$

$$q_1(v) = \frac{1-7v^2}{1-6v^2}, \quad q_2(v) = \frac{1-11v^2+26v^4}{2(1-6v^2)(1-2v^2)}, \tag{7·8}$$

$$\begin{aligned} p_1(v) &= \frac{2(1-3v^2)}{(1-2v^2)\sqrt{1-6v^2}}, \quad p_2(v) = \frac{2(1-3v^2)(1-7v^2)}{(1-2v^2)(1-6v^2)^{3/2}}, \\ p_3(v) &= \frac{5-64v^2+250v^4-300v^6}{4(1-2v^2)^2(1-6v^2)^{3/2}}. \end{aligned} \tag{7·9}$$

As is well known, since $\Omega_r \neq \Omega_\varphi$, the orbit does not close.

Now we evaluate the source term of the Teukolsky equation. In the present case, $A_{n\,n\,0}$, etc., given in Eqs. (2·25) reduce to

$$A_{n\,n\,0} = -\frac{E}{2\sqrt{2}\Delta}S^{(0)}_{\ell m}\left(1 + \frac{r^2}{\Delta}\frac{dr}{dt}\right)^2, \tag{7·10}$$

$$A_{\bar{m}\,n\,0} = \frac{i l_z}{\sqrt{2\pi}}S^{(1)}_{\ell m}\left(\frac{i\omega}{\Delta} + \frac{2}{r^3}\right)\left(1 + \frac{r^2}{\Delta}\frac{dr}{dt}\right), \tag{7·11}$$

$$A_{\bar{m}\,\bar{m}\,0}=\frac{l_z^2}{2\sqrt{2\pi}E}S^{(2)}_{\ell m}\left(\frac{2i\omega(r-M)}{\Delta r^2}-\frac{\omega^2}{\Delta}\right), \tag{7·12}$$

$$A_{\bar{m}\,n\,1}=\frac{il_z}{\sqrt{2\pi}r^2}S^{(1)}_{\ell m}\left(1+\frac{r^2}{\Delta}\frac{dr}{dt}\right), \tag{7·13}$$

$$A_{\bar{m}\,\bar{m}\,1}=\frac{l_z^2}{\sqrt{2\pi}E}S^{(2)}_{\ell m}\left(\frac{i\omega}{r^2}+\frac{\Delta}{r^5}\right), \tag{7·14}$$

$$A_{\bar{m}\,\bar{m}\,2}=\frac{l_z^2\Delta}{2\sqrt{2\pi}Er^4}S^{(2)}_{\ell m}, \tag{7·15}$$

where

$$S^{(0)}_{\ell m}=L_1^\dagger L_2^\dagger S_{\ell m}(\theta=\pi/2), \tag{7·16}$$

$$S^{(1)}_{\ell m}=L_2^\dagger S_{\ell m}(\theta=\pi/2), \tag{7·17}$$

$$S^{(2)}_{\ell m}=S_{\ell m}(\theta=\pi/2). \tag{7·18}$$

Then noting that the orbits of our interest have the properties,

$$r(t+\Delta t_r)=r(t), \qquad \left.\frac{d\varphi}{dt}\right|_{t=t+\Delta t_r}=\left.\frac{d\varphi}{dt}\right|_{t=t}, \tag{7·19}$$

where $\Delta t_r=2\pi/\Omega_r$, Eq. (2·26) can be rewritten as

$$\begin{aligned}\tilde{Z}_{\ell m\omega}&=\frac{\mu}{2i\omega B^{\rm inc}_{\ell m\omega}}\int_{-\infty}^{\infty}dte^{i\omega t-im\varphi(t)}W_{\ell m\omega}\\ &=\frac{\mu}{2i\omega B^{\rm inc}_{\ell m\omega}}\frac{2\pi}{\Delta t_r}\int_0^{\Delta t_r}dte^{i\omega t-im\varphi(t)}W_{\ell m\omega}\sum_n\delta(\omega-\omega_n),\end{aligned} \tag{7·20}$$

where

$$\omega_n=n\Omega_r+m\Omega_\varphi. \quad (n=0,\pm1,\pm2,\cdots) \tag{7·21}$$

We see that $\tilde{Z}_{\ell m\omega}$ takes the form as given in Eq. (2·28) with $Z_{\ell m\omega}$ given by

$$Z_{\ell m\omega}=\frac{\mu\Omega_r}{2i\omega B^{\rm inc}_{\ell m\omega}}\int_0^{\Delta t_r}dte^{i\omega t-im\varphi(t)}W_{\ell m\omega}. \tag{7·22}$$

When $Z_{\ell m\omega_n}$ are obtained, the energy and angular momentum fluxes averaged over $t\gg\Delta t_r$ are calculated by using Eqs. (2·31) and (2·32), respectively. Here, we show these fluxes accurate to $O(e^2)$ and to $O(v^8)$ beyond Newtonian:

$$\left\langle\frac{dE}{dt}\right\rangle=\dot{E}^{(0)}+e^2\dot{E}^{(2)}+O(e^3), \tag{7·23}$$

$$\left\langle\frac{dJ_z}{dt}\right\rangle=\dot{J}_z^{(0)}+e^2\dot{J}_z^{(2)}+O(e^3). \tag{7·24}$$

We note that $\dot{J}_z^{(0)}=\dot{E}^{(0)}/\Omega$ where $\Omega=v^3/M$. We have already given the 5.5PN formula for $\dot{E}^{(0)}$ in §4, Eq. (4·18). Hence our task here is to evaluate the $O(e^2)$

corrections. From the forms of $r(t)$ and $\varphi(t)$ given in Eqs. (7·5), we readily see that $Z_{\ell m\omega_n} = O(e^{|n|})$ for $n = \pm 1, \pm 2$. Thus, we only need the modes $n = 0, \pm 1$: As for the $n = 0$ modes, we must retain terms up to $O(e^2)$, while for the $n = \pm 1$ modes, we only need terms up to $O(e)$. Then the 4PN formulas for $\dot{E}^{(2)}$ and $\dot{J}_z^{(2)}$ are found as

$$\begin{aligned} e^2 \dot{E}^{(2)}_{\rm PN} =& e^2 \left(\frac{dE}{dt}\right)_{\rm N} \\ & \times \Bigg\{ \frac{37}{24} - \frac{65\, v^2}{21} + \frac{1087\, \pi\, v^3}{48} - \frac{465337\, v^4}{9072} - \frac{118607\, \pi\, v^5}{1344} - \frac{17328779\, \pi\, v^7}{48384} \\ & + \left(\frac{98546617999}{69854400} - \frac{65056\, \gamma}{315} + \frac{608\, \pi^2}{9} + \frac{1712 \ln 2}{315} - \frac{234009 \ln 3}{560} \right. \\ & \left. - \frac{65056 \ln v}{315} \right) v^6 + \left(- \frac{6653525574791}{2118916800} + \frac{118015\, \gamma}{98} - \frac{34093\, \pi^2}{126} \right. \\ & \left. - \frac{1035547 \ln 2}{4410} + \frac{3986901 \ln 3}{1120} + \frac{118015 \ln v}{98} \right) v^8 \Bigg\}, \end{aligned} \tag{7·25}$$

and

$$\begin{aligned} e^2 \dot{J}^{(2)}_{z\,\rm PN} =& \frac{e^2}{\Omega} \left(\frac{dE}{dt}\right)_{\rm N} \\ & \times \Bigg\{ -\frac{5}{8} + \frac{749\, v^2}{96} + \frac{49\, \pi\, v^3}{8} - \frac{232181\, v^4}{6048} + \frac{773\, \pi\, v^5}{336} - \frac{300637\, \pi\, v^7}{1008} \\ & + \left(\frac{8017536229}{12700800} - \frac{19367\, \gamma}{210} + \frac{181\, \pi^2}{6} \right. \\ & \left. + \frac{20009 \ln 2}{210} - \frac{78003 \ln 3}{280} - \frac{19367 \ln v}{210} \right) v^6 \\ & + \left(- \frac{12713730793}{61122600} + \frac{3463711\, \gamma}{8820} - \frac{14675\, \pi^2}{252} \right. \\ & \left. - \frac{2312441 \ln 2}{980} + \frac{35449083 \ln 3}{15680} + \frac{3463711 \ln v}{8820} \right) v^8 \Bigg\}. \end{aligned} \tag{7·26}$$

In Appendix G, we show each (ℓ, m, n) component of the energy and angular momentum fluxes. Note that there was an error in the coefficients of the $e^2 v^4$ terms in Ref. 17). This error is corrected in Eqs. (7·25) and (7·26) above.

To express the energy and angular momentum fluxes in terms of the variable $x = (M\Omega_\varphi)^{1/3}$, we use Eq. (7·7). To $O(e^2)$, it can be easily solved for v as

$$v = x \left[1 + \frac{1}{3} f(x) e^2 + O(e^3) \right]. \tag{7·27}$$

Then to $O(x^8)$ the energy and angular momentum fluxes are expressed as

$$\left\langle \frac{dE}{dt} \right\rangle = \left(\frac{\widetilde{dE}}{dt}\right)_{\rm N} \Bigg[1 + (e\text{-independent terms}) + e^2 \left(\frac{157}{24} - \frac{6781\,x^2}{168} + \frac{2335\,\pi\,x^3}{48} - \frac{14929\,x^4}{189} - \frac{773\,\pi\,x^5}{3} + \frac{156066596771\,x^6}{69854400} - \frac{106144\,\gamma\,x^6}{315} + \frac{992\,\pi^2\,x^6}{9} - \frac{80464\,x^6\,\ln 2}{315} - \frac{234009\,x^6\,\ln 3}{560} - \frac{106144\,x^6\,\ln x}{315} - \frac{32443727\,\pi\,x^7}{48384} - \frac{3045355111074427\,x^8}{671272842240} + \frac{507208\,\gamma\,x^8}{245} - \frac{31271\,\pi^2\,x^8}{63} - \frac{151336\,x^8\,\ln 2}{441} + \frac{12887991\,x^8\,\ln 3}{3920} + \frac{507208\,x^8\,\ln x}{245} \right) \Bigg], \tag{7·28}$$

and

$$\left\langle \frac{dJ_z}{dt} \right\rangle = \left(\frac{\widetilde{dJ_z}}{dt}\right)_{\rm N} \Bigg[1 + (e\text{-independent terms}) + e^2 \left(\frac{23}{8} - \frac{3259\,x^2}{168} + \frac{209\,\pi\,x^3}{8} - \frac{1041349\,x^4}{18144} - \frac{785\,\pi\,x^5}{6} + \frac{91721955203\,x^6}{69854400} - \frac{41623\,\gamma\,x^6}{210} + \frac{389\,\pi^2\,x^6}{6} - \frac{24503\,x^6\,\ln 2}{210} - \frac{78003\,x^6\,\ln 3}{280} - \frac{41623\,x^6\,\ln x}{210} - \frac{91565\,\pi\,x^7}{168} - \frac{105114325363\,x^8}{72648576} + \frac{696923\,\gamma\,x^8}{630} - \frac{4387\,\pi^2\,x^8}{18} - \frac{7051\,x^8\,\ln 2}{10} + \frac{3986901\,x^8\,\ln 3}{1960} + \frac{696923\,x^8\,\ln x}{630} \right) \Bigg], \tag{7·29}$$

where $(\widetilde{dJ_z/dt})_{\rm N}$ is the Newtonian angular momentum flux expressed in terms of x,

$$\left(\frac{\widetilde{dJ_z}}{dt}\right)_{\rm N} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 M x^7, \tag{7·30}$$

and the e-independent terms in both $\langle dE/dt \rangle$ and $\langle dJ_z/dt \rangle$ are the same and are given by the terms in the case of circular orbit, Eq. (4·18), with the replacement $v \to x$.

Finally, we consider the stability of circular orbits. We note that the following relation holds:

$$\left\langle \frac{dE}{dt} \right\rangle - \Omega_\varphi \left\langle \frac{dJ_z}{dt} \right\rangle = \sum_{\ell,m} \frac{m\Omega_r}{4\pi} \left(\frac{|Z_{\ell m}^{(+)}|^2}{\omega_+^3} - \frac{|Z_{\ell m}^{(-)}|^2}{\omega_-^3} \right) e^2 + O(e^4) \equiv H(v)e^2 + O(e^4), \tag{7·31}$$

where $\omega_\pm = m\Omega_\varphi \pm \Omega_r$ and $Z_{\ell m}^{(\pm)} = Z_{\ell m \omega_\pm}$. Here $H(v)$ is an important quantity which determines the stability of circular orbits under the radiation reaction. Assuming the adiabatic evolution of the orbit, the evolution equations for r_0 and e due to the gravitational radiation reaction are written as[43]

$$\mu \frac{dr_0}{dt} = -\frac{2M(1-3v^2)^{3/2}}{v^4(1-6v^2)} \dot{E}^{(0)} + O(e^2), \tag{7·32}$$

$$\mu \frac{d\ln e}{dt} = -\frac{(1-2v^2)(1-3v^2)^{1/2}}{v^2(1-6v^2)} \left[g(v)\dot{E}^{(0)} + G(v) \right] + O(e), \tag{7·33}$$

where

$$g(v) = \frac{2 - 27v^2 + 72v^4 - 36v^6}{2(1-2v^2)^2(1-6v^2)}, \tag{7·34}$$

$$G(v) = \dot{E}^{(2)} - \Omega \dot{J}_z^{(2)} = H(v) - f(v)\dot{E}^{(0)}. \tag{7·35}$$

Using Eqs. (7·25) and (7·26), the 4PN formula of $G(v)$ is calculated as

$$\begin{aligned} G_{\rm PN}(v) = \left(\frac{dE}{dt}\right)_{\rm N} \Bigg[& \frac{13}{6} - \frac{2441\,v^2}{224} + \frac{793\,\pi\,v^3}{48} - \frac{234131\,v^4}{18144} - \frac{121699\,\pi\,v^5}{1344} \\ & - \frac{414029\,\pi\,v^7}{6912} + \left(\frac{36300112493}{46569600} - \frac{72011\,\gamma}{630} + \frac{673\,\pi^2}{18} \right. \\ & \left. - \frac{56603\,\ln 2}{630} - \frac{78003\,\ln 3}{560} - \frac{72011\,\ln v}{630} \right) v^6 \\ & + \left(- \frac{18638348721901}{6356750400} + \frac{7157639\,\gamma}{8820} - \frac{17837\,\pi^2}{84} \right. \\ & \left. + \frac{1085179\,\ln 2}{1764} + \frac{20367531\,\ln 3}{15680} - \frac{133120\,\ln 4}{441} + \frac{7157639\,\ln v}{8820} \right) v^8 \Bigg]. \end{aligned} \tag{7·36}$$

Note that $[g(v)\dot{E}^{(0)} + G(v)]/\dot{E}^{(0)} \to 19/6$ for $v \to 0$; i.e., in the Newtonian limit, the radiation reaction always reduces the eccentricity.[44] By a numerical calculation, Apostolatos et al.[43] found that there exists a critical radius r_c below which the circular orbit becomes unstable; $r_c \simeq 6.6792M$. On the other hand, we find the use of the 4PN formulas for $\dot{E}^{(0)}$ and $G(v)$ gives $r_c \sim 7.38M$. This indicates that a much higher order PN formula will be necessary to determine r_c with good accuracy.

§8. Slightly eccentric orbit around a rotating black hole

In this section, we consider a slightly eccentric orbit on the equatorial plane of a Kerr black hole and calculate the leading order corrections of the eccentricity to the energy and angular momentum fluxes up to $O(v^5)$ beyond Newtonian. The calculation is parallel to the one given in the previous section.

We consider the motion of a particle in the equatorial plane $\theta = \pi/2$, hence we have $C = 0$. We define the radius $r = r_0$ as the one at which the potential R/r^4 is minimum; $\partial(R/r^4)/\partial r \mid_{r=r_0} = 0$. We define the eccentricity e such that $r = r_0(1+e)$ is a turning point of the radial motion at which $R(r = r_0(1+e)) = 0$. We assume $e \ll 1$. Using these definitions of r_0 and e, E and l_z are expressed as

$$E = E^{(0)} + eE^{(1)} + e^2E^{(2)} + O(e^3),$$
$$l_z = l_z^{(0)} + el_z^{(1)} + e^2 l_z^{(2)} + O(e^3),$$

where $E^{(n)}$ and $l_z^{(n)}$ $(n = 0, 1, 2)$ are given by

$$E^{(0)} = \frac{1 - 2v^2 + qv^3}{(1 - 3v^2 + 2qv^3)^{(1/2)}},$$
$$E^{(1)} = 0,$$
$$E^{(2)} = \frac{v^2(1 - 3v^2 + qv^3 + q^2v^4)(-1 + 6v^2 - 8qv^3 + 3q^2v^4)}{2(1 - 3v^2 + 2qv^3)^{3/2}(-1 + 2v^2 - q^2v^4)},$$
$$l_z^{(0)} = \frac{r_0 v(1 - 2qv^3 + q^2v^4)}{(1 - 3v^2 + 2qv^3)^{(1/2)}},$$
$$l_z^{(1)} = 0,$$
$$l_z^{(2)} = \frac{q\, r_0\, v^5(q - 3v + qv^2 + q^2v^3)(-1 + 6v^2 - 8qv^3 + 3q^2v^4)}{2(1 - 3v^2 + 2qv^3)^{3/2}(-1 + 2v^2 - q^2v^4)},$$

where $v = (M/r_0)^{1/2}$. The post-Newtonian expansions of $E^{(n)}$ and $l_z^{(n)}$ up to the required order are

$$E = 1 - \frac{M}{2r_0} + \frac{3M^2}{8r_0^2} - \frac{qM^{5/2}}{r_0^{5/2}} + e^2\left(\frac{M}{2r_0} - \frac{5M^2}{4r_0^2} + \frac{3qM^{5/2}}{r_0^{5/2}}\right) + O(v^6), \quad (8{\cdot}1)$$

$$l_z = (Mr_0)^{1/2}\left[1 + \frac{3M}{2r_0} - \frac{3qM^{3/2}}{r_0^{3/2}} + \left(\frac{27}{8} + q^2\right)\frac{M^2}{r_0^2} - \frac{15qM^{5/2}}{2r_0^{5/2}} + e^2\left(\frac{q^2M^2}{2r_0^2} - \frac{3qM^{5/2}}{2r_0^{5/2}}\right) + O(v^6)\right]. \quad (8{\cdot}2)$$

Now we solve the geodesic equations for a slightly eccentric orbit. The radial equation is

$$\left(\frac{dr}{dt}\right)^2 = \frac{R}{T^2}. \quad (8{\cdot}3)$$

We expand $r(t)$ as

$$r(t) = r_0 \left[1 + e r^{(1)}(t) + e^2 r^{(2)}(t) + O(e^3)\right], \tag{8·4}$$

and R/T^2 in terms of e and v using Eqs. (8·1) and (8·2). Collecting terms of the equal order in e, we obtain

$$\left(\frac{dr^{(1)}}{dt}\right)^2 = \Omega_r^2(1 - (r^{(1)})^2), \tag{8·5}$$

and

$$\frac{1}{\Omega_r^2}\frac{dr^{(1)}}{dt}\frac{dr^{(2)}}{dt} = -r^{(1)}r^{(2)} + q_0 + q_1 r^{(1)} + q_2 (r^{(1)})^2, \tag{8·6}$$

where Ω_r, q_0, q_1 and q_2 are given in the post-Newtonian series forms as

$$\Omega_r = \frac{M^{1/2}}{r_0^{3/2}}\left[1 - \frac{3M}{r_0} + \frac{3qM^{3/2}}{r_0^{3/2}} - \frac{(9+3q^2)M^2}{2r_0^2} + \frac{15qM^{5/2}}{r_0^{5/2}} + O(v^6)\right], \tag{8·7}$$

$$q_0 = -1 + \frac{M}{r_0} - \frac{2qM^{3/2}}{r_0^{3/2}} + \frac{(6+q^2)M^2}{r_0^2} - \frac{20qM^{5/2}}{r_0^{5/2}} + O(v^6), \tag{8·8}$$

$$q_1 = \frac{2M}{r_0}\left[1 + \frac{2M}{r_0} - \frac{3qM^{3/2}}{r_0^{3/2}} + \frac{4M^2}{r_0^2} - \frac{6qM^{5/2}}{r_0^{5/2}} + O(v^6)\right], \tag{8·9}$$

$$q_2 = 1 - \frac{3M}{r_0} + \frac{2qM^{3/2}}{r_0^{3/2}} - \frac{(10+q^2)M^2}{r_0^2} + \frac{26qM^{5/2}}{r_0^{5/2}} + O(v^6). \tag{8·10}$$

We obtain $r^{(1)}(t)$ from Eq. (8·5) as

$$r^{(1)}(t) = \cos \Omega_r t, \tag{8·11}$$

where we set $r(t=0) = r_0(1+e)$. Substitution of Eq. (8·11) into Eq. (8·6) and yields after integration

$$r^{(2)}(t) = q_3(1 - \cos \Omega_r t) + q_4(1 - \cos 2\Omega_r t), \tag{8·12}$$

where $q_3 = -q_0$ and $q_4 = q_2/2$.

In the same way, we can solve the angular motion $\varphi(t)$. From Eq. (2·18), we have $d\varphi/dt = \Phi/T$, which can be expanded in terms of e using Eqs. (8·1), (8·2), (8·4), (8·11) and (8·12). Integrating the resulting equation, we obtain

$$\varphi(t) = \Omega_\varphi t + e p_1 \sin \Omega_r t + e^2 p_2 \sin \Omega_r t + e^2 p_3 \sin 2\Omega_r t + O(e^3), \tag{8·13}$$

where

$$p_1 = -2 - \frac{4M}{r_0} + \frac{6qM^{3/2}}{r_0^{3/2}} - \frac{(17+q^2)M^2}{r_0^2} + \frac{48qM^{5/2}}{r_0^{5/2}} + O(v^6),$$

$$p_2 = 2 + \frac{2M}{r_0} - \frac{2qM^{3/2}}{r_0^{3/2}} + \frac{(1-q^2)M^2}{r_0^2} + \frac{6qM^{5/2}}{r_0^{5/2}} + O(v^6),$$
$$p_3 = \frac{5}{4} + \frac{M}{4r_0} - \frac{2qM^{3/2}}{r_0^{3/2}} - \frac{(9+7q^2)M^2}{8r_0^2} + \frac{59(-1+q^2)M^3}{8r_0^3} + O(v^6), \tag{8·14}$$

and

$$\Omega_\varphi = \left(\frac{M}{r_0}\right)^{1/2}\left[1 - \frac{q\,M^{3/2}}{{r_0}^{3/2}} + e^2\left(-\frac{3}{2} + \frac{9M}{2r_0} - \frac{9qM^{3/2}}{2r_0^{3/2}} + \frac{3\,(6+q^2)\,M^2}{r_0^2} - \frac{60qM^{5/2}}{r_0^{5/2}}\right) + O(v^6)\right]. \tag{8·15}$$

As in the case of the previous section, the fact that $\Omega_r \neq \Omega_\varphi$ implies that these eccentric orbits are not closed.

Using the above solution of the geodesic equations, we evaluate the source term of the Teukolsky equation. We set $\theta = \pi/2$ in the expressions of $A_{n\,n\,0}$, etc., in Eqs. (2·25). Again, parallel to the discussion in §7, the orbits of our interest have the properties,

$$r(t+\Delta t_r) = r(t), \qquad \left.\frac{d\varphi}{dt}\right|_{t=t+\Delta t_r} = \left.\frac{d\varphi}{dt}\right|_{t=t}, \tag{8·16}$$

where $\Delta t_r = 2\pi/\Omega_r$. Hence Eq. (2·26) reduces to the form

$$\tilde{Z}_{\ell m\omega} = Z_{\ell m\omega}\delta(\omega - \omega_n), \tag{8·17}$$

where

$$\omega_n = n\Omega_r + m\Omega_\varphi, \quad (n = 0, \pm1, \pm2, \cdots) \tag{8·18}$$

and

$$Z_{\ell m\omega} = \frac{\mu\Omega_r}{2i\omega B^{\rm inc}_{\ell m\omega}}\int_0^{\Delta t_r} dt e^{i\omega t - im\varphi(t)} W_{\ell m\omega} \tag{8·19}$$

with $W_{\ell m\omega}$ given by Eq. (2·27).

Using the solution of the geodesic equation for $r(t)$, we expand $W_{\ell m\omega}$ in terms of e. The result takes the form

$$W_{\ell m\omega} = f_0 + e\left(f_1 r^{(1)} + f_2\frac{dr^{(1)}}{dt}\right) + e^2\left(f_3 r^{(2)} + f_4\frac{dr^{(2)}}{dt} + f_5(r^{(1)})^2 + f_6\left(\frac{dr^{(1)}}{dt}\right)^2 + f_7 r^{(1)}\frac{dr^{(1)}}{dt} + f_8\right) + O(e^3), \tag{8·20}$$

where $f_0 \sim f_8$ are time-independent coefficients. Inserting this form to Eq. (8·19) we obtain

$$
\begin{aligned}
Z_{\ell m\omega_n} = \frac{\mu\pi}{i\omega_n B^{\rm inc}_{\ell m\omega_n}} \Bigg[\Bigg\{ & f_0 + e^2 \left(\frac{f_5}{2} + f_8 - \frac{m^2 p_1^2}{4} f_0 \right. \\
& \left. +(q_3+q_4) f_3 + \frac{i m \Omega_r p_1}{2} f_2 + \frac{\Omega_r^2}{2} f_6 \right) \Bigg\} \delta_{n,0} \\
& + e \left(\frac{f_1}{2} + \frac{m p_1}{2} f_0 - \frac{i\Omega_r}{2} f_2 \right) \delta_{n,1} \\
& + e \left(\frac{f_1}{2} - \frac{m p_1}{2} f_0 + \frac{i\Omega_r}{2} f_2 \right) \delta_{n,-1} \\
& + e^2 \left(\frac{3 f_5}{4} + f_8 + \frac{f_1\, m\, p_1}{4} - \frac{f_0\, m^2\, {p_1}^2}{8} + \frac{f_0\, m\, p_3}{2} + f_3\, q_3 \right. \\
& \left. + \frac{f_3\, q_4}{2} - \frac{i}{4} f_7\, w + \frac{i}{4} f_2\, m\, p_1\, w + i\, f_4\, q_4\, w + \frac{f_6\, w^2}{4} \right) \delta_{n,2} \\
& + e^2 \left(\frac{3 f_5}{4} + f_8 - \frac{f_1\, m\, p_1}{4} - \frac{f_0\, m^2\, {p_1}^2}{8} - \frac{f_0\, m\, p_3}{2} + f_3\, q_3 \right. \\
& \left. + \frac{f_3\, q_4}{2} + \frac{i}{4} f_7\, w + \frac{i}{4} f_2\, m\, p_1\, w - i\, f_4\, q_4\, w + \frac{f_6\, w^2}{4} \right) \delta_{n,-2} \Bigg] , \qquad (8·21)
\end{aligned}
$$

where $\delta_{n,n'}$ is the Kronecker delta. We see from this equation that $Z_{\ell m\omega_n} = O(e^{|n|})$ just as in the Schwarzschild case. Therefore we only need to retain the $n = 0, \pm 1$ modes to evaluate the luminosity up to $O(e^2)$.

We calculate the energy and angular momentum fluxes to $O(v^5)$ beyond the quadrupole formula and to $O(e^2)$ in the eccentricity. The time-averaged energy and angular momentum fluxes are given by Eqs. (2·31) and (2·32), respectively. In order to express the post-Newtonian corrections to the luminosity, we define $\eta_{\ell mn}$ as

$$\left(\frac{dE}{dt}\right)_{\ell mn} \equiv \frac{1}{2}\left(\frac{dE}{dt}\right)_{\rm N} \eta_{\ell,m,n}\,, \qquad (8·22)$$

where $(dE/dt)_{\rm N}$ is the Newtonian quadrupole luminosity given by Eq. (4·19). In the following, we show $\eta_{\ell mn}$ for $m \ge 0$ modes. $\eta_{\ell,m,n}$ for $m < 0$ are obtained from the symmetry $\eta_{\ell,m,n} = \eta_{\ell,-m,-n}$, which follows from the property of ω_n in the present case, given by Eq. (7·21).

For $\ell = 2$, the 2.5PN formulas for $\eta_{\ell,m,n}$ are found to be*)

$$
\begin{aligned}
\eta_{2,2,0} = 1 & - \frac{107\, v^2}{21} + 4\,\pi\, v^3 - 6\, q\, v^3 + \frac{4784\, v^4}{1323} + 2\, q^2\, v^4 \\
& - \frac{428\,\pi\, v^5}{21} + \frac{4216\, q\, v^5}{189}
\end{aligned}
$$

*) As mentioned in the previous section, we have detected an error in the formula for $\eta_{2,2,0}$ in Ref. 17). The term $-14270/147 e^2 v^4$ there is in correct. Accordingly, formulas for dE/dt and dJ_z/dt below are also corrected in this paper.

$$+e^2\left(-10+\frac{932\,v^2}{21}-46\,\pi\,v^3+84\,q\,v^3-\frac{14270\,v^4}{147}-23\,q^2\,v^4\right.$$
$$\left.+\frac{4748\,\pi\,v^5}{21}+\frac{2675\,q\,v^5}{189}\right),$$
$$\eta_{2,2,1}=e^2\left(\frac{729}{64}-\frac{3645\,v^2}{64}+\frac{2187\,\pi\,v^3}{32}-\frac{3645\,q\,v^3}{32}\right.$$
$$\left.+\frac{24057\,v^4}{256}+\frac{2187\,q^2\,v^4}{64}-\frac{6561\,\pi\,v^5}{16}+\frac{9477\,q\,v^5}{112}\right),$$
$$\eta_{2,2,-1}=e^2\left(\frac{9}{64}+\frac{1041\,v^2}{448}+\frac{9\,\pi\,v^3}{32}-\frac{153\,q\,v^3}{32}\right.$$
$$\left.+\frac{2224681\,v^4}{112896}+\frac{99\,q^2\,v^4}{64}+\frac{615\,\pi\,v^5}{112}-\frac{27857\,q\,v^5}{336}\right),$$
$$\eta_{2,1,0}=\frac{v^2}{36}-\frac{q\,v^3}{12}-\frac{17\,v^4}{504}+\frac{q^2\,v^4}{16}+\frac{\pi\,v^5}{18}-\frac{793\,q\,v^5}{9072}$$
$$+e^2\left(\frac{-2\,v^2}{9}+\frac{2\,q\,v^3}{3}+\frac{93\,v^4}{112}-\frac{q^2\,v^4}{2}-\frac{19\,\pi\,v^5}{36}-\frac{27113\,q\,v^5}{18144}\right),$$
$$\eta_{2,1,1}=e^2\left(\frac{4\,v^2}{9}-\frac{4\,q\,v^3}{3}-\frac{172\,v^4}{63}+q^2\,v^4+\frac{16\,\pi\,v^5}{9}+\frac{2794\,q\,v^5}{567}\right),$$
$$\eta_{2,0,\pm1}=e^2\left(\frac{1}{96}-\frac{145\,v^2}{672}+\frac{\pi\,v^3}{48}+\frac{3\,q\,v^3}{16}+\frac{282521\,v^4}{169344}-\frac{3\,q^2\,v^4}{32}\right.$$
$$\left.-\frac{83\,\pi\,v^5}{168}-\frac{1255\,q\,v^5}{504}\right),$$

and $\eta_{2,1,-1}$ becomes $O(v^6)$. Putting together the above results, we obtain $(dE/dt)_\ell \equiv \sum_{mn}(dE/dt)_{\ell mn}$ for $\ell=2$ as

$$\left(\frac{dE}{dt}\right)_2=\left(\frac{dE}{dt}\right)_{\rm N}\left\{1-\frac{1277\,v^2}{252}+4\,\pi\,v^3-\frac{73\,q\,v^3}{12}+\frac{37915\,v^4}{10584}\right.$$
$$+\frac{33\,q^2\,v^4}{16}-\frac{2561\,\pi\,v^5}{126}+\frac{201575\,q\,v^5}{9072}$$
$$+e^2\left(\frac{37}{24}-\frac{2581\,v^2}{252}+\frac{1087\,\pi\,v^3}{48}-\frac{211\,q\,v^3}{6}+\frac{325393\,v^4}{21168}+\frac{105\,q^2\,v^4}{8}\right.$$
$$\left.\left.-\frac{29857\,\pi\,v^5}{168}+\frac{11293\,q\,v^5}{672}\right)\right\}. \qquad (8\cdot23)$$

For $\ell=3$, we obtain

$$\eta_{3,3,0}=\frac{1215\,v^2}{896}-\frac{1215\,v^4}{112}+\frac{3645\,\pi\,v^5}{448}-\frac{1215\,q\,v^5}{112}$$

$$+e^2\left(\frac{-10935\,v^2}{448}+\frac{37665\,v^4}{256}-\frac{142155\,\pi\,v^5}{896}+\frac{134865\,q\,v^5}{448}\right),$$

$$\eta_{3,3,1}=e^2\left(\frac{640\,v^2}{21}-\frac{46720\,v^4}{189}+\frac{5120\,\pi\,v^5}{21}-\frac{1280\,q\,v^5}{3}\right),$$

$$\eta_{3,3,-1}=e^2\left(\frac{15\,v^2}{14}+\frac{1055\,v^4}{126}+\frac{30\,\pi\,v^5}{7}-\frac{435\,q\,v^5}{14}\right),$$

$$\eta_{3,2,0}=\frac{5\,v^4}{63}-\frac{40\,q\,v^5}{189}+e^2\left(\frac{-65\,v^4}{63}+\frac{520\,q\,v^5}{189}\right),$$

$$\eta_{3,2,1}=e^2\left(\frac{3645\,v^4}{1792}-\frac{1215\,q\,v^5}{224}\right),$$

$$\eta_{3,2,-1}=e^2\left(\frac{5\,v^4}{1792}-\frac{5\,q\,v^5}{672}\right),$$

$$\eta_{3,1,0}=\frac{v^2}{8064}-\frac{v^4}{1512}+\frac{\pi\,v^5}{4032}-\frac{17\,q\,v^5}{9072}$$
$$+e^2\left(\frac{-v^2}{4032}+\frac{65\,v^4}{16128}-\frac{\pi\,v^5}{1152}+\frac{199\,q\,v^5}{36288}\right),$$

$$\eta_{3,1,1}=e^2\left(\frac{v^2}{126}-\frac{23\,v^4}{126}+\frac{2\,\pi\,v^5}{63}+\frac{122\,q\,v^5}{567}\right),$$

$$\eta_{3,0,\pm1}=e^2\left(\frac{v^4}{2688}-\frac{q\,v^5}{1008}\right),$$

and $\eta_{3,1,-1}$ becomes $O(v^6)$. Thus we obtain

$$\left(\frac{dE}{dt}\right)_3=\left(\frac{dE}{dt}\right)_{\rm N}\left\{\frac{1367\,v^2}{1008}-\frac{32567\,v^4}{3024}+\left(\frac{16403\,\pi}{2016}-\frac{896\,q}{81}\right)v^5\right.$$
$$\left.+e^2\left(\frac{1801\,v^2}{252}-\frac{78509\,v^4}{864}+\left(\frac{40083\,\pi}{448}-\frac{8913\,q}{56}\right)v^5\right)\right\}. \quad (8\cdot24)$$

For $\ell=4$, we have

$$\eta_{4,4,0}=\frac{1280\,v^4}{567}-\frac{37120\,e^2\,v^4}{567},$$
$$\eta_{4,4,1}=\frac{48828125\,e^2\,v^4}{580608},$$
$$\eta_{4,4,-1}=\frac{32805\,e^2\,v^4}{7168},$$
$$\eta_{4,2,0}=\frac{5\,v^4}{3969}-\frac{25\,e^2\,v^4}{3969},$$
$$\eta_{4,2,-1}=\frac{5\,e^2\,v^4}{254016},$$

$$\eta_{4,0,\pm1} = \frac{e^2\,v^4}{225792},$$

and $\eta_{4,2,1}$ becomes $O(v^6)$. Hence we have

$$\left(\frac{dE}{dt}\right)_4 = \left(\frac{dE}{dt}\right)_{\rm N}\left\{\frac{8965\,v^4}{3969} + \frac{2946739\,e^2\,v^4}{127008}\right\}. \tag{8·25}$$

Finally, gathering all the above results, we have the luminosity up to $O(v^5)$ as

$$\begin{aligned}\left\langle\frac{dE}{dt}\right\rangle = \left(\frac{dE}{dt}\right)_{\rm N}\Bigg\{&1 - \frac{1247\,v^2}{336} + 4\,\pi\,v^3 - \frac{73\,q\,v^3}{12} - \frac{44711\,v^4}{9072} + \frac{33\,q^2\,v^4}{16}\\ &- \frac{8191\,\pi\,v^5}{672} + \frac{3749\,q\,v^5}{336} + e^2\left(\frac{37}{24} - \frac{65\,v^2}{21} + \frac{1087\,\pi\,v^3}{48} - \frac{211\,q\,v^3}{6}\right.\\ &\left.- \frac{465337\,v^4}{9072} + \frac{105\,q^2\,v^4}{8} - \frac{118607\,\pi\,v^5}{1344} - \frac{95663\,q\,v^5}{672}\right)\Bigg\}. \end{aligned}\tag{8·26}$$

If we set $q = 0$, the e^2 correction terms in the above formula completely agree with the corresponding terms in Eq. (7·25) in the previous section.

To compare our results with those derived in the standard post-Newtonian method, it is convenient to change the parameter from v to $x \equiv (M\Omega_\varphi)^{1/3}$. The relation between v and x is given by

$$v = x\left[1 + \frac{q}{3}x^3 + e^2\left\{\frac{1}{2} - \frac{3}{2}x^2 + \frac{8}{3}qx^3 - 6x^4 - q^2x^4 + \frac{31}{2}qx^5\right\}\right]. \tag{8·27}$$

Then we obtain

$$\begin{aligned}\left\langle\frac{dE}{dt}\right\rangle = \left(\widetilde{\frac{dE}{dt}}\right)_{\rm N}\Bigg\{&1 - \frac{1247}{336}x^2 - \frac{11}{4}qx^3 + 4\,x^3\pi - \frac{44711}{9072}x^4 + \frac{33}{16}x^4q^2\\ &- \frac{59}{16}qx^5 - \frac{8191}{672}x^5\pi + \Big(\frac{157}{24} - \frac{6781}{168}x^2 - \frac{2009}{72}qx^3 + \frac{2335}{48}x^3\pi\\ &- \frac{14929}{189}x^4 + \frac{281}{16}x^4q^2 - \frac{2399}{56}qx^5 - \frac{773}{3}x^5\pi\Big)e^2\Bigg\}, \end{aligned}\tag{8·28}$$

where $(\widetilde{dE/dt})_{\rm N}$ is the quadrupole flux expressed in terms of x, Eq. (6·6). We find that the terms which are proportional to e^2 agree with the formulas derived by Peters and Mathews[45] at leading order, Galt'sov et al.[7] and Blanchet and Schäfer at v^2 order,[46] Blanchet and Schäfer at v^3 order for $q = 0$[47] and Shibata at v^3 order for $q \neq 0$,[48] if we expand their formulas by e assuming $e \ll 1$ and $\mu/M \ll 1$.

From Eq. (2·32), the partial mode contributions to the angular momentum fluxes for $\ell = 2$, 3 and 4 are calculated to be

$$\left(\frac{dJ_z}{dt}\right)_2 = \left(\frac{dJ_z}{dt}\right)_{\rm N}\left\{1 - \frac{1277\,v^2}{252} + 4\,\pi\,v^3 - \frac{61\,q\,v^3}{12} + \frac{37915\,v^4}{10584} + \frac{33\,q^2\,v^4}{16}\right.$$

$$-\frac{2561\,\pi\,v^5}{126}+\frac{22229\,q\,v^5}{1296}+e^2\left(-\frac{5}{8}+\frac{137\,v^2}{24}+\frac{49\,\pi\,v^3}{8}-\frac{57\,q\,v^3}{4}\right.$$
$$\left.\left.-\frac{235675\,v^4}{14112}+\frac{203\,q^2\,v^4}{32}-\frac{20437\,\pi\,v^5}{504}-\frac{164449\,q\,v^5}{4536}\right)\right\},$$
$$\left(\frac{dJ_z}{dt}\right)_3=\left(\frac{dJ_z}{dt}\right)_{\rm N}\left\{\frac{1367\,v^2}{1008}-\frac{32567\,v^4}{3024}+\left(\frac{16403\,\pi}{2016}-\frac{88049\,q}{9072}\right)v^5\right.$$
$$\left.+e^2\left(\frac{67\,v^2}{32}-\frac{66497\,v^4}{2016}+\left(\frac{43193\,\pi}{1008}-\frac{1675571\,q}{18144}\right)v^5\right)\right\},$$
$$\left(\frac{dJ_z}{dt}\right)_4=\left(\frac{dJ_z}{dt}\right)_{\rm N}\left\{\frac{8965\,v^4}{3969}+\frac{478195\,e^2\,v^4}{42336}\right\},$$

where $(dJ_z/dt)_{\rm N}$ is defined by

$$\left(\frac{dJ_z}{dt}\right)_{\rm N}=\frac{32\mu^2M^{5/2}}{5r_0^{7/2}}=\frac{32}{5}\left(\frac{\mu}{M}\right)^2Mv^7. \tag{8·29}$$

Total angular momentum luminosity is then given by

$$\left\langle\frac{dJ_z}{dt}\right\rangle=\left(\frac{dJ_z}{dt}\right)_{\rm N}\left\{1-\frac{1247\,v^2}{336}+\left(4\,\pi-\frac{61\,q}{12}\right)v^3+\left(-\frac{44711}{9072}+\frac{33\,q^2}{16}\right)v^4\right.$$
$$+\left(\frac{-8191\,\pi}{672}+\frac{417\,q}{56}\right)v^5+e^2\left(-\frac{5}{8}+\frac{749\,v^2}{96}+\left(\frac{49\,\pi}{8}-\frac{57\,q}{4}\right)v^3\right.$$
$$\left.\left.+\left(-\frac{232181}{6048}+\frac{203\,q^2}{32}\right)v^4+\left(\frac{773\,\pi}{336}-\frac{28807\,q}{224}\right)v^5\right)\right\}. \tag{8·30}$$

The e^2 terms in the above also agree with the corresponding terms in Eq. (7·26). The angular momentum flux expressed in terms of $x=(M\Omega_\varphi)^{1/3}$ is given by

$$\left\langle\frac{dJ_z}{dt}\right\rangle=\left(\widetilde{\frac{dJ_z}{dt}}\right)_{\rm N}\left\{1-\frac{1247}{336}x^2-\frac{11}{4}qx^3+4x^3\pi-\frac{44711}{9072}x^4+\frac{33}{16}x^4q^2\right.$$
$$-\frac{59}{16}qx^5-\frac{8191}{672}x^5\pi+\left(\frac{23}{8}-\frac{3259}{168}x^2-\frac{371}{24}qx^3+\frac{209}{8}x^3\pi\right.$$
$$\left.\left.-\frac{1041349}{18144}x^4+\frac{171}{16}x^4q^2-\frac{243}{8}qx^5-\frac{785}{6}x^5\pi\right)e^2\right\}, \tag{8·31}$$

where $(\widetilde{dJ_z/dt})_{\rm N}$ is the Newtonian flux expressed in terms of x, Eq. (7·30).

§9. Circular orbit with small inclination from the equatorial plane

In this section, we consider the case of a circular orbit at $r=r_0$ with small inclination from the equatorial plane. We evaluate $\langle dE/dt\rangle$ and $\langle dJ_z/dt\rangle$ to $O(v^5)$ beyond Newtonian.

In this case, the orbital plane precesses around the symmetric axis. The degree of precession is determined by the value of the Carter constant C. If r_0 and C are given, the energy E and the z-component of the angular momentum l_z are obtained by the two equations, $R = 0$ and $\partial R/\partial r = 0$, where R is a function defined by Eq. (2·19). We introduce a dimensionless parameter y defined by

$$y = \frac{C}{Q^2}\,; \quad Q^2 = l_z^2 + a^2(1 - E^2). \tag{9·1}$$

We assume y is a small number. Since $Q^2 \sim l_z^2$ and $C \sim l_x^2 + l_y^2$, this is physically equivalent to assuming $l_x^2 + l_y^2 \ll l_z^2$. Since we do not need the exact expressions for E and l_z in terms of r_0 and y, we show them to the first order in y as well as to $O(v^5)$. They are given by

$$E = 1 - \frac{M}{2r_0} + \frac{3M^2}{8r_0^2} - \frac{M^{3/2}a}{r_0^{5/2}}\left(1 - \frac{y}{2}\right) + O(v^6), \tag{9·2}$$

$$l_z = (Mr_0)^{1/2}\left[\left(1 - \frac{y}{2}\right) + \frac{3M}{2r_0}\left(1 - \frac{y}{2}\right) - \frac{3M^{1/2}a}{r_0^{3/2}}(1 - y) \right.$$
$$\left. + \frac{27M^2}{8r_0^2}\left(1 - \frac{y}{2}\right) + \frac{a^2}{r_0^2}(1 - 2y) - \frac{15M^{3/2}a}{2r_0^{5/2}}(1 - y) + O(v^6)\right], \tag{9·3}$$

where note that $a = Mq$ ($|q| < 1$).

To solve the geodesic equations under the assumption $y \ll 1$, we first set $\theta = \pi/2 + y^{1/2}\theta'$ and consider the geodesic equation for θ. It then becomes

$$\left(\frac{d\theta'}{d\tau}\right)^2 = \frac{1}{\Sigma^2}\left[Q^2 - \frac{\sin^2(y^{1/2}\theta')}{y}\left\{a^2(1 - E^2) + \frac{l_z^2}{\cos^2(y^{1/2}\theta')}\right\}\right]. \tag{9·4}$$

Since the right-hand side of Eq. (9·4) contains only even-functions of $y^{1/2}\theta'$, we can solve it iteratively by expanding θ' as

$$\theta' = \theta_{(0)} + y\theta_{(1)} + y^2\theta_{(2)} + \cdots. \tag{9·5}$$

This method is similar to the one we have used in §7 or 8. However, here we only consider the lowest order solution $\theta_{(0)}$. This means we take into account the effect of inclination up to $O(y)$, as seen from the structure of the geodesic equations. The equation for $\theta_{(0)}$ is

$$\left(\frac{d\theta_{(0)}}{d\tau}\right)^2 = \frac{Q^2}{\Sigma^2}\left(1 - \theta_{(0)}^2\right), \tag{9·6}$$

or dividing it by $(dt/d\tau)^2$,

$$\left(\frac{d\theta_{(0)}}{dt}\right)^2 = \frac{Q^2}{\sigma^2}\left(1 - \theta_{(0)}^2\right), \tag{9·7}$$

where

$$\sigma \equiv -a(aE - l_z) + \frac{a^2 + r_0^2}{\Delta(r_0)}\left\{E(r_0^2 + a^2) - al_z\right\}. \tag{9·8}$$

Then the solution is easily obtained as

$$\theta_{(0)} = \sin(\Omega_\theta t); \quad \Omega_\theta = \frac{Q}{\sigma}, \tag{9·9}$$

where we have chosen $\theta_{(0)} = 0$ at $t = 0$. Thus we have

$$\theta = \frac{\pi}{2} + y^{1/2}\sin(\Omega_\theta t). \tag{9·10}$$

Note that the solution (9·10) implies that the inclination angle θ_i is indeed given by $\theta_i = y^{1/2}$ in the present approximation.

Next, we consider the geodesic equation for φ. Taking account of the terms up to $O(y)$, it becomes

$$\begin{aligned}\frac{d\varphi}{dt} &= \frac{\kappa}{\sigma}\left[1 + \left(\frac{l_z}{\kappa} - \frac{a^2E}{\sigma}\right)y\theta_{(0)}^2\right] \\ &= \Omega_\varphi - y\frac{\Omega_2}{2}\cos(2\Omega_\theta t),\end{aligned} \tag{9·11}$$

where

$$\kappa \equiv -(aE - l_z) + \frac{a}{\Delta(r_0)}\{E(r_0^2 + a^2) - al_z\} \tag{9·12}$$

and

$$\Omega_\varphi = \frac{\kappa}{\sigma} + \frac{1}{2}y\Omega_2, \quad \Omega_2 = \frac{\kappa}{\sigma}\left(\frac{l_z}{\kappa} - \frac{a^2E}{\sigma}\right). \tag{9·13}$$

The solution to Eq. (9·11) with $\varphi = 0$ at $t = 0$ is

$$\varphi = \Omega_\varphi t - y\frac{\Omega_2}{4\Omega_\theta}\sin(2\Omega_\theta t). \tag{9·14}$$

Note that $\Omega_\varphi \neq \Omega_\theta$. This means the precession of a test particle orbit around the spin axis of the black hole. Specifically, to the order required for the present purpose, we have

$$\begin{aligned}\Omega_\varphi &= \frac{M^{1/2}}{r_0^{3/2}}\left[1 - \frac{M^{1/2}a}{r_0^{3/2}} + \frac{3}{2}y\left(\frac{M^{1/2}a}{r_0^{3/2}} - \frac{a^2}{r_0^2}\right) + O(v^6)\right], \\ \Omega_\theta &= \frac{M^{1/2}}{r_0^{3/2}}\left[1 - \frac{3M^{1/2}a}{r_0^{3/2}} + \frac{3a^2}{2r_0^2} + O(v^6) + O(y)\right].\end{aligned} \tag{9·15}$$

We see that $\Omega_\varphi - \Omega_\theta \to 2Ma/r_0^3$ for $r_0 \to \infty$ and $y \to 0$, which is just the Lense-Thirring precessional frequency.[49)]

Now we are ready to calculate the source integral for the amplitude $\tilde{Z}_{\ell m\omega}$. Analogous to the case of an eccentric orbit considered in §7 or 8, Eq. (2·26) can be simplified further by noting that the orbits of our interest have the properties,

$$\theta(t + \Delta t_\theta) = \theta(t), \qquad \varphi(t + \Delta t_\theta) = \varphi(t) + \Delta\varphi, \tag{9·16}$$

where Δt_θ is the orbital period of the motion in the θ-direction and $\Delta\varphi$ is the phase advancement during Δt_θ. In other words, we have

$$\Omega_\theta = \frac{2\pi}{\Delta t_\theta}, \quad \Omega_\varphi = \frac{\Delta\varphi}{\Delta t_\theta}. \tag{9·17}$$

Then we obtain

$$\tilde{Z}_{\ell m\omega} = \sum_n \delta(\omega - \omega_n) Z_{\ell m\omega_n}, \tag{9·18}$$

where

$$\omega_n = n\Omega_\theta + m\Omega_\varphi, \quad (n = 0, \pm 1, \pm 2, \cdots) \tag{9·19}$$

and

$$Z_{\ell m\omega_n} = \frac{\mu\Omega_\theta}{2i\omega_n B^{\rm inc}_{\ell m\omega_n}} \int_0^{\Delta t} dt e^{i\omega t - im\varphi(t)} W_{\ell m\omega_n} \tag{9·20}$$

with $W_{\ell m\omega_n}$ being given by Eq. (2·27).

Let us discuss the final form of $Z_{\ell m\omega_n}$. In the present case, up to $O(y)$ the integrand $W_{\ell m\omega_n}$ has the form,

$$W_{\ell m\omega_n} = g_0 + y^{1/2} g_1 \theta_{(0)} + y g_2 \theta_{(0)}^2 + y^{1/2} g_3 \frac{d\theta_{(0)}}{dt} + y g_4 \theta_{(0)} \frac{d\theta_{(0)}}{dt} + y g_5 \left(\frac{d\theta_{(0)}}{dt}\right)^2 + O(y^{3/2}), \tag{9·21}$$

where $g_0 \sim g_5$ are complicated functions of r_0. Using an approximation,

$$e^{i\omega_n t - im\varphi(t)} = e^{in\Omega_\theta t}\left(1 + y\frac{m\Omega_2}{8\Omega_\theta}(e^{2i\Omega_\theta t} - e^{-2i\Omega_\theta t}) + O(y^{3/2})\right), \tag{9·22}$$

we have

$$\begin{aligned}
&\int_0^{\Delta t} dt e^{i\omega t - im\varphi(t)} W_{\ell m\omega_n} \\
&= \frac{2\pi}{\Omega_\theta}\Bigg[\left\{\delta_{n0} + y\frac{m\Omega_2}{8\Omega_\theta}(\delta_{n,-2} - \delta_{n,2})\right\} g_0 + y^{1/2}\frac{1}{2i}(\delta_{n,-1} - \delta_{n,1}) g_1 \\
&\quad + y\frac{1}{4}(2\delta_{n,0} - \delta_{n,-2} - \delta_{n,2}) g_2 + y^{1/2}\frac{\Omega_\theta}{2}(\delta_{n,-1} + \delta_{n,1}) g_3 \\
&\quad + y\frac{\Omega_\theta}{4i}(\delta_{n,-2} - \delta_{n,2}) g_4 + y\frac{\Omega_\theta^2}{4}(2\delta_{n,0} + \delta_{n,-2} + \delta_{n,2}) g_5\Bigg] \\
&\quad + O(y^{3/2}).
\end{aligned} \tag{9·23}$$

Thus the amplitude $Z_{\ell m\omega_n}$ is found to have the form,

$$\begin{aligned}
Z_{\ell m\omega_n} = \Big[&\left(Z^{0,0} + yZ^{0,2}\right)\delta_{n,0} + y^{1/2}\left(Z^{1,1}\delta_{n,1} + Z^{1,-1}\delta_{n,-1}\right) \\
&+ y\left(Z^{2,2}\delta_{n,2} + Z^{2,-2}\delta_{n,-2}\right) + O(y^{3/2})\Big],
\end{aligned} \tag{9·24}$$

where $Z^{i,j}$ are functions of r_0. Here, it is worth noting the symmetry of $Z_{\ell m\omega_n}$. The spin weighted spheroidal harmonics have a property ${}_{-2}S^{a\omega_n}_{\ell m}(\theta) = (-1)^\ell {}_{-2}S^{a\omega_{-n}}_{\ell -m}(\pi - \theta)$. Then, from Eqs. (9·20) and (9·22), we have $Z_{\ell -m\omega_{-n}} = (-1)^{n+\ell}\overline{Z}_{\ell m\omega_n}$.

Now we evaluate the energy and angular momentum fluxes at infinity. The energy and angular momentum fluxes averaged over $t \gg \Delta t_\theta$ are given by Eqs. (2·31) and (2·32), respectively. Then we see from Eq. (9·24) that the $n = \pm 2$ modes contribute to the luminosity at $O(y^2)$. Thus, when we calculate the luminosity to $O(y)$, we need to include only the $n = 0, \pm 1$ modes. In order to express the post-Newtonian corrections to the luminosity, we define $\eta_{\ell mn}$ as

$$\left(\frac{dE}{dt}\right)_{\ell mn} \equiv \frac{1}{2}\left(\frac{dE}{dt}\right)_{\rm N} \eta_{\ell mn}\,, \tag{9·25}$$

where $(dE/dt)_{\rm N}$ is the Newtonian quadrupole luminosity given by Eq. (4·19).

For $\ell = 2$, the results are as follows. If $|m+n| > 2$ or $m+n=0$, $\eta_{\ell mn}$ becomes of $O(v^6)$ or higher. The remaining $\eta_{\ell mn}$ which contribute to the 2.5PN luminosity formula are given by

$$\begin{aligned}
\eta_{2\pm2\ 0} &= 1 - \frac{107}{21}v^2 + 4\pi v^3 - 6qv^3 + \frac{4784}{1323}v^4 + 2q^2v^4 - \frac{428\pi}{21}v^5 + \frac{4216}{189}qv^5 \\
&\quad + y\biggl(-1 + \frac{170}{21}v^2 - 4\pi v^3 + 15qv^3 - \frac{4784}{1323}v^4 - 11q^2v^4 \\
&\quad + \frac{428\pi}{21}v^5 - \frac{13186}{189}qv^5\biggr), \\
\eta_{2\pm2\mp1} &= y\left(\frac{1}{36}v^2 - \frac{17}{504}v^4 + \frac{\pi}{18}v^5 + \frac{17}{1134}qv^5\right), \\
\eta_{2\pm1\ 0} &= \frac{1}{36}v^2 - \frac{1}{12}qv^3 - \frac{17}{504}v^4 + \frac{1}{16}q^2v^4 + \frac{\pi}{18}v^5 - \frac{793}{9072}qv^5 \\
&\quad + y\left(-\frac{5}{72}v^2 + \frac{1}{8}qv^3 + \frac{85}{1008}v^4 - \frac{1}{32}q^2v^4 - \frac{5\pi}{36}v^5 + \frac{13931}{18144}qv^5\right), \\
\eta_{2\pm1\pm1} &= y\biggl(1 - \frac{170}{21}v^2 + 4\pi v^3 - 12qv^3 + \frac{4784}{1323}v^4 + \frac{11}{2}q^2v^4 \\
&\quad - \frac{428\pi}{21}v^5 + \frac{11078}{189}qv^5\biggr), \\
\eta_{2\ 0\pm1} &= y\left(\frac{1}{24}v^2 - \frac{1}{12}qv^3 - \frac{17}{336}v^4 + \frac{1}{24}q^2v^4 + \frac{\pi}{12}v^5 - \frac{745}{1008}qv^5\right).
\end{aligned} \tag{9·26}$$

Putting together the above results, we obtain $(dE/dt)_\ell \equiv \sum_{mn}(dE/dt)_{\ell mn}$ for $\ell = 2$ as

$$\begin{aligned}
\left(\frac{dE}{dt}\right)_2 = \left(\frac{dE}{dt}\right)_{\rm N}\biggl\{&1 - \frac{1277}{252}v^2 + 4\pi v^3 - \frac{73}{12}qv^3\left(1 - \frac{y}{2}\right) + \frac{37915}{10584}v^4 \\
&+ \frac{33}{16}q^2v^4 - \frac{527}{96}q^2v^4y - \frac{2561\pi}{126}v^5 + \frac{201575}{9072}qv^5\left(1 - \frac{y}{2}\right)\biggr\}.
\end{aligned} \tag{9·27}$$

For $\ell = 3$, the non-trivial $\eta_{\ell mn}$ are given by

$$\eta_{3\pm3\ 0} = \frac{1215}{896}v^2 - \frac{1215}{112}v^4 + \frac{3645\pi}{448}v^5 - \frac{1215}{112}qv^5$$

$$
\begin{aligned}
&\quad +y\left(-\frac{3645}{1792}v^2+\frac{3645}{224}v^4-\frac{10935\pi}{896}v^5+\frac{3645}{112}qv^5\right),\\
\eta_{3\pm3\mp1} &= \frac{5}{42}v^4y,\\
\eta_{3\pm2\ \ 0} &= \frac{5}{63}v^4-\frac{40}{189}qv^5+y\left(-\frac{20}{63}v^4+\frac{100}{189}qv^5\right),\\
\eta_{3\pm2\pm1} &= y\left(\frac{3645}{1792}v^2-\frac{3645}{224}v^4+\frac{10935\pi}{896}v^5-\frac{6075}{224}qv^5\right),\\
\eta_{3\pm2\mp1} &= y\left(\frac{5}{16128}v^2-\frac{5}{3024}v^4+\frac{5\pi}{8064}v^5+\frac{25}{18144}qv^5\right),\\
\eta_{3\pm1\ \ 0} &= \frac{1}{8064}v^2-\frac{1}{1512}v^4+\frac{\pi}{4032}v^5-\frac{17}{9072}qv^5\\
&\quad +y\left(-\frac{11}{16128}v^2+\frac{11}{3024}v^4-\frac{11\pi}{8064}v^5+\frac{95}{9072}qv^5\right),\\
\eta_{3\pm1\pm1} &= y\left(\frac{25}{126}v^4-\frac{80}{189}qv^5\right),\\
\eta_{3\ \ 0\pm1} &= y\left(\frac{1}{2688}v^2-\frac{1}{504}v^4+\frac{\pi}{1344}v^5-\frac{11}{1008}qv^5\right).
\end{aligned}
\tag{9·28}
$$

The other $\eta_{\ell mn}$ are of $O(v^6)$ or higher. Then we obtain

$$
\left(\frac{dE}{dt}\right)_3=\left(\frac{dE}{dt}\right)_{\mathrm{N}}\left\{\frac{1367}{1008}v^2-\frac{32567}{3024}v^4+\frac{16403\pi}{2016}v^5-\frac{896}{81}qv^5\left(1-\frac{y}{2}\right)\right\}. \tag{9·29}
$$

For $\ell=4$, we have

$$
\begin{aligned}
\eta_{4\pm4\ \ 0} &= \frac{1280}{567}v^4(1-2y),\\
\eta_{4\pm3\pm1} &= \frac{2560}{567}v^4y,\\
\eta_{4\pm3\mp1} &= \frac{5}{1134}v^4y,\\
\eta_{4\pm2\ \ 0} &= \frac{5}{3969}v^4(1-8y),\\
\eta_{4\pm1\pm1} &= \frac{5}{882}v^4y,
\end{aligned}
\tag{9·30}
$$

and the others are of $O(v^6)$ or higher. Hence we obtain

$$
\left(\frac{dE}{dt}\right)_4=\left(\frac{dE}{dt}\right)_{\mathrm{N}}\times\frac{8965}{3969}v^4. \tag{9·31}
$$

Finally, gathering all the terms, the total energy flux up to $O(v^5)$ is found to be

$$
\begin{aligned}
\left\langle\frac{dE}{dt}\right\rangle &= \left(\frac{dE}{dt}\right)_{\mathrm{N}}\left(1-\frac{1247}{336}v^2+4\pi v^3-\frac{73}{12}qv^3\left(1-\frac{y}{2}\right)-\frac{44711}{9072}v^4\right.\\
&\quad \left.+\frac{33}{16}q^2v^4-\frac{527}{96}q^2v^4y-\frac{8191\pi}{672}v^5+\frac{3749}{336}qv^5\left(1-\frac{y}{2}\right)\right).
\end{aligned}
\tag{9·32}
$$

Using the above results for $\eta_{\ell mn}$, the time-averaged angular momentum flux is calculated from Eq. (2·32). The partial mode contributions of the $\ell = 2$, 3 and 4 modes are calculated to give

$$\begin{aligned}
\left(\frac{dJ_z}{dt}\right)_2 &= \left(\frac{dJ_z}{dt}\right)_{\rm N}\left[1-\frac{y}{2}-\frac{1277}{252}v^2\left(1-\frac{y}{2}\right)+4\pi v^3\left(1-\frac{y}{2}\right)\right.\\
&\quad -qv^3\left(\frac{61}{12}-\frac{61}{8}y\right)+\frac{37915}{10584}v^4\left(1-\frac{y}{2}\right)+q^2v^4\left(\frac{33}{16}-\frac{229}{32}y\right)\\
&\quad \left.-\pi v^5\frac{2561}{126}\left(1-\frac{y}{2}\right)+qv^5\left(\frac{22229}{1296}-\frac{27809}{864}y\right)\right],\\
\left(\frac{dJ_z}{dt}\right)_3 &= \left(\frac{dJ_z}{dt}\right)_{\rm N}\left[\frac{1367}{1008}v^2\left(1-\frac{y}{2}\right)-\frac{32567}{3024}v^4\left(1-\frac{y}{2}\right)\right.\\
&\quad \left.+\pi v^5\frac{16403}{2016}\left(1-\frac{y}{2}\right)-qv^5\left(\frac{88049}{9072}-\frac{9817}{756}y\right)\right],\\
\left(\frac{dJ_z}{dt}\right)_4 &= \left(\frac{dJ_z}{dt}\right)_{\rm N}\left[\frac{8965}{3969}v^4\left(1-\frac{y}{2}\right)\right],
\end{aligned} \tag{9·33}$$

where $(dJ_z/dt)_{\rm N}$ is defined in Eq. (8·29). The total angular momentum flux is then given by

$$\begin{aligned}
\left\langle\frac{dJ_z}{dt}\right\rangle &= \left(\frac{dJ_z}{dt}\right)_{\rm N}\left[\left(1-\frac{y}{2}\right)-\frac{1247}{336}v^2\left(1-\frac{y}{2}\right)+4\pi v^3\left(1-\frac{y}{2}\right)\right.\\
&\quad -\frac{61}{12}qv^3\left(1-\frac{y}{2}\right)-\frac{44711}{9072}v^4\left(1-\frac{y}{2}\right)+q^2v^4\left(\frac{33}{16}-\frac{229}{32}y\right)\\
&\quad \left.-\frac{8191}{672}\pi v^5\left(1-\frac{y}{2}\right)+qv^5\left(\frac{417}{56}-\frac{4301}{224}y\right)\right].
\end{aligned} \tag{9·34}$$

We note that the result is proportional to $(1-y/2)$ in the limit $q \to 0$. This is simply because the orbital plane is slightly tilted from the equatorial plane by an angle $\theta_i \sim y^{1/2}$, hence $dJ_z/dt \sim (dJ_{\rm tot}/dt)\cos\theta_i$.

§10. Adiabatic backreaction

In the preceding sections, we have evaluated the energy flux $\langle dE/dt\rangle$ and the z-component of the angular momentum flux $\langle dJ_z/dt\rangle$ emitted to infinity by a particle for various cases. By emitting gravitational waves, a particle orbit will suffer from radiation reaction. In the limit of small μ/M, the reaction time scale will be much longer than the characteristic orbital time scale; $t_{\rm react} \sim M^2/\mu \gg \Delta t$. Hence the evolution of the orbit will be well described by the adiabatic backreaction.

In the case of orbits around a Schwarzschild black hole or orbits confined on the equatorial plane around a Kerr black hole, it is straightforward to calculate the evolutionary path under radiation reaction because the orbits are completely specified by the energy E and the z-component of the angular momentum l_z, hence their time derivatives can be simply evaluated by equating them with $-\langle dE/dt\rangle$ and $-\langle dJ_z/dt\rangle$, respectively. However, once we consider motions off the equatorial plane of a Kerr black hole, the orbits cannot be specified by E and l_z alone but the specification of the Carter constant C becomes necessary. Unlike E or l_z, since C is not associated with the Killing vector of the spacetime, one cannot calculate the radiation reaction to C by simply calculating the gravitational waves at infinity. This implies that we have to derive a local radiation reaction force term to the geodesic equation by evaluating the metric perturbations around the particle, as is done in the derivation of radiation reaction force in the standard post-Newtonian method. For almost Newtonian orbits, applying a post-Newtonian radiation reaction force, Ryan derived the evolution equation for the Carter constant.[50)] However, no relativistic treatment has been done so far. This is a challenging issue. An approach to this issue is discussed in Chapter 7.

In this section, instead of attacking this very difficult problem, we discuss some general properties of the adiabatic radiation reaction in a restricted class of orbits. Namely we consider orbits which are circular or those having small eccentricity. We clarify the conditions for circular orbits to remain circular under radiation reaction. A detailed discussion on this matter has been given by Kennefick and Ori.[51)] We give a less detailed but more general discussion below.

We recall that the radial velocity $u^r = dr/d\tau$ is written in terms of the first integrals of motion in the test particle limit as

$$(\Sigma u^r)^2 = R(I^i, r), \tag{10·1}$$

where $I^i = (E, l_z, C)$, and $R(I^i, r)$ is independent of θ and ϕ. First let us consider orbits which are circular in the test particle limit. These orbits are determined by the conditions,

$$R(I^i, r) = 0, \quad \frac{\partial R}{\partial r}(I^i, r) = 0. \tag{10·2}$$

Eliminating r from these equations gives an implicit relation among the I^i. For example,

$$f(I^i) = R(I^i, r(I^i)) = 0, \tag{10·3}$$

where $r(I^i)$ is obtained by solving the second of Eq. (10·2) for r. This equation determines a two-dimensional hypersurface S in the 3-dimensional space M of the I^i. The adiabatic evolution of an orbit is characterized by slow evolution of I^i, i.e., $\dot{I}^i = O(\mu)$, where μ is the mass of the particle. Then a necessary condition for circular orbits to remain circular under radiation reaction is that we have $\dot{f}(I^i) = f_{,i}\dot{I}^i = O(\mu^2)$. In other words, the vector $\dot{I}^i$ on S is tangent to S to $O(\mu)$. This condition can be shown to hold by the following theorem.

Theorem: If the radiation reaction to the r-component of the acceleration is of order μ; $a^r := du^r/d\tau = O(\mu)$, i.e., the r-component of the radiation reaction force is well-defined and finite, then for orbits which are circular in the test particle limit; i.e., $u^r = O(\mu)$, the radiation reaction to I^i is constrained by the equation,

$$\frac{\partial R}{\partial I^i}(I^i, r)\dot{I}^i = O(\mu^2), \tag{10·4}$$

where the argument r is to be replaced by $r(I^i)$ after differentiation.

Proof: It is almost trivial. Just taking the τ-derivatives of both hand sides of Eq. (10·1) gives Eq. (10·4). Q.E.D.

Thus, since

$$\dot{f}(I^i) = \left(\frac{\partial R}{\partial I^i}(I^i, r(I^i)) + \frac{\partial R}{\partial r}(I^i, r(I^i))\frac{\partial r}{\partial I^i}\right)\dot{I}^i, \tag{10·5}$$

and the second term in the parentheses vanishes by definition, we have $\dot{f} = O(\mu^2)$.

This theorem alone, however, does not mean that circular orbits remain circular, since we have constrained the first integrals to be those for circular orbits from the beginning. Let us explain the reason. Since we may regard $\dot{I}^i$ a vector field in M, what we need for circular orbits to remain circular is the regularity of $\dot{I}^i$ in the vicinity of the hypersurface S. In other words, if the vector field $\dot{I}^i$ is not differentiable on S, an orbit on S may spontaneously deviate away from S. A simple illustrative example is the case $\dot{I}_\perp = \sqrt{I_\perp}$ at $I_\perp = 0$ where $I_\perp$ is the component of I^i perpendicular to S.

Thus, provided $\dot{I}^i$ is regular in an open neighborhood of S, the above theorem implies that a circular orbit in the test particle limit remains circular under adiabatic radiation reaction. In this case, the radiation reaction to the Carter constant, $\dot{C}$, is determined by the radiation reaction to the energy, $\dot{E}$, and the z-component of the angular momentum, $\dot{l}_z$. Specifically we have

$$\dot{C} = \left(\frac{2}{\Delta}\left(E\left(r^2+a^2\right)-al_z\right)\left(r^2+a^2\right) - 2a\left(Ea-l_z\right)\right)\dot{E}$$
$$+\left(-2\frac{a}{\Delta}\left(E\left(r^2+a^2\right)-al_z\right)+2\left(Ea-l_z\right)\right)\dot{l}_z. \tag{10·6}$$

Yet this is not the end of the story. What we have shown is that $\dot{I}^i$ lies on S. But if $\dot{I}^i$ slightly off the hypersurface S is diverging away from S, circular orbits will be unstable. Thus the condition for the stability of circular orbits is that S is an attractor plane of the vector field $\dot{I}^i$. However, the notion of divergence or convergence of a vector depends on the metric of the space M, but we have no guiding principle to determine the metric. This implies that the notion of the attractor or the stability is ambiguous.

Nevertheless, extrapolating from the case of Newtonian orbits, there seems to exist a natural choice of the metric. Namely, as the distance of the orbit from the hypersurface S of circular orbits, we define the eccentricity of an orbit as given in §§7 and 8. With this choice of the metric, let us consider the adiabatic radiation reaction problem in more specific terms.

Let us parametrize an orbit in terms of the mean radius r_0, the eccentricity e and the square root of the Carter constant $y := C^{1/2}$, instead of the energy E, the angular momentum l_z and the Carter constant C. The mean radius r_0 is defined by the equation,

$$0 = R'(I^i, r_0), \tag{10·7}$$

where the prime denotes the partial derivative with respect to r. This definition says that $\dot{r}$ is maximum at $r = r_0$. The eccentricity e is defined by setting the maximum radius to $r = r_0(1+e)$, i.e.,

$$0 = R(I^i, r_0(1+e)). \tag{10·8}$$

This definition guarantees that $e = 0$ corresponds to a circular orbit. Assuming $e \ll 1$, the above equation can be expanded in powers of e as

$$0 = R(I^i, r_0) + \frac{1}{2}R''(I^i, r_0)(r_0 e)^2 + \frac{1}{3!}R^{(3)}(I^i, r_0)(r_0 e)^3 + \frac{1}{4!}R^{(4)}(I^i, r_0)(r_0 e)^4 + \cdots, \tag{10·9}$$

where $R^{(n)}$ is the n-th derivative of R with respect to r. The parameters (r_0, e, y) are chosen because the geodesic trajectory $x^\mu = z^\mu(\tau)$ allows perturbative expansion in powers of e and y at least for $e \ll 1$ and $y \ll 1$. Therefore the first integrals of motion $I^i = (E, l_z, C)$ will be regular functions of the parameters (r_0, e, y). On the other hand, if we consider (r_0, e, y) as functions of I^i, it should be noted that e is not a regular function of I^i in a neighborhood of circular orbits because of the absence of a term linear in e on the right-hand side of Eq. (10·9).

Now we consider the adiabatic evolution of e under the radiation reaction. Taking the τ-derivative of Eqs. (10·7) and (10·9), we obtain

$$0 = R'_{0,i}\dot{I}^i + R''_0\dot{r}_0\,, \tag{10·10}$$

$$\begin{aligned} 0 &= R_{0,i}I^i + R''_0\left(r_0\dot{r}_0e^2 + r_0^2e\dot{e}\right) + \frac{1}{2}\left(R''_{0,i}\dot{I}^i + R_0^{(3)}\dot{r}_0\right)r_0^2e^2 \\ &\quad + \frac{1}{2}R_0^{(3)}r_0^3e^2\dot{e} + \frac{1}{3!}R_0^{(4)}r_0^4e^3\dot{e} + O\left(e^3, \dot{e}e^4\right), \end{aligned} \tag{10·11}$$

where $R'_{0,i} = \partial^2 R/\partial I^i\partial r$, etc. Equation (10·10) determines $\dot{r}_0$ as

$$\dot{r}_0 = -\frac{R'_{0,i}}{R''_0}\dot{I}^i. \tag{10·12}$$

Substituting this into Eq. (10·11), we obtain the expression for $\dot{e}$ as

$$\begin{aligned} \dot{e} = \frac{1}{er_0^2R''_0}\Bigg[&-R_{0,i} + \frac{e}{2}\frac{r_0R_0^{(3)}}{R''_0}R_{0,i} \\ &+e^2\left\{-\frac{1}{2}{r_0}^2R''_{0,i} + \left(r_0 + \frac{1}{2}\frac{r_0^2R_0^{(3)}}{R''_0}\right)R'_{0,i} + r_0^2\left(\frac{1}{6}\frac{R_0^{(4)}}{R''_0} - \frac{1}{4}\left(\frac{R_0^{(3)}}{R''_0}\right)^2\right)R_{0,i}\right\} \\ &+O(e^3)\Bigg]\dot{I}^i. \end{aligned} \tag{10·13}$$

Since the trajectory $z^\mu(\tau)$ is assumed to be analytic in (r_0, e, y), it is reasonable to further assume that $\dot{I}^i$ are regular functions of (r_0, e, y). Then we can expand $\dot{I}^i$ with respect to e as

$$\dot{I}^i(r_0, e, y) = \dot{I}^{i(0)}(r_0, y) + e\dot{I}^{i(1)}(r_0, y) + e^2\dot{I}^{i(2)}(r_0, y) + \cdots. \tag{10·14}$$

Then we obtain

$$\dot{e} = \frac{1}{r_0^2 R_0''}\left[-\frac{1}{e}R_{0,i}\dot{I}^{i(0)} + \left(-R_{0,i}\dot{I}^{i(1)} + \frac{1}{2}\frac{r_0 R_0'''}{R_0''}R_{0,i}\dot{I}^{i(0)}\right)\right.$$
$$+e\left(-R_{0,i}\dot{I}^{i(2)} + \frac{1}{2}\frac{r_0 R_0'''}{R_0''}R_{0,i}\dot{I}^{i(1)}\right.$$
$$\left. + \left\{-\frac{1}{2}{r_0}^2 R_{0,i}'' + \left(r_0 + \frac{1}{2}\frac{{r_0}^2 R_0'''}{R_0''}\right)R_{0,i}' + r_0^2\left(\frac{1}{6}\frac{R_0^{(4)}}{R_0''} - \frac{1}{4}\left(\frac{R_0'''}{R_0''}\right)^2\right)R_{0,i}\right\}\dot{I}^{i(0)}\right)$$
$$\left. +O(e^2)\right]. \qquad (10\cdot15)$$

As shown by the theorem above, Eq. (10·4), the leading term of order e^{-1} vanishes provided the radiation reaction force is finite:

$$R_{0,i}\dot{I}^{i(0)} = 0. \qquad (10\cdot16)$$

The next order term determines whether the circular orbit remains circular or not. If it does not vanish, the eccentricity will spontaneously develop as

$$\dot{e} = -R_{0,i}\dot{I}^{i(1)}. \qquad (10\cdot17)$$

Here the regularity of $\dot{I}^i$ comes into play. As noted above, e is singular on the hypersurface S. Hence if $\dot{I}^i$ is regular on S, $\dot{I}^{(1)}(r_0, y)$ should vanish. By a detailed analysis, it is shown in Ref. 51) that this is indeed the case. The physical reason is rather simple: If one considers a slightly eccentric orbit, there appears a frequency of wobbling motion due to the eccentricity, say Ω_e. In general the ratio of Ω_e to the frequency of the motion in the θ or φ direction is an irrational number. Hence the part of the metric perturbation which is proportional to e will have frequencies that are integer multiples of Ω_e, and the same property is shared by the corresponding term of the backreaction force linear in e. Since any sinusoidal oscillation has zero mean when averaged over time longer than its period, this implies there will be no term linear in e in the adiabatic expression of $\dot{I}^i$.

Thus we have

$$\dot{e} = \left[-R_{0,i}\dot{I}^{i(2)} + \left\{-\frac{1}{2}{r_0}^2 R_{0,i}'' + \left(r_0 + \frac{1}{2}\frac{r_0^2 R_0'''}{R_0''}\right)R_{0,i}'\right\}\dot{I}^{i(0)}\right]e + O(e^2), \quad (10\cdot18)$$

and circular orbits will remain circular under radiation reaction. As for the stability of circular orbits, whether the eccentricity decreases or increases is determined by the sign of the coefficient of e on the right-hand side. Thus it is necessary to calculate the radiation reaction to the Carter constant to determine the stability. As mentioned in the beginning of this section, this is a challenging issue. Finally, we should again note that the meaning of stability does depend on the definition of the eccentricity, i.e., how we define the distance from the hypersurface of circular orbits S.

§11. Spinning particle

So far we have considered only a monopole particle orbiting a black hole. However, in a realistic binary system of compact bodies such as a neutron star-neutron star, black hole-neutron star or black hole-black hole binary, both bodies may have non-negligible spin angular momenta. Hence it is desirable to take into account not only the spin of a black hole but also the spin of a particle in the calculations of gravitational waves from a particle orbiting a black hole.

To incorporate the spin of a particle, one must know (1) the equations of motion and (2) the energy momentum tensor of a spinning particle. Fortunately, we know that (1) have been derived by Papapetrou,[19] Dixon[20] and Wald[52] and (2) has also been derived by Dixon.[20] Hence, by using the expression for the energy momentum tensor of a spinning particle as the source term in the Teukolsky formalism,[3] we can calculate the gravitational waves emitted by a spinning small mass particle orbiting a rotating black hole. One may regard this particle as a model of a small Kerr black hole, but it may be appropriate here to give a word of caution. A Kerr black hole of mass μ and the spin parameter S, where S is defined so that μS gives the spin angular momentum, has quadrupole ($\ell = 2$) and higher multipole moments ($\ell > 2$) proportional to μS^ℓ as well. Since we neglect the contributions of these higher multipole moments here, our treatment will be valid only up to $O(S)$ if we regard the particle as a Kerr black hole. To incorporate the contributions of all higher multipole moments to represent the Kerr black hole is a future problem to be investigated.

Here we review the results obtained by Tanaka et al..[18] We concentrate on the leading effect due to the spin of the small mass particle. We consider a class of circular orbits which stay near the equatorial plane with the inclination solely due to the spin of the particle, i.e., those orbits which would be confined in the equatorial plane if the spin were zero. Then we calculate the gravitational wave luminosity to $O(v^5)$ with linear corrections due to the spin.

11.1. *Equation of motion and source term of a spinning particle*

To give the source term of the Teukolsky equation, we need to solve the equations of motion of a spinning particle and also to give an expression for the energy momentum tensor. In this section we give the necessary expressions, following Refs. 20),52) and 53).

Neglecting the effect of the higher multipole moments, the equations of motion of a spinning particle are given by

$$\begin{aligned}\frac{D}{d\tau}p^\mu(\tau) &= -\frac{1}{2}R^\mu{}_{\nu\rho\sigma}(z(\tau))v^\nu(\tau)S^{\rho\sigma}(\tau),\\ \frac{D}{d\tau}S^{\mu\nu}(\tau) &= 2p^{[\mu}(\tau)v^{\nu]}(\tau),\end{aligned}\tag{11·1}$$

where $v^\mu(\tau) = dz^\mu(\tau)/d\tau$, τ is a parameter which is not necessarily the proper time of the particle, and, as we will see later, the vector $p^\mu(\tau)$ and the antisymmetric tensor $S^{\mu\nu}(\tau)$ represent the linear and spin angular momenta of the particle, respectively.

Here $D/d\tau$ denotes the covariant derivative along the particle trajectory.

We do not have the evolution equation for $v^\mu(\tau)$ yet. In order to determine $v^\mu(\tau)$, we need to impose a supplementary condition which determines the center of mass of the particle,[20]

$$S^{\mu\nu}(\tau)p_\nu(\tau) = 0. \tag{11·2}$$

Then one can show that $p_\mu p^\mu$ = const. and $S_{\mu\nu}S^{\mu\nu}$ = const. along the particle trajectory.[52] Therefore we may set

$$\begin{aligned} p^\mu &= \mu u^\mu, \quad u_\mu u^\mu = -1, \\ S^{\mu\nu} &= \epsilon^{\mu\nu}{}_{\rho\sigma} p^\rho S^\sigma, \quad p_\mu S^\mu = 0, \\ S^2 &= S_\mu S^\mu = \frac{1}{2\mu^2} S_{\mu\nu}S^{\mu\nu}, \end{aligned} \tag{11·3}$$

where μ is the mass of the particle, u^μ is the specific linear momentum, and S^μ is the specific spin vector with S its magnitude. Note that if we use S^μ instead of $S^{\mu\nu}$ in the equations of motion, the center of mass condition (11·2) will be replaced by the condition

$$p_\mu S^\mu = 0. \tag{11·4}$$

Since the above equations of motion are invariant under reparametrization of the orbital parameter τ, we can fix τ to satisfy

$$u^\mu(\tau)v_\mu(\tau) = -1. \tag{11·5}$$

Then, from Eqs. (11·1), (11·2) and (11·5), $v^\mu(\tau)$ is determined as[20]

$$v^\mu(\tau) - u^\mu(\tau) = \frac{1}{2}\left(\mu^2 + \frac{1}{4}R_{\chi\xi\zeta\eta}(z(\tau))S^{\chi\xi}(\tau)S^{\zeta\eta}(\tau)\right)^{-1} S^{\mu\nu}(\tau)R_{\nu\rho\sigma\kappa}(\tau)u^\rho(\tau)S^{\sigma\kappa}(\tau). \tag{11·6}$$

With this equation, the equations of motion (11·1) completely determine the evolution of the orbit and the spin. Note that $v^\mu = u^\mu + O(S^2)$, hence v^μ and u^μ are identical to each other to $O(S)$.

As for the energy momentum tensor, Dixon[20] gives it in terms of the Dirac delta-function on the tangent space at $x^\mu = z^\mu(\tau)$. For later convenience, in this paper we use an equivalent but alternative form of the energy momentum tensor, given in terms of the Dirac delta-function on the coordinate space:

$$\begin{aligned} T^{\alpha\beta}(x) = \int d\tau \Bigg\{ & p^{(\alpha}(x,\tau)v^{\beta)}(x,\tau)\frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \\ & - \nabla_\gamma\left(S^{\gamma(\alpha}(x,\tau)v^{\beta)}(x,\tau)\frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}}\right)\Bigg\}, \end{aligned} \tag{11·7}$$

where $v^\alpha(x,\tau)$, $p^\alpha(x,\tau)$ and $S^{\alpha\beta}(x,\tau)$ are bi-tensors which are spacetime extensions of $v^\mu(\tau)$, $p^\mu(\tau)$ and $S^{\mu\nu}(\tau)$ which are defined only along the world line, $x^\mu = z^\mu(\tau)$.*)

*) In the rest of this section, we use $\mu, \nu, \sigma, \cdots$ as the tensor indices associated with the world line $z(\tau)$ and $\alpha, \beta, \gamma, \cdots$ as those with a field point x, and suppress the coordinate indices of $z(\tau)$ and x for notational simplicity.

To define $v^\alpha(x, z(\tau))$, $p^\alpha(x, z(\tau))$ and $S^{\alpha\beta}(x, z(\tau))$ we introduce a bi-tensor $\bar{g}^\alpha{}_\mu(x, z)$ which satisfies

$$\lim_{x\to z} \bar{g}^\alpha{}_\mu(x, z(\tau)) = \delta^\alpha{}_\mu\,,$$
$$\lim_{x\to z} \nabla_\beta \bar{g}^\alpha{}_\mu(x, z(\tau)) = 0. \qquad (11{\cdot}8)$$

For the present purpose, further specification of $\bar{g}^\alpha{}_\mu(x, z)$ is not necessary. Using this bi-tensor $\bar{g}^\alpha{}_\mu(x, z)$, we define $p^\alpha(x, \tau)$, $v^\alpha(x, \tau)$ and $S^{\alpha\beta}(x, \tau)$ as

$$p^\alpha(x, \tau) = \bar{g}^\alpha{}_\mu\Big(x, z(\tau)\Big)p^\mu(\tau),$$
$$v^\alpha(x, \tau) = \bar{g}^\alpha{}_\mu\Big(x, z(\tau)\Big)v^\mu(\tau),$$
$$S^{\alpha\beta}(x, \tau) = \bar{g}^\alpha{}_\mu\Big(x, z(\tau)\Big)\bar{g}^\beta{}_\nu\Big(x, z(\tau)\Big)S^{\mu\nu}(\tau). \qquad (11{\cdot}9)$$

It is easy to see that the divergence free condition of this energy momentum tensor gives the equations of motion (11·1). Noting the relations,

$$\nabla_\beta \bar{g}^\alpha{}_\mu(x, z(\tau))\delta^{(4)}(x, z(\tau)) = 0,$$
$$v^\alpha(x)\nabla_\alpha\left(\frac{\delta^{(4)}\Big(x, z(\tau)\Big)}{\sqrt{-g}}\right) = -\frac{d}{d\tau}\left(\frac{\delta^{(4)}\Big(x, z(\tau)\Big)}{\sqrt{-g}}\right), \qquad (11{\cdot}10)$$

the divergence of Eq. (11·7) becomes

$$\nabla_\beta T^{\alpha\beta}(x) = \int d\tau \bar{g}^\alpha{}_\mu\Big(x, z(\tau)\Big)\frac{\delta^{(4)}\Big(x - z(\tau)\Big)}{\sqrt{-g}}\left(\frac{d}{d\tau}p^\mu(\tau) + \frac{1}{2}R^\mu{}_{\nu\sigma\kappa}\Big(z(\tau)\Big)v^\nu(\tau)S^{\sigma\kappa}(\tau)\right)$$
$$+\frac{1}{2}\int d\tau \nabla_\beta\left(\bar{g}^\alpha{}_\mu\Big(x, z(\tau)\Big)\bar{g}^\beta{}_\nu\Big(x, z(\tau)\Big)\frac{\delta^{(4)}\Big(x - z(\tau)\Big)}{\sqrt{-g}}\right)$$
$$\times\left(\frac{d}{d\tau}S^{\mu\nu}(\tau) - 2p^{[\mu}(\tau)v^{\nu]}(\tau)\right). \qquad (11{\cdot}11)$$

Since the first and second terms on the right-hand side must vanish separately, we obtain the equations of motion (11·1).

In order to clarify the meaning of p^μ and $S^{\mu\nu}$, we consider the volume integral of this energy momentum tensor such as $\int_{\Sigma(\tau_0)} \bar{g}_\alpha{}^\mu T^{\alpha\beta} d\Sigma_\beta$, where we take the surface $\Sigma(\tau_0)$ to be perpendicular to $u^\alpha(\tau_0)$. It is convenient to introduce a scalar function $\tau(x)$ which determines the surface $\Sigma(\tau_0)$ by the equation $\tau(x) = \tau_0$, and $\partial\tau/\partial x^\beta = -u_\beta$ at $x = z(\tau_0)$. Then we have

$$\int_{\Sigma(\tau_0)} \bar{g}_\alpha{}^\mu T^{\alpha\beta} d\Sigma_\beta = \int d^4x\sqrt{-g}\frac{\partial\tau}{\partial x^\beta}\delta(\tau(x) - \tau_0)\bar{g}_\alpha{}^\mu T^{\alpha\beta}(x)$$
$$= \int d\tau'\left\{\delta(\tau' - \tau_0)\left[p^\mu + p^{[\mu}v^{\nu]}u_\nu - \frac{1}{2}\frac{Du_\nu}{d\tau}S^{\nu\mu}\right]\right\}$$
$$= p^\mu(\tau_0), \qquad (11{\cdot}12)$$

where we used the center of mass condition and the equation of motion for $S^{\mu\nu}$. We clearly see p^μ indeed represents the linear momentum of the particle.

In order to clarify the meaning of $S^{\mu\nu}$, following Dixon,[20] we introduce the relative position vector

$$X^\mu := -g^{\mu\nu}\partial_\nu \sigma(x,z), \tag{11·13}$$

where $\sigma(x,z)$ is the squared geodetic interval between z and x defined by using the parametric form of a geodesic $y(u)$ joining $z=y(0)$ and $x=y(1)$ as

$$\sigma(x,z) := \frac{1}{2}\int_0^1 g_{\alpha\beta}\frac{dy^\alpha}{du}\frac{dy^\beta}{du}du. \tag{11·14}$$

Then noting the relations

$$\lim_{x\to z} X^\mu = 0, \quad \lim_{x\to z} X^\mu{}_{,\beta} = \delta^\mu_\beta, \tag{11·15}$$

it is easy to see that

$$S^{\mu\nu} = 2\int_{\Sigma_{\tau_0}} X^{[\mu}\bar{g}_\alpha{}^{\nu]}T^{\alpha\beta}d\Sigma_\beta. \tag{11·16}$$

Now that the meaning of $S^{\mu\nu}$ is manifest. From the above equation, it is also easy to see that the center of mass condition (11·2) is the generalization of the Newtonian counterpart,

$$\int d^3x\rho(x)x^i = 0, \tag{11·17}$$

where ρ is the matter density.

Before closing this subsection, we mention several conserved quantities of the present system. We have already noted that $p_\mu p^\mu = -\mu^2$ and $S_\mu S^\mu = S^2$ are constant along the particle trajectory on an arbitrary spacetime. There will be an additional conserved quantity if the spacetime admits a Killing vector field ξ_μ,

$$\xi_{(\mu;\nu)} = 0. \tag{11·18}$$

Namely, the quantity

$$Q_\xi := p^\mu\xi_\mu - \frac{1}{2}S^{\mu\nu}\xi_{\mu;\nu}, \tag{11·19}$$

is conserved along the particle trajectory.[20] It is easy to verify that Q_ξ is conserved by directly using the equations of motion.

11.2. *Circular orbits near the equatorial plane*

Let us consider circular orbits around a Kerr black hole with a fixed Boyer-Lindquist radial coordinate, $r = r_0$. We consider a class of orbits that would stay on the equatorial plane if the particle were spinless. Hence we assume that $\tilde{\theta} := \theta - \pi/2 \sim O(S/M) \ll 1$. Under this assumption, we write down the equations of motion and solve them up to the linear order in S.

In order to find a solution representing a circular orbit, it is convenient to introduce the tetrad frame defined by

$$e^0{}_\mu = \left(\sqrt{\frac{\Delta}{\Sigma}},\ 0,\ 0, -a\sin^2\theta\sqrt{\frac{\Delta}{\Sigma}}\right),$$

$$e^1{}_\mu = \left(0,\ \sqrt{\frac{\Sigma}{\Delta}},\ 0,\ 0\right),$$
$$e^2{}_\mu = \left(0,\ 0,\ \sqrt{\Sigma},\ 0\right),$$
$$e^3{}_\mu = \left(-\frac{a}{\sqrt{\Sigma}}\sin\theta,\ 0,\ 0,\ \frac{r^2+a^2}{\sqrt{\Sigma}}\sin\theta\right), \qquad (11\cdot20)$$

where $\Sigma = r^2 + a^2\cos^2\theta$, and $e^a{}_\mu = (e^a{}_t, e^a{}_r, e^a{}_\theta, e^a{}_\varphi)$ for $a = 0 \sim 3$. Hereafter, we use the Latin letters to denote the tetrad indices.

For convenience, we introduce $\omega_1 \sim \omega_6$ to represent the tetrad components of the spin coefficients, $\omega_{ab}{}^c = e_a{}^\mu e_b{}^\nu e^c{}_{\nu;\mu}$, near the equatorial plane:

$$\omega_{01}{}^0 = \omega_{00}{}^1 = \omega_1 + O(\tilde\theta^2), \quad \omega_1 = \frac{a^2 - Mr}{r^2\Delta^{1/2}},$$
$$\omega_{31}{}^0 = \omega_{30}{}^1 = \omega_{13}{}^0 = \omega_{10}{}^3 = \omega_{03}{}^1 = -\omega_{01}{}^3 = \omega_2 + O(\tilde\theta^2), \quad \omega_2 := \frac{a}{r^2}$$
$$\omega_{22}{}^1 = -\omega_{21}{}^2 = \omega_{33}{}^1 = -\omega_{31}{}^3 = \omega_3 + O(\tilde\theta^2), \quad \omega_3 := \frac{\Delta^{1/2}}{r^2},$$
$$\omega_{02}{}^0 = \omega_{00}{}^2 = \omega_{12}{}^1 = -\omega_{11}{}^2 = \tilde\theta\,\omega_4 + O(\tilde\theta^2), \quad \omega_4 := -\frac{a^2}{r^3},$$
$$\omega_{32}{}^0 = \omega_{30}{}^2 = -\omega_{23}{}^0 = -\omega_{20}{}^3 = \omega_{03}{}^2 = -\omega_{02}{}^3 = \tilde\theta\,\omega_5 + O(\tilde\theta^2), \quad \omega_5 := -\frac{a\Delta^{1/2}}{r^3},$$
$$\omega_{33}{}^2 = -\omega_{32}{}^3 = \tilde\theta\,\omega_6 + O(\tilde\theta^2), \quad \omega_6 := -\frac{(r^2+a^2)}{r^3}. \qquad (11\cdot21)$$

Since the following relation holds for an arbitrary vector f^μ,

$$e^a{}_\mu \frac{D}{d\tau} f^\mu = \frac{d}{d\tau} f^a - \omega_{bc}{}^a v^b f^c,$$

the tetrad components of $Df^\mu/d\tau$ along a circular orbit are given explicitly as

$$e^0{}_\mu \frac{D}{d\tau} f^\mu = \dot f^0 - \left(Af^1 + \tilde\theta C f^2\right) + O(\tilde\theta^2),$$
$$e^1{}_\mu \frac{D}{d\tau} f^\mu = \dot f^1 - \left(Af^0 + Bf^3 + Ef^2\right) + O(\tilde\theta^2),$$
$$e^2{}_\mu \frac{D}{d\tau} f^\mu = \dot f^2 - \left(\tilde\theta C f^0 + \tilde\theta D f^3 - Ef^1\right) + O(\tilde\theta^2),$$
$$e^3{}_\mu \frac{D}{d\tau} f^\mu = \dot f^3 - \left(-Bf^1 - \tilde\theta D f^2\right) + O(\tilde\theta^2), \qquad (11\cdot22)$$

where A, B, C, D and E are defined by*)

$$A := \omega_1 v^0 + \omega_2 v^3,$$

*) The symbols $A \sim E$ used here to define the auxiliary variables are applicable only in this subsection, and not to be confused with quantities defined with the same symbols such as E for energy, in the other sections.

$$
\begin{aligned}
B &:= \omega_2 v^0 + \omega_3 v^3, \\
C &:= \omega_4 v^0 + \omega_5 v^3, \\
D &:= \omega_5 v^0 + \omega_6 v^3, \\
E &:= \omega_3 v^2,
\end{aligned} \tag{11·23}
$$

and we have assumed that $v^1 = 0$ and $v^2 = O(\tilde{\theta})$.

For convenience, we rewrite the equations of motion by changing the spin variable. Instead of the spin tensor, we introduce a unit vector parallel to the spin, ζ^a, defined by

$$\zeta^a := \frac{S^a}{S} = -\frac{1}{2\mu S}\epsilon^a{}_{bcd} u^b S^{cd}, \tag{11·24}$$

or equivalently by

$$S^{ab} = \mu S \epsilon^{ab}{}_{cd} u^c \zeta^d, \tag{11·25}$$

where ϵ_{abcd} is the completely antisymmetric symbol with the sign convention $\epsilon_{0123} = 1$. As noted in the previous subsection, if we use the spin vector as an independent variable, the center of mass condition (11·2) is replaced by Eq. (11·4), that is

$$\zeta^a u_a = 0. \tag{11·26}$$

Then the equations of motion reduce to

$$
\begin{aligned}
\frac{du^a}{d\tau} &= \omega_{bc}{}^a v^b u^c - S R^a, \\
\frac{d\zeta^a}{d\tau} &= \omega_{bc}{}^a v^b \zeta^c - S u^a \zeta^b R_b,
\end{aligned} \tag{11·27}
$$

where

$$R^a := R^{*a}{}_{bcd} v^b u^c \zeta^d = \frac{1}{2\mu S} R^a{}_{bcd} v^b S^{cd}, \tag{11·28}$$

and $R^*_{abcd} = \frac{1}{2} R_{abef}\epsilon^{ef}{}_{cd}$ is the right dual of the Riemann tensor. It will be convenient to write explicitly the tetrad components of $R^*{}_{abcd}$. Since $\tilde{\theta} = O(S)$, we only need R^*_{abcd} at $O(\tilde{\theta}^0)$. Then the non-vanishing components of $R^*{}_{abcd}$ are given by

$$-\frac{1}{2}R^*_{0123} = -R^*_{0213} = R^*_{0312} = R^*_{1203} = -R^*_{1302} = -\frac{1}{2}R^*_{2301} = -\frac{M}{r^3} + O(\tilde{\theta}^2). \tag{11·29}$$

Although we do not need them, we note that the following components are not identically zero but are of $O(\tilde{\theta})$:

$$R^*_{1212},\ R^*_{1313},\ R^*_{1010},\ R^*_{2323},\ R^*_{2020} \text{ and } R^*_{3030}\,.$$

Further, we may set $u^\mu = v^\mu$ in the equations of motion (11·27).

11.2.1. Lowest order in S

We first solve the equations of motion for a circular orbit at $r = r_0$ at the lowest order in S. For notational simplicity, we omit the suffix 0 of r_0 in the following.

For the class of orbits we have assumed, we have $v^1 = 0$ and $v^2 = O(\tilde{\theta})$. Then the non-trivial equations are

$$\frac{d}{d\tau}v^1 = Av^0 + Bv^3 = 0, \tag{11·30}$$

$$\frac{d}{d\tau}\zeta^2 = 0, \quad \frac{d}{d\tau}\begin{pmatrix} \zeta^0 \\ \zeta^1 \\ \zeta^3 \end{pmatrix} = \begin{pmatrix} 0 & A & 0 \\ A & 0 & B \\ 0 & -B & 0 \end{pmatrix}\begin{pmatrix} \zeta^0 \\ \zeta^1 \\ \zeta^3 \end{pmatrix}. \tag{11·31}$$

The equation (11·30) determines the rotation velocity of the orbital motion. By setting $\xi := v^3/v^0$, we obtain the equation

$$\omega_1 + 2\omega_2\xi + \omega_3\xi^2 = 0, \tag{11·32}$$

which is solved to give

$$\xi = \frac{\pm\sqrt{Mr} - a}{\sqrt{\Delta}}. \tag{11·33}$$

The upper (lower) sign corresponds to the case that v^3 is positive (negative). Then, with the aid of the normalization condition of the four momentum, $v^\mu v_\mu = -1 + O(S^2)$, we find

$$v^0 = \frac{1}{\sqrt{1-\xi^2}}, \quad v^3 = \frac{\xi}{\sqrt{1-\xi^2}}. \tag{11·34}$$

Note that, in this case, the orbital angular frequency Ω_φ is given by a well-known formula,

$$\Omega_\varphi = \frac{\pm\sqrt{M}}{r^{3/2} \pm \sqrt{M}a}. \tag{11·35}$$

On the other hand, the equations of spin (11·31) are solved to give

$$\zeta^2 = -\zeta_\perp, \qquad \begin{pmatrix} \zeta^0 \\ \zeta^1 \\ \zeta^3 \end{pmatrix} = \zeta_\parallel \begin{pmatrix} \alpha\sin(\phi + c_1) + \beta c_2 \\ \cos(\phi + c_1) \\ -\beta\sin(\phi + c_1) - \alpha c_2 \end{pmatrix}, \tag{11·36}$$

where $\zeta_\perp$, $\zeta_\parallel$, c_1 and c_2 are constants, and

$$\alpha = \frac{A}{\sqrt{B^2 - A^2}} = \mp v^3, \quad \beta = \frac{B}{\sqrt{B^2 - A^2}} = \pm v^0,$$
$$\phi = \Omega_p\tau, \quad \Omega_p = \sqrt{B^2 - A^2} = \sqrt{\frac{M}{r^3}}. \tag{11·37}$$

The supplementary condition $v^a\zeta_a = 0$ requires that $c_2 = 0$. The condition $\zeta_a\zeta^a = 1$ implies $\zeta_\perp^2 + \zeta_\parallel^2 = 1$. Further since the origin of the time τ can be chosen arbitrarily, we set $c_1 = 0$. Thus, we obtain

$$\zeta^2 = -\zeta_\perp, \qquad \begin{pmatrix} \zeta^0 \\ \zeta^1 \\ \zeta^3 \end{pmatrix} = \zeta_\parallel \begin{pmatrix} \alpha\sin\phi \\ \cos\phi \\ -\beta\sin\phi \end{pmatrix}. \tag{11·38}$$

Here, we should note that $\Omega_p \neq \Omega_\varphi$ in general if $a \neq 0$ or $S \neq 0$ (see below).

11.2.2. First order in S

Having obtained the leading order solution with respect to S, we now turn to the equations of motion up to the linear order in S. We assume that the spin vector components are expressed in the same form as were in the leading order but consider corrections of $O(S)$ to the coefficients α, β and Ω_p. As we have noted, Eq. (11·6) tells us that v^a can be identified with u^a to $O(S)$. In order to write down the equations of motion up to the linear order in S, we need the explicit form of R^a, which can be evaluated by using the knowledge of the lowest order solution. They are given as

$$\begin{aligned} R^0 &= R^3 = O(\tilde{\theta}), \\ R^1 &= 3\frac{M}{r^3}v^0v^3\zeta^2 + O(\tilde{\theta}), \\ R^2 &= 3\frac{M}{r^3}v^0v^3\zeta^1 + O(\tilde{\theta}). \end{aligned} \tag{11·39}$$

First we consider the orbital equations of motion. With the assumption that $v^1 = 0$ and $v^2 = O(\tilde{\theta})$, the non-trivial equations of the orbital motion are

$$\dot{v}^1 = Av^0 + Bv^3 - SR^1 = 0, \tag{11·40}$$

$$\dot{v}^2 = (Cv^0 + Dv^3)\tilde{\theta} - SR^2. \tag{11·41}$$

The first equation gives the rotation velocity as before, while the second equation determines the motion in the θ-direction.

Again using the variable $\xi = v^3/v^0$, Eq. (11·40) is rewritten as

$$\omega_1 + 2\omega_2\xi + \omega_3\xi^2 + 3\frac{S_\perp M}{r^3}\xi = 0, \tag{11·42}$$

where $S_\perp := S\zeta_\perp$. The solution of this equation is

$$\xi = \left(\frac{\pm\sqrt{Mr} - a}{\sqrt{\Delta}}\right)\left(1 \mp \frac{3S_\perp\sqrt{M}}{2r^{3/2}}\right) + O(S^2). \tag{11·43}$$

Using the relations (11·34), it immediately gives v^0 and v^3. From the definition of the tetrad, we have the following relations,

$$\begin{aligned} v^0 &= \sqrt{\frac{\Delta}{\Sigma}}\left[\frac{dt}{d\tau} - a\sin^2\theta\frac{d\varphi}{d\tau}\right], \\ v^3 &= \frac{\sin\theta}{\sqrt{\Sigma}}\left[-a\frac{dt}{d\tau} + (r^2 + a^2)\frac{d\varphi}{d\tau}\right]. \end{aligned} \tag{11·44}$$

Thus, the orbital angular velocity observed at infinity is calculated to be

$$\begin{aligned} \Omega_\varphi :=& \frac{d\varphi}{dt} = \frac{a + \xi\sqrt{\Delta}}{r^2 + a^2 + a\xi\sqrt{\Delta}} + O(\tilde{\theta}^2) \\ =& \pm\frac{\sqrt{M}}{r^{3/2} \pm a\sqrt{M}}\left[1 - \frac{3S_\perp}{2}\frac{\pm\sqrt{Mr} - a}{r^2 \pm a\sqrt{Mr}}\right] + O(\tilde{\theta}^2). \end{aligned} \tag{11·45}$$

In order to solve the second equation (11·41), we note that $v^2 = \sqrt{\Sigma}\dot{\theta} \simeq r\dot{\tilde{\theta}}$ and

$$Cv^0 + Dv^3 = -\frac{M}{r^2}\frac{1+2\xi^2}{1-\xi^2} + O(S). \tag{11·46}$$

Then we find that Eq. (11·41) reduces to

$$r\ddot{\tilde{\theta}} = -\frac{M}{r^2}\frac{1+2\xi^2}{1-\xi^2}\tilde{\theta} - 3\frac{S_{\|}M}{r^3}\frac{\xi}{1-\xi^2}\cos\phi\,, \tag{11·47}$$

where $S_{\|} = S\zeta_{\|}$. This equation can be solved easily by setting $\tilde{\theta} = \theta_0 \cos\phi$. Recalling that $\Omega_p^2 = M/r^3 + O(S)$, we obtain

$$\theta_0 = -\frac{S_{\|}}{r\xi}. \tag{11·48}$$

Thus we see that the orbit will remain in the equatorial plane if $S_{\|} = 0$, but deviates from it if $S_{\|} \neq 0$. We note that there exists a degree of freedom to add a homogeneous solution of Eq. (11·47), whose frequency, $\Omega_\theta = \sqrt{\frac{M}{r^3}\frac{1+2\xi^2}{1-\xi^2}}$, is different from Ω_p and which corresponds to giving a small inclination angle to the orbit, indifferent to the spin. Here, we only consider the case when this homogeneous solution to $\tilde{\theta}$ is zero, i.e., those orbits which would be on the equatorial plane if the spin were zero. Schematically speaking, the orbits under consideration are those with the total angular momentum $\boldsymbol{J}$ being parallel to the z-direction, which is sum of the orbital and spin angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ (see Fig. 3).

Next we consider the evolution of the spin vector. To the linear order in S, the equations to be solved are

$$\begin{aligned}
\dot{\zeta}^0 &= A\zeta^1 + C\zeta^2\tilde{\theta} - Sv^0\zeta^a R_a,\\
\dot{\zeta}^1 &= A\zeta^0 + B\zeta^3 + E\zeta^2,\\
\dot{\zeta}^2 &= \left(C\zeta^0 + D\zeta^3\right)\tilde{\theta} - E\zeta^1,\\
\dot{\zeta}^3 &= -B\zeta^1 - D\zeta^2\tilde{\theta} - Sv^3\zeta^a R_a.
\end{aligned} \tag{11·49}$$

The third equation is written down explicitly as

$$\dot{\zeta}^2 = -\bar{\theta}\zeta_{\|}\kappa\sin\phi\cos\phi, \tag{11·50}$$

where

$$\kappa := \alpha D - \beta C - \Omega_p\omega_3 r. \tag{11·51}$$

Thus we find

$$\zeta^2 = -\zeta_\perp + \frac{\theta_0\zeta_{\|}\kappa}{4\Omega_p}\cos 2\phi. \tag{11·52}$$

Since the spin vector S^a is itself of $O(S)$ already, the effect of the second term is always unimportant as long as we neglect corrections of $O(S^2)$ to the orbit.

The remaining three equations determine α, β and Ω_p. Corrections of $O(S)$ to α and β are less interesting because they remain to be small however long the

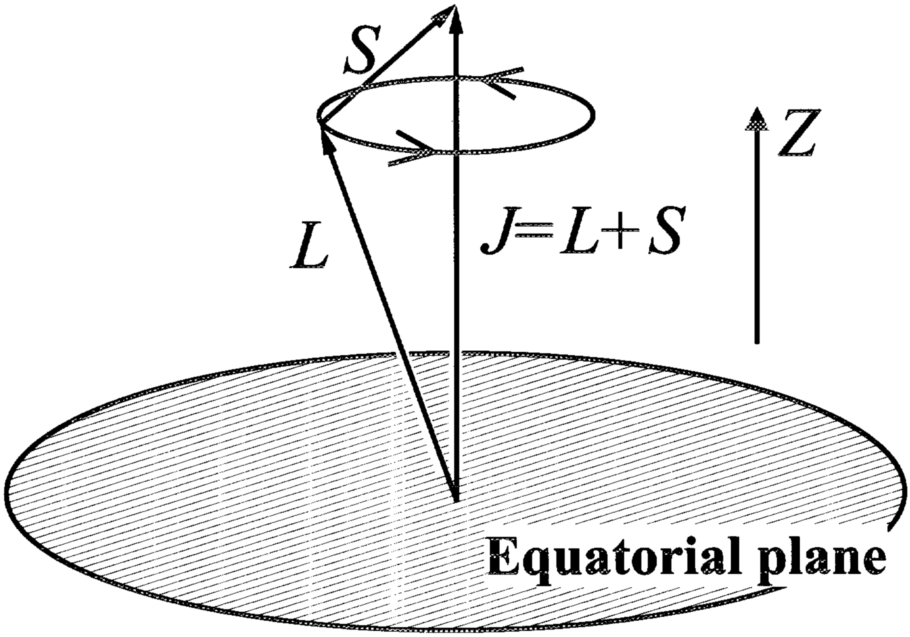

Fig. 3. A schematic picture of the precession of orbit and spin vector, to the leading order in S. The vector $\boldsymbol{J}$ represents the total angular momentum of the particle. The vector $\boldsymbol{L}$ is orthogonal to the orbital plane and reduces to the orbital angular momentum in the Newtonian limit. In the relativistic case, however, these vectors should not be regarded as well-defined.

time passes. On the other hand, the correction to Ω_p will cause a big effect after a sufficiently long lapse of time because it appears in the combination of $\Omega_p\tau$. The small phase correction will be accumulated to become large. Hence, we solve Ω_p to the next leading order. Eliminating ζ^0 and ζ^3 from these three equations, we obtain

$$\left[(B^2 - A^2) - \Omega_p^2\right] = \frac{S_\perp}{\xi}\left(\frac{AC - BD}{r} - \Omega_p^2\omega_3\right). \qquad (11\cdot53)$$

Then after a straightforward calculation, we find

$$\Omega_p^2 = \frac{M}{r^3}\left\{1 - \frac{3S_\perp}{r^{3/2}}\frac{\pm\sqrt{M}\,(2r^2 - 3Mr + a^2) + ar^{1/2}(M - r)}{r^2 - 3Mr \pm 2a\sqrt{Mr}}\right\}. \qquad (11\cdot54)$$

As noted above, $\Omega_p \neq \Omega_\varphi$ for $S_\perp \neq 0$. The difference $\Omega_p - \Omega_\varphi$ gives the angular velocity of the precession of the spin vector, as depicted in Fig. 3.

11.3. *Gravitational waves and energy loss rate*

We now proceed to the calculation of the source terms in the Teukolsky equation and evaluate the gravitational wave flux. For this purpose, we must write down the expression of the energy momentum tensor of the spinning particle explicitly. We rewrite the tetrad components of the energy momentum tensor in the following way:

$$T^{ab} = \mu \int d\tau \left\{ u^{(a}v^{b)}\frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} - e^{(a}_{\ \nu}e^{b)}_{\ \rho}\nabla_\mu S^{\mu\nu}v^\rho\frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \right\}$$

$$= \mu \int d\tau \Biggl\{ \left[u^{(a} v^{b)} + \omega_{dc}{}^{(a} v^{b)} S^{dc} - \omega_{dc}{}^{(a} S^{b)d} v^{c} \right] \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}}$$

$$- \frac{1}{\sqrt{-g}} \partial_\mu \left(S^{\mu(a} v^{b)} \delta^{(4)}(x - z(\tau)) \right) \Biggr\}$$

$$=: \mu \int d\tau \left\{ A^{ab} \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} + \frac{1}{\sqrt{-g}} \partial_\mu \left(B^{\mu ab} \delta^4 (x - z(\tau)) \right) \right\}. \quad (11{\cdot}55)$$

The last line gives the definition of A^{ab} and $B^{\mu ab}$. Then the source term of the Teukolsky equation is given by Eq. (2·14) with Eqs. (2·15).

As we will see shortly, the terms proportional to $S_{\|}$ in the energy momentum tensor do not contribute to the energy or angular momentum fluxes at linear order in S. In other words, the energy and angular momentum fluxes are the same for all orbits having the same $S_\perp$. Thus, we ignore these terms in the following discussion. Further we recall that the particle can stay in the equatorial plane if $S_{\|} = 0$. Hence we fix $\theta = \pi/2$ in the following calculations.

Using the formula (2·13), we obtain the amplitude of gravitational waves at infinity as

$$\tilde{Z}^{\infty}_{\ell m\omega} = \tilde{Z}^{nn}_{\ell m\omega} + \tilde{Z}^{\bar{m}n}_{\ell m\omega} + \tilde{Z}^{\bar{m}\bar{m}}_{\ell m\omega}, \quad (11{\cdot}56)$$

where

$$\tilde{Z}^{nn}_{\ell m\omega} = \frac{i\sqrt{2\pi}}{\omega B^{\rm inc}_{\ell m\omega}} \delta(\omega - m\Omega) \left(\frac{dt}{d\tau} \right)^{-1} \left[A_{nn} - i\omega B^t_{nn} + imB^\varphi_{nn} - B^r_{nn} \frac{\partial}{\partial r} \right]$$
$$\times \left[L^\dagger_1 \rho^{-4} \left(L^\dagger_2 \rho^3 {}_{-2}S^{a\omega}_{\ell m} \right) \right]_{\theta=\pi/2} \frac{1}{r\Delta} R^{\rm in}_{\ell m\omega} \Bigg|_{r=r_0},$$

$$\tilde{Z}^{\bar{m}n}_{\ell m\omega} = \frac{i\sqrt{\pi}}{\omega B^{\rm inc}_{\ell m\omega}} \delta(\omega - m\Omega) \left(\frac{dt}{d\tau} \right)^{-1} \left[A_{\bar{m}n} - i\omega B^t_{\bar{m}n} + imB^\varphi_{\bar{m}n} - B^r_{\bar{m}n} \frac{\partial}{\partial r} \right]$$
$$\times \left(L^\dagger_2 {}_{-2}S^{a\omega}_{\ell m} \right)_{\theta=\pi/2} \frac{1}{\sqrt{\Delta}} \left[2\frac{\partial}{\partial r} - \frac{2iK}{\Delta} - \frac{4}{r} \right] R^{\rm in}_{\ell m\omega} \Bigg|_{r=r_0},$$

$$\tilde{Z}^{\bar{m}\bar{m}}_{\ell m\omega} = \frac{i\sqrt{\pi}}{\omega B^{\rm inc}_{\ell m\omega}} \delta(\omega - m\Omega) \left(\frac{dt}{d\tau} \right)^{-1} \left[A_{\bar{m}\bar{m}} - i\omega B^t_{\bar{m}\bar{m}} + imB^\varphi_{\bar{m}\bar{m}} - B^r_{\bar{m}\bar{m}} \frac{\partial}{\partial r} \right]$$
$$\times \left({}_{-2}S^{a\omega}_{\ell m} \right)_{\theta=\pi/2} \left[\frac{\partial^2}{\partial r^2} - 2\left(\frac{1}{r} + \frac{iK}{\Delta} \right) \frac{\partial}{\partial r} - \left(\frac{iK}{\Delta} \right)_{,r} + \frac{2iK}{\Delta r} - \frac{K^2}{\Delta^2} \right] R^{\rm in}_{\ell m\omega} \Bigg|_{r=r_0}, \quad (11{\cdot}57)$$

and

$$A_{nn} = \frac{1}{4} \frac{1}{1-\xi^2} \left\{ 1 - S_\perp \left((2\omega_1 + \omega_3)\xi + \omega_2 \right) \right\},$$

$$B^\mu_{nn} = \frac{1}{4r} S_\perp \frac{1}{1-\xi^2} \left(\frac{r^2 + a^2}{\sqrt{\Delta}} \xi + a,\ -\sqrt{\Delta}\xi,\ 0,\ \frac{a}{\sqrt{\Delta}} \xi + 1 \right),$$

$$
\begin{aligned}
A_{\bar{m}n} &= \frac{i}{4\sqrt{2}}\frac{1}{1-\xi^2}\left\{2\xi - S_\perp\left(\omega_1\xi^2 - 4\omega_2\xi - \omega_3\right)\right\}, \\
B^\mu_{\bar{m}n} &= \frac{i}{4\sqrt{2}r}S_\perp\frac{1}{1-\xi^2}\left(\frac{r^2+a^2}{\sqrt{\Delta}}\xi^2 + a\xi,\ -\sqrt{\Delta}(1+\xi^2),\ 0,\ \frac{a}{\sqrt{\Delta}}\xi^2+\xi\right), \\
A_{\bar{m}\bar{m}} &= -\frac{1}{2}\frac{1}{1-\xi^2}\left\{\xi^2 + S_\perp\left(\omega_2(1+2\xi^2)+\omega_3\xi\right)\right\}, \\
B^\mu_{\bar{m}\bar{m}} &= \frac{1}{2r}S_\perp\frac{1}{1-\xi^2}\left(0,\ \sqrt{\Delta}\xi,\ 0,\ 0\right). \qquad (11\cdot58)
\end{aligned}
$$

The Lorentz factor $dt/d\tau$ which appears in Eqs. (11·57) is calculated from Eqs. (11·44) as

$$\frac{dt}{d\tau} = \frac{1}{r\sqrt{1-\xi^2}}\left(a\xi + \frac{r^2+a^2}{\sqrt{\Delta}}\right). \qquad (11\cdot59)$$

In general, as we have seen in the preceding sections, when the orbit is quasi-periodic the Fourier components of gravitational waves will have a discrete spectrum,

$$\tilde{Z}_{\ell m\omega} = \sum_n \delta(\omega-\omega_n)Z_{\ell m\omega_n}. \qquad (11\cdot60)$$

Then the time-averaged energy flux and the z-component of the angular momentum flux are given by the formulas (2·31) and (2·32), respectively. In the present case, since we may regard the orbits to be on the equatorial plane, the index n degenerates to the angular index m and ω_n is simply given by $m\Omega_\varphi$ ($n=m$). Hence we eliminate the index n in the following discussion. Here we mention the effect of nonzero $S_\parallel$. If we recall that all the terms which are proportional to $S_\parallel$ have the time dependence of $e^{\pm i\Omega_p\tau}$, we find that they give the contribution to the side bands. That is to say, their contributions in $\tilde{Z}_{\ell m\omega}$ are all proportional to $\delta(\omega - m\Omega \pm \Omega_p)$. Then, since the energy and angular momentum fluxes are quadratic in $Z_{\ell m\omega_n}$, they are not affected by the presence of $S_\parallel$ as long as we are working only up to linear order in S.

As before, in order to express the post-Newtonian corrections to the energy flux, we define $\eta_{\ell m\omega}$ as

$$\left(\frac{dE}{dt}\right)_{\ell m} = \frac{1}{2}\left(\frac{dE}{dt}\right)_{\rm N}\eta_{\ell m}, \qquad (11\cdot61)$$

where $(dE/dt)_{\rm N}$ is the Newtonian quadrupole formula defined by Eq. (4·19).

We calculate $\eta_{\ell m}$ up to 2.5PN order. Keeping the S-dependent terms, the results are

$$
\begin{aligned}
\eta^{(s)}_{2\pm2} &= \left(-\frac{19}{3}v^3 + 9qv^4 + \frac{2134}{63}v^5\right)\hat{s}, \\
\eta^{(s)}_{2\pm1} &= \left(-\frac{1}{12}v^3 - \frac{1}{8}qv^4 - \frac{535}{1008}v^5\right)\hat{s}, \\
\eta^{(s)}_{3\pm3} &= -\frac{10935}{896}v^5\hat{s}, \\
\eta^{(s)}_{3\pm2} &= \frac{20}{63}v^5\hat{s}, \\
\eta^{(s)}_{3\pm1} &= -\frac{1}{8064}v^5\hat{s}, \qquad (11\cdot62)
\end{aligned}
$$

where $q = a/M$ and $\hat{s} := S_\perp/M$. The rest of $\eta_{\ell m}^{(s)}$ are all of higher order. We should mention that if we regard the spinning particle as a model of a black hole or neutron star, S is of order μ. Therefore the correction due to S is small compared with the S-independent terms in the test particle limit $\mu/M \ll 1$.

Putting all together, we obtain

$$\left\langle \frac{dE}{dt} \right\rangle = \left(\frac{dE}{dt}\right)_{\rm N} \left[1 - \frac{1247}{336} v^2 + \left(4\pi - \frac{73}{12} q - \frac{25}{4}\hat{s}\right) v^3 + \left(-\frac{44711}{9072} + \frac{33}{16} q^2 + \frac{71}{8} q\hat{s}\right) v^4 + \left(-\frac{8191}{672}\pi + \frac{3749}{336} q + \frac{2403}{112}\hat{s}\right) v^5 \right]. \quad (11\cdot63)$$

Since v is defined in terms of the coordinate radius of the orbit, the expansion with respect to v does not have a clear gauge-invariant meaning. In particular, for the purpose of the comparison with the standard post-Newtonian calculations it is better to write the result by means of the angular velocity observed at infinity. Using the post-Newtonian expansion of Eq. (11·45)

$$M\Omega_\varphi = v^3 \left(1 - \left(\frac{3}{2}\hat{s} + q\right) v^3 + \frac{3}{2} q\hat{s} v^4 + O\left(v^6\right)\right), \quad (11\cdot64)$$

Eq. (11·63) can be rewritten as

$$\left\langle \frac{dE}{dt} \right\rangle = \left(\widetilde{\frac{dE}{dt}}\right)_{\rm N} \left[1 - \frac{1247}{336} x^2 + \left(4\pi - \frac{11}{4} q - \frac{5}{4}\hat{s}\right) x^3 + \left(-\frac{44711}{9072} + \frac{33}{16} q^2 + \frac{31}{8} q\hat{s}\right) x^4 + \left(-\frac{8191}{672}\pi + \frac{59}{16} q - \frac{13}{16}\hat{s}\right) x^5 \right], \quad (11\cdot65)$$

where $x = (M\Omega_\varphi)^{1/3}$, and $(\widetilde{dE/dt})_{\rm N}$ is the Newtonian quadrupole formula expressed in terms of x, Eq. (6·6). Since there is no side-band contribution in the present case, the angular momentum flux is simply given by $\langle dJ_z/dt \rangle = \Omega_\varphi^{-1} \langle dE/dt \rangle_{\rm GW}$. The result (11·65) is consistent with the one obtained by the standard post-Newtonian approach[41),54)] to the 2PN order in the limit $\mu/M \to 0$. The $\hat{s}$-dependent term of order x^5 is the one which is newly obtained by the black hole perturbation approach.[18)]

§12. Black hole absorption

When a particle moves around a Kerr black hole, it radiates gravitational waves. Some of those waves are absorbed by the black hole. We calculate such absorption of gravitational waves induced by a particle of mass μ in circular orbit on the equatorial plane around a Kerr black hole of Mass M.

The post-Newtonian approximation of the absorption of gravitational waves into the black hole horizon was first calculated by Poisson and Sasaki in the case when a test particle is in a circular orbit around a Schwarzschild black hole.[21] In this case, the effect of the black hole absorption is found to appear at $O(v^8)$ compared to the flux emitted to infinity and it turns out to be negligible for the orbital evolution of coalescing compact binaries in the near future laser interferometer's band. On the other hand, the black hole absorption appears at $O(v^5)$ if a black hole is rotating. That calculation was done by Tagoshi, Mano and Takasugi.[24]

In order to calculate the post-Newtonian expansion of ingoing gravitational waves into a Schwarzschild black hole, Poisson and Sasaki[21] used two types of representations of a solution of the homogeneous Teukolsky equation. One is expressed in terms of the spherical Bessel functions which can be used at large radius, and the other is expressed in terms of a hypergeometric function which can be used near the horizon. Then two types of expressions are matched at some region where both formulas can be applied. They obtained formulas for a solution of the Teukolsky equation which can be used to calculate ingoing gravitational waves to $O(v^{13})$, although they gave formulas for ingoing waves only to $O(v^8)$.

Here, we first review the method found by Mano, Suzuki and Takasugi,[23] since it is the only existing method by which higher order post-Newtonian terms of the gravitational waves absorbed into a rotating black hole can be calculated. We note that this method is also the only existing method that can be used to calculate the gravitational waves emitted to infinity to an arbitrarily high post-Newtonian order. Then we calculate the energy flux absorbed into the horizon to $O(v^{13})$, i.e., $O(v^8)$ beyond the lowest order flux absorbed into the horizon, for circular orbits on the equatorial plane of a Kerr black hole.

12.1. *Analytic solutions of the homogeneous Teukolsky equation*

Analytic series solutions of the homogeneous Teukolsky equation were found by Mano, Suzuki and Takasugi[23] and various properties of the solution were discussed by Mano and Takasugi.[55] Here, we follow the notation of Ref. 55) except that we focus on the case of spin weight $s=-2$. In this method, the solution of the radial Teukolsky equation (2·3) is represented by two kinds of expansion. One is given by a series of hypergeometric functions and the other by a series of Coulomb wave functions. The former is convergent at horizon and the latter at infinity. Then the matching of these two solutions is done exactly in the overlapping region of convergence.

First we consider the solution expressed in terms of hypergeometric functions. The solution which satisfies the ingoing wave boundary condition at horizon is expressed as

$$\begin{aligned} R^{\mathrm{in}\,\nu} =& e^{i\epsilon\kappa x}(-x)^{2-i\epsilon_+}(1-x)^{i\epsilon_-} \\ &\times \sum_{n=-\infty}^{\infty} a_n^{\nu} F(n+\nu+1-i\tau, -n-\nu-i\tau, 3-2i\epsilon_+; x), \end{aligned} \tag{12·1}$$

where $F(a,b,c,x)$ is the hypergeometric function and

$$x = -\frac{\omega(r - r_+)}{\epsilon\kappa}, \quad \kappa = \sqrt{1 - q^2}, \quad \tau = \frac{\epsilon - mq}{\kappa}, \quad \epsilon_\pm = \frac{\epsilon \pm \tau}{2}. \qquad (12{\cdot}2)$$

The coefficients a_n^ν obey a three terms recurrence relation,

$$\alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0, \qquad (12{\cdot}3)$$

where

$$\begin{aligned}
\alpha_n^\nu &= \frac{i\epsilon\kappa(n+\nu-1+i\epsilon)(n+\nu-1-i\epsilon)(n+\nu+1+i\tau)}{(n+\nu+1)(2n+2\nu+3)}, \\
\beta_n^\nu &= -\lambda - 2 + (n+\nu)(n+\nu+1) + \epsilon^2 + \epsilon(\epsilon - mq) \\
&\quad + \frac{\epsilon(\epsilon - mq)(4+\epsilon^2)}{(n+\nu)(n+\nu+1)}, \\
\gamma_n^\nu &= -\frac{i\epsilon\kappa(n+\nu+2+i\epsilon)(n+\nu+2-i\epsilon)(n+\nu-i\tau)}{(n+\nu)(2n+2\nu-1)}.
\end{aligned} \qquad (12{\cdot}4)$$

The series converges if ν satisfies the equation,

$$R_n(\nu)L_{n-1}(\nu) = 1, \qquad (12{\cdot}5)$$

where $R_n(\nu)$ and $L_n(\nu)$ are the continued fractions defined by

$$\begin{aligned}
R_n(\nu) &\equiv \frac{a_n^\nu}{a_{n-1}^\nu} = -\frac{\gamma_n^\nu}{\beta_n^\nu + \alpha_n^\nu R_{n+1}(\nu)}, \\
L_n(\nu) &\equiv \frac{a_n^\nu}{a_{n+1}^\nu} = -\frac{\alpha_n^\nu}{\beta_n^\nu + \gamma_n^\nu L_{n-1}(\nu)}.
\end{aligned} \qquad (12{\cdot}6)$$

The range of convergence is $0 \le (-x) < \infty$ for physical x. We can prove that if ν is a solution to Eq. (12·5), then so is $-\nu - 1$. Since we can set n in Eq. (12·5) to an arbitrary integer, a convenient choice is to put $n = 1$. Further, for convenience, we may set $a_0^\nu = a_0^{-\nu-1} = 1$. It then follows that we have

$$a_{-n}^{-\nu-1} = a_n^\nu. \qquad (12{\cdot}7)$$

This implies $R^{\mathrm{in}\,\nu} = R^{\mathrm{in}\,-\nu-1} \equiv R^{\mathrm{in}}$. Consequently, for $(-x) > 0$, Eq. (12·1) can be rewritten as

$$R^{\mathrm{in}} = e^{i\epsilon\kappa}(R_0^\nu + R_0^{-\nu-1}), \qquad (12{\cdot}8)$$

where

$$\begin{aligned}
R_0^\nu &= e^{-i\epsilon\kappa\tilde{x}}(\tilde{x})^{\nu+i\epsilon_+}(\tilde{x}-1)^{-s-i\epsilon_+} \\
&\quad \times \sum_{n=-\infty}^{\infty} \frac{\Gamma(3-2i\epsilon_+)\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-i\tau)\Gamma(n+\nu+3-i\epsilon)} a_n^\nu \\
&\quad \times \tilde{x}^n F\left(-n-\nu-i\tau, -n-\nu+2-i\epsilon, -2n-2\nu; \frac{1}{\tilde{x}}\right),
\end{aligned} \qquad (12{\cdot}9)$$

where

$$\tilde{x} = 1 - x = \frac{\omega(r - r_-)}{\epsilon\kappa}. \tag{12·10}$$

Since $\nu - (-\nu - 1) = 2\nu + 1$ is not an integer in general, the solutions R_0^ν and $R_0^{-\nu-1}$ form a pair of independent solutions.

The other series solution which is convergent at infinity is expressed in terms of the Coulomb wave functions[56)]

$$R_C^\nu = \tilde{z}\left(1 - \frac{\epsilon\kappa}{\tilde{z}}\right)^{2-i\epsilon_+} \sum_{n=-\infty}^{\infty} (-i)^n \frac{(\nu - 1 - i\epsilon)_n}{(\nu + 3 + i\epsilon)_n} a_n^\nu F_{n+\nu}(\tilde{z}), \tag{12·11}$$

where $\tilde{z} = \omega(r - r_-) = \epsilon\kappa\tilde{x}$, $(a)_n = \Gamma(n + a)/\Gamma(a)$, and $F_{n+\nu}(z)$ is the Coulomb wave function given by

$$F_{n+\nu}(z) = e^{-iz}(2z)^{n+\nu} z \frac{\Gamma(n + \nu + 3 + i\epsilon)}{\Gamma(2n + 2\nu + 2)} \times\Phi(n + \nu + 3 + i\epsilon, 2n + 2\nu + 2; 2iz),$$

where $\Phi(a, b, z)$ is the regular confluent hypergeometric function.[57)] A crucial observation made by Mano, Suzuki and Takasugi[23)] is that the coefficients a_n^ν obey the same recurrence relation as that for the hypergeometric type solution, Eq. (12·3). The series (12·11) converges in the range $\tilde{z} > \epsilon\kappa$ if ν is a solution of Eq. (12·5). The solution R_C^ν can be decomposed into a pair of solutions, a purely incoming wave at infinity R_+^ν and a purely outgoing wave at infinity R_-^ν. Explicitly, we have

$$R_C^\nu = R_+^\nu + R_-^\nu, \tag{12·12}$$

where

$$R_+^\nu = 2^\nu e^{-\pi\epsilon} e^{i\pi(\nu+3)} \frac{\Gamma(\nu + 3 + i\epsilon)}{\Gamma(\nu - 1 - i\epsilon)} e^{-i\tilde{z}} \tilde{z}^{\nu+i\epsilon_+} (\tilde{z} - \epsilon\kappa)^{2-i\epsilon_+} \times \sum_{n=-\infty}^{\infty} i^n a_n^\nu (2\tilde{z})^n \Psi(n + \nu + 3 + i\epsilon, 2n + 2\nu + 2; 2i\tilde{z}), \tag{12·13}$$

$$R_-^\nu = 2^\nu e^{-\pi\epsilon} e^{-i\pi(\nu-1)} e^{i\tilde{z}} \tilde{z}^{\nu+i\epsilon_+} (\tilde{z} - \epsilon\kappa)^{2-i\epsilon_+} \sum_{n=-\infty}^{\infty} i^n \times \frac{(\nu - 1 - i\epsilon)_n}{(\nu + 3 + i\epsilon)_n} a_n^\nu (2\tilde{z})^n \Psi(n + \nu - 1 - i\epsilon, 2n + 2\nu + 2; -2i\tilde{z}), \tag{12·14}$$

where $\Psi(a, b, z)$ is the irregular confluent hypergeometric function.[57)] By definition, the upgoing solution $R^{\rm up}$ is given by

$$R^{\rm up} = R_-^\nu. \tag{12·15}$$

We see that the above two kinds of solutions are convergent in the common region $1 < \tilde{x} < \infty$. Then comparing the asymptotic behavior of R_0^ν and R_C^ν for $\tilde{x} \to \infty$, we find that they have the same characteristic exponent, $\sim \tilde{x}^\nu$, hence describe the same solution up to the normalization factor. Therefore by comparing each power of $\tilde{x}$ we have

$$R_0^\nu = K_\nu R_C^\nu, \tag{12·16}$$

where

$$K_\nu = \frac{(2\epsilon\kappa)^{-\nu-2-\tilde{r}} 2^2 i^{\tilde{r}} \Gamma(3-2i\epsilon_+)\Gamma(\tilde{r}+2\nu+1)\Gamma(\tilde{r}+2\nu+2)}{\Gamma(\nu+1-i\tau)\Gamma(\nu+3-i\epsilon)\Gamma(\tilde{r}+\nu+3+i\epsilon)}$$
$$\times \frac{\Gamma(\nu+1+i\tau)\Gamma(\nu-1+i\epsilon)}{\Gamma(\tilde{r}+\nu+1+i\tau)\Gamma(\tilde{r}+\nu-1+i\epsilon)}$$
$$\times \left(\sum_{n=\tilde{r}}^{\infty} \frac{(\tilde{r}+2\nu+1)_n}{(n-\tilde{r})!} \frac{(\nu-1-i\epsilon)_n}{(\nu+3+i\epsilon)_n} a_n^\nu\right)$$
$$\times \left(\sum_{n=-\infty}^{\tilde{r}} \frac{(-1)^n}{(\tilde{r}-n)!(\tilde{r}+2\nu+2)_n} \frac{(\nu-1-i\epsilon)_n}{(\nu+3+i\epsilon)_n} a_n^\nu\right)^{-1}, \tag{12·17}$$

where $\tilde{r}$ can be any integer and K_ν is independent of the choice of $\tilde{r}$.

The gravitational wave absorbed into the black hole is expressed by Eq. (2·12). Hence we need to know the amplitudes $B^{\rm inc}$ and $B^{\rm trans}$ of $R^{\rm in}$ and $C^{\rm trans}$ of $R^{\rm up}$, defined in Eq. (2·7). The asymptotic ingoing amplitude at horizon, $B^{\rm trans}$ of $R^{\rm in}$ is readily obtained from Eq. (12·1) as

$$B^{\rm trans} = \left(\frac{\epsilon\kappa}{\omega}\right)^{2s} e^{i\epsilon_+ \ln\kappa} \sum_{n=-\infty}^{\infty} a_n^\nu . \tag{12·18}$$

Similarly, the asymptotic outgoing amplitude at infinity, $C^{\rm trans}$ of $R^{\rm up}$ is obtained from Eq. (12·14) as

$$C^{\rm trans} = 2^{1+i\epsilon} e^{-\pi\epsilon/2} e^{-\frac{\pi}{2}i(\nu-1)} \omega^3 e^{i\epsilon\ln\epsilon}$$
$$\times \left[\sum_{n=-\infty}^{\infty} \frac{(\nu-1-i\epsilon)_n}{(\nu+3+i\epsilon)_n} (-1)^n a_n^\nu\right]. \tag{12·19}$$

On the other hand, a bit of work is necessary to obtain the asymptotic incoming amplitude at infinity, $B^{\rm inc}$ of $R^{\rm in}$. Setting the asymptotic behavior of R_C^ν at $z \to \infty$ as

$$R_C^\nu \to A_+^\nu z^{-1} e^{-i(z+\epsilon\ln z)} + A_-^\nu z^3 e^{i(z+\epsilon\ln z)}, \tag{12·20}$$

we find

$$A_+^\nu = 2^{-3-i\epsilon} e^{i(\pi/2)(\nu+3)} e^{-\pi\epsilon/2} \frac{\Gamma(\nu+3+i\epsilon)}{\Gamma(\nu-1-i\epsilon)} \sum_{n=-\infty}^{\infty} a_n^\nu,$$
$$A_-^\nu = 2^{1+i\epsilon} e^{-i(\pi/2)(\nu-1)} e^{-\pi\epsilon/2} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(\nu-1-i\epsilon)_n}{(\nu+3+i\epsilon)_n} a_n^\nu. \tag{12·21}$$

Because of Eq. (12·7), $A^{\nu}_{\pm}$ and $A^{-\nu-1}_{\pm}$ are related to each other as

$$A^{-\nu-1}_{+} = -ie^{-i\pi\nu}\frac{\sin\pi(\nu+i\epsilon)}{\sin\pi(\nu-i\epsilon)}A^{\nu}_{+},$$
$$A^{-\nu-1}_{-} = ie^{i\pi\nu}A^{\nu}_{-}. \qquad (12·22)$$

With the help of the above relations, we find from Eqs. (12·8) and (12·16) the asymptotic amplitudes of $R^{\rm in}$ at infinity as

$$B^{\rm inc} = \frac{e^{i\epsilon\kappa}}{\omega}\left[K_{\nu} - ie^{-i\pi\nu}\frac{\sin\pi(\nu+i\epsilon)}{\sin\pi(\nu-i\epsilon)}K_{-\nu-1}\right]A^{\nu}_{+}e^{-i\epsilon\ln\epsilon},$$
$$B^{\rm ref} = e^{i\epsilon\kappa}\omega^{3}\left[K_{\nu} + ie^{i\pi\nu}K_{-\nu-1}\right]A^{\nu}_{-}e^{i\epsilon\ln\epsilon}. \qquad (12·23)$$

So far our discussion has been on exact analytic series expressions for the homogeneous Teukolsky functions. Now we consider their post-Minkowski expansion by assuming $\epsilon \ll 1$. Provided we set $a^{\nu}_{0} = 1$, we see from Eqs. (12·4) that α^{ν}_{n}, $\gamma^{\nu}_{n} = O(\epsilon)$ and $\beta^{\nu}_{n} = O(1)$ unless the value of ν is such that the denominator in the expression of α^{ν}_{n} or γ^{ν}_{n} happens to vanish or β^{ν}_{n} happens to vanish in the limit $\epsilon \to 0$. Except for such an exceptional case, it is easy to see from Eq. (12·3) that the order of a^{ν}_{n} in ϵ increases as $|n|$ increases. Thus the series solution naturally gives the post-Minkowski expansion.

For the moment, let us assume that the above mentioned exceptional case does not happen for $n = 0$. Then we have $R_1(\nu) = O(\epsilon)$ and $L_0(\nu) = O(1/\epsilon)$. This implies $\beta^{\nu}_{0} + \gamma^{\nu}_{0}L_{-1}(\nu) = O(\epsilon^2)$. Then assuming $L_{-1}(\nu) = O(\epsilon)$, we must have $\beta^{\nu}_{0} = O(\epsilon^2)$. Using the expansion of λ given by Eq. (3·1), we then find $\nu = \ell + O(\epsilon^2)$ or $\nu = -\ell - 1 + O(\epsilon^2)$. Since we know that $-\nu - 1$ is a solution if ν is so, we may take the solution $\nu = \ell + O(\epsilon^2)$ without loss of generality. Then the assumptions that $R_1(\nu) = 1/L_0(\nu) = O(\epsilon)$ and $L_{-1}(\nu) = O(\epsilon)$ are justified. Further it is easily seen that $R_n(\nu) = O(\epsilon)$ for all $n > 0$. On the other hand, for $n < 0$, $\alpha^{\nu}_{n} = O(1)$ at $n = -\ell - 1$ and $\beta^{\nu}_{n} = O(\epsilon^2)$ at $n = -2\ell - 1$. Thus we have $L_{-\ell-1}(\nu) = O(1)$ and $L_{-2\ell-1}(\nu) = O(1/\epsilon)$. To summarize, we have

$$R_n(\nu) = \frac{a^{\nu}_{n}}{a^{\nu}_{n-1}} = O(\epsilon) \quad \text{for all } n > 0,$$
$$L_{-\ell-1}(\nu) = \frac{a^{\nu}_{-\ell-1}}{a^{\nu}_{-\ell}} = O(1), \quad L_{-2\ell-1}(\nu) = \frac{a^{\nu}_{-2\ell-1}}{a^{\nu}_{-2\ell}} = O(1/\epsilon),$$
$$L_n(\nu) = \frac{a^{\nu}_{n}}{a^{\nu}_{n+1}} = O(\epsilon) \quad \text{for all the other } n < 0. \qquad (12·24)$$

With the above results, the post-Minkowski expansion of the homogeneous Teukolsky functions can be obtained with arbitrary accuracy by solving Eq. (12·5) to a desired order and by summing up the terms to a sufficiently large $|n|$. For our present purpose, we need ν which is accurate to $O(\epsilon^2)$. Solving Eq. (12·5) to this order, we find

$$\nu = \ell + \frac{\epsilon^2}{2\ell+1}\left[-2 - \frac{4}{\ell(\ell+1)}\right.$$

$$+\frac{(\ell-1)^2(\ell+3)^2}{(2\ell+1)(2\ell+2)(2\ell+3)}-\frac{(\ell-2)^2(\ell+2)^2}{(2\ell-1)2\ell(2\ell+1)}\Bigg]. \qquad (12\cdot25)$$

Interestingly, ν is found to be independent of the azimuthal eigenvalue m to $O(\epsilon^2)$.

The post-Newtonian expansion in the near zone is given by further assuming $\epsilon \ll z \ll 1$ in the series solution (12·11) and expand it in powers z. For evaluation of the black hole absorption, we need the post-Newtonian expansion of $R^{\rm up}$ which is obtained from Eq. (12·14). The explicit post-Newtonian formula for $R^{\rm up}$ and the asymptotic amplitudes $B^{\rm inc}$, $B^{\rm trans}$ and $C^{\rm trans}$ to $O(\epsilon^2)$ are given in Appendix H.

12.2. *Absorption rate to* $O(v^8)$

In this subsection, we evaluate the energy absorption rate by a black hole. The energy flux formula is given by Teukolsky and Press[58] as

$$\left(\frac{dE_{\rm hole}}{dtd\Omega}\right)=\sum_{\ell m}\int d\omega \frac{{}_2S^2_{\ell m}}{2\pi}\frac{128\omega k(k^2+4\tilde\epsilon^2)(k^2+16\tilde\epsilon^2)(2Mr_+)^5}{|C|^2}|\tilde Z^{\rm H}_{\ell m\omega}|^2, \qquad (12\cdot26)$$

where $\tilde\epsilon=\kappa/(4r_+)$ and

$$|C|^2=\left((\lambda+2)^2+4a\omega m-4a^2\omega^2\right)\left[\lambda^2+36a\omega m-36a^2\omega^2\right] \\ +(2\lambda+3)(96a^2\omega^2-48a\omega m)+144\omega^2(M^2-a^2). \qquad (12\cdot27)$$

The calculation of $\tilde Z^{\rm H}_{\ell m\omega}$ is parallel to the calculation of $\tilde Z^{\infty}_{\ell m\omega}$ except that $R^{\rm in}$ is replaced by $R^{\rm up}$. The solution of the geodesic equations are given in §6. Using that solution, we have the amplitude of the Teukolsky function at the horizon in Eq. (2·12) as

$$\tilde Z^{\rm H}_{\ell m\omega}=\frac{2\pi B^{\rm trans}\delta(\omega-m\Omega)}{2i\omega B^{\rm inc}C^{\rm trans}}\Bigg[R^{\rm up}_{\ell m\omega}\{A_{n\,n\,0}+A_{\bar m\,n\,0}+A_{\bar m\,\bar m\,0}\} \\ -\frac{dR^{\rm up}_{\ell m\omega}}{dr}\{A_{\bar m\,n\,1}+A_{\bar m\,\bar m\,1}\}+\frac{d^2R^{\rm up}_{\ell m\omega}}{dr^2}A_{\bar m\,\bar m\,2}\Bigg]_{r=r_0,\theta=\pi/2} \\ \equiv \delta(\omega-m\Omega)Z^{\rm H}_{\ell m}. \qquad (12\cdot28)$$

From Eqs. (12·26) and (12·28), the time averaged energy absorption rate becomes

$$\left\langle\frac{dE}{dt}\right\rangle_{\rm H}=\sum_{\ell m}\left[\frac{128\omega k(k^2+4\tilde\epsilon^2)(k^2+16\tilde\epsilon^2)(2Mr_+)^5}{|C|^2}|Z^{\rm H}_{\ell m}|^2\right]_{\omega=m\Omega} \\ \equiv\sum_{\ell m}\left(\frac{dE}{dt}\right)_{\ell,m}. \qquad (12\cdot29)$$

As in the case of the Teukolsky function at infinity, we can show that $\bar Z^{\rm H}_{\ell,-m,-\omega} = (-1)^\ell Z^{\rm H}_{\ell,m,\omega}$. Then, from Eq. (12·29), we have $(dE/dt)_{\ell,-m}=(dE/dt)_{\ell,m}$.

In order to express the post-Newtonian corrections to the black hole absorption, we define $\eta^{\rm H}_{\ell,m}$ as

$$\left(\frac{dE}{dt}\right)_{\ell,m}\equiv\frac{1}{2}\left(\frac{dE}{dt}\right)_{\rm N}v^5\eta^{\rm H}_{\ell,m}, \qquad (12\cdot30)$$

where $(dE/dt)_{\rm N}$ is the Newtonian quadrupole luminosity at infinity, Eq. (4·19). In Appendix I, we show $\eta_{\ell m}$.

The total absorption rate to $O(v^8)$ beyond the lowest order is given by

$$\left\langle\frac{dE}{dt}\right\rangle_{\rm H} = \left(\frac{dE}{dt}\right)_{\rm N} v^5 \Bigg[-\frac{3}{4}q^3 - \frac{1}{4}q + \left(-q - \frac{33}{16}q^3\right)v^2$$

$$+\left(\frac{7}{2}q^4 + 2qB_2 + \frac{1}{2} + 6q^3B_2 + \frac{85}{12}q^2 + 3q^4\kappa + \frac{1}{2}\kappa + \frac{13}{2}\kappa q^2\right)v^3$$

$$+\left(-\frac{4651}{336}q^3 - \frac{43}{7}q - \frac{17}{56}q^5\right)v^4$$

$$+\Bigg(\frac{569}{24}q^2 + \frac{371}{48}q^4 + 18q^3B_2 - \frac{3}{4}q^3B_1 + 2\kappa + 2 + \frac{33}{4}q^4\kappa$$

$$+6qB_2 + \frac{163}{8}\kappa q^2 + qB_1\Bigg)v^5$$

$$+\Bigg(-\frac{2718629}{44100}q - 4B_2 + \frac{428}{105}\gamma q + \frac{2}{3}\pi^2 q + \frac{428}{105}q\ln 2 - 4qC_2$$

$$-12q^3C_2 - 36q^4B_2 - 56q^2B_2 + \frac{428}{35}q^3\gamma + \frac{428}{35}q^3\ln 2 + 2q^3\pi^2$$

$$+\frac{428}{105}q\ln\kappa + \frac{428}{105}qA_2 + \frac{428}{35}q^3\ln\kappa + 6\frac{q^7}{\kappa} + \frac{428}{35}q^3A_2 - 8q{B_2}^2$$

$$-24q^3{B_2}^2 + \frac{856}{105}q\ln v + \frac{856}{35}q^3\ln v - 4\frac{B_2}{\kappa} - 32\frac{q^3}{\kappa} - 31\frac{q}{\kappa}$$

$$+57\frac{q^5}{\kappa} + q^5\kappa + q^4B_1 - \frac{2}{3}\kappa q - \frac{7}{6}\kappa q^3 - \frac{4}{3}q^2B_1$$

$$-48\frac{q^2B_2}{\kappa} + 28\frac{q^4B_2}{\kappa} - 24\frac{q^3C_2}{\kappa} - 8\frac{qC_2}{\kappa} + 24\frac{q^6B_2}{\kappa}$$

$$-\frac{2400247}{19600}q^3 + \frac{299}{16}q^5\Bigg)v^6$$

$$+\Bigg(\frac{225}{28}q^5B_3 - \frac{41}{28}\kappa q^6 + \frac{86}{7} + \frac{8741}{56}q^2 + \frac{3485}{42}q^4 + \frac{167}{112}q^6$$

$$+\frac{86}{7}\kappa + \frac{45}{56}qB_3 - \frac{9}{28}q^5B_1 + \frac{899}{168}qB_1 - \frac{803}{224}q^3B_1 + \frac{1665}{224}q^3B_3$$

$$+\frac{2372}{21}q^3B_2 - \frac{16}{7}q^5B_2 + \frac{719}{12}q^4\kappa + \frac{22201}{168}\kappa q^2 + \frac{796}{21}qB_2\Bigg)v^7$$

$$+\Bigg(-\frac{20542807}{88200}q - 12B_2 - 2B_1 + \frac{1061}{35}\gamma q + \frac{13}{6}\pi^2 q + \frac{995}{21}q\ln 2$$

$$-12qC_2 - 36q^3C_2 - \frac{308}{3}q^4B_2 - \frac{1496}{9}q^2B_2 + \frac{12197}{140}q^3\gamma + \frac{3873}{28}q^3\ln 2$$

$$+\frac{47}{8}q^3\pi^2 + \frac{1391}{105}q\ln\kappa + \frac{428}{35}qA_2 + \frac{5029}{140}q^3\ln\kappa + \frac{37}{6}\frac{q^7}{\kappa} + \frac{1284}{35}q^3A_2$$

$$-24q{B_2}^2 - 72q^3{B_2}^2 + \frac{4574}{105}q\ln v + \frac{8613}{70}q^3\ln v - 12\frac{B_2}{\kappa} - \frac{341}{4}\frac{q^3}{\kappa}$$

$$-\frac{637}{6}\frac{q}{\kappa}+\frac{741}{4}\frac{q^5}{\kappa}+\frac{73}{6}q^4B_1-qC_1-\frac{283}{18}q^2B_1$$
$$+\frac{3}{2}q^3{B_1}^2+\frac{107}{105}qA_1-\frac{107}{140}q^3A_1-3\frac{q^6B_1}{\kappa}+\frac{13}{2}\frac{q^4B_1}{\kappa}-\frac{3}{2}\frac{q^2B_1}{\kappa}$$
$$-2\frac{qC_1}{\kappa}+\frac{3}{2}\frac{q^3C_1}{\kappa}-2\,q{B_1}^2+\frac{3}{4}q^3C_1-2\frac{B_1}{\kappa}-144\frac{q^2B_2}{\kappa}$$
$$+84\frac{q^4B_2}{\kappa}-72\frac{q^3C_2}{\kappa}-24\frac{qC_2}{\kappa}+72\frac{q^6B_2}{\kappa}-\frac{2945984497}{6350400}q^3$$
$$\left.\left.+\frac{1385}{24}q^5+\frac{25}{252}q^7\right)v^8\right], \tag{12·31}$$

where

$$A_n=\frac{1}{2}\left[\psi^{(0)}\left(3+\frac{niq}{\sqrt{1-q^2}}\right)+\psi^{(0)}\left(3-\frac{niq}{\sqrt{1-q^2}}\right)\right],$$
$$B_n=\frac{1}{2i}\left[\psi^{(0)}\left(3+\frac{niq}{\sqrt{1-q^2}}\right)-\psi^{(0)}\left(3-\frac{niq}{\sqrt{1-q^2}}\right)\right],$$
$$C_n=\frac{1}{2}\left[\psi^{(1)}\left(3+\frac{niq}{\sqrt{1-q^2}}\right)+\psi^{(1)}\left(3-\frac{niq}{\sqrt{1-q^2}}\right)\right], \tag{12·32}$$

and $\psi^{(n)}(z)$ is the polygamma function. We see that the absorption effect starts at $O(v^5)$ beyond the quadrapole formula in the case $q \neq 0$, while for $q=0$, the above formula reduced to

$$\left(\frac{dE}{dt}\right)_{\rm H}=\left(\frac{dE}{dt}\right)_{\rm N}\left(v^8+O(v^{10})\right), \tag{12·33}$$

as was found by Poisson and Sasaki.[21] We note that the leading terms in $\langle dE/dt\rangle_{\rm H}$ are negative for $q>0$, i.e., the black hole loses the energy if the particle is corotating. This is because of the superradiance for modes with $k<0$. In Appendix J, we also show $\langle dE/dt\rangle_{\rm H}$ written in terms of $x\equiv(M\Omega_\varphi)^{1/3}$.

It is not manifest from Eq. (12·31) that if it has a finite limit for $|q|\to 1$. But by using the formulas,

$$\lim_{q\to\pm1}\psi^{(0)}\left(3+\frac{niq}{\sqrt{1-q^2}}\right)=\ln n-\ln\kappa+i\frac{q}{|q|}\frac{\pi}{2}, \tag{12·34}$$

$$\lim_{q\to\pm1}\psi^{(k)}\left(3+\frac{niq}{\sqrt{1-q^2}}\right)=0,\quad (k\neq 0) \tag{12·35}$$

we obtain the limit of $\langle dE/dt\rangle_{\rm H}$ as

$$\lim_{q\to\pm1}\left(\frac{dE}{dt}\right)_{\rm H}=\left(\frac{dE}{dt}\right)_{\rm N}v^5\left[-\frac{q}{|q|}-\frac{49}{16}\frac{q}{|q|}v^2+\left(4\pi+\frac{133}{12}\right)v^3\right.$$
$$-\frac{6817}{336}\frac{q}{|q|}v^4+\left(\frac{535}{16}+\frac{97}{8}\pi\right)v^5$$

$$
+\left(\frac{3424}{105}\frac{q}{|q|}\ln(2)+\frac{1712}{105}\gamma\frac{q}{|q|}+\frac{3424}{105}\frac{q}{|q|}\ln(v)\right.
$$
$$
\left.-\frac{3647533}{22050}\frac{q}{|q|}-\frac{289}{6}\frac{q}{|q|}\pi-16/3\,\pi^2\frac{q}{|q|}\right)v^6
$$
$$
+\left(\frac{84955}{336}+\frac{55873}{672}\pi\right)v^7
$$
$$
+\left(\frac{14077}{60}\frac{q}{|q|}\ln(2)+\frac{16441}{140}\gamma\frac{q}{|q|}+\frac{34987}{210}\frac{q}{|q|}\ln(v)-\frac{193}{12}\pi^2\frac{q}{|q|}\right.
$$
$$
\left.\left.-\frac{4057965601}{6350400}\frac{q}{|q|}-\frac{1289}{9}\frac{q}{|q|}\pi\right)v^8\right]. \qquad (12\cdot36)
$$

Appendix A
—— *Spheroidal Harmonics* ——

In this appendix, we describe the expansion of the spheroidal harmonics ${}_{-2}S^{a\omega}_{\ell m}$ to $O((a\omega)^2)$.

The spheroidal harmonics of spin weight $s=-2$ obey the equation,

$$
\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{d}{d\theta}\right\}-a^2\omega^2\sin^2\theta-\frac{(m-2\cos\theta)^2}{\sin^2\theta}\right.
$$
$$
\left.+4a\omega\cos\theta-2+2ma\omega+\lambda\right]{}_{-2}S^{a\omega}_{\ell m}=0. \qquad (\mathrm{A}\cdot1)
$$

We expand ${}_{-2}S^{a\omega}_{\ell m}$ and λ as

$$
{}_{-2}S^{a\omega}_{\ell m}={}_{-2}P_{\ell m}+a\omega S^{(1)}_{\ell m}+(a\omega)^2S^{(2)}_{\ell m}+O((a\omega)^3),
$$
$$
\lambda=\lambda_0+a\omega\lambda_1+a^2\omega^2\lambda_2+O((a\omega)^3), \qquad (\mathrm{A}\cdot2)
$$

where ${}_{-2}P_{\ell m}$ are the spherical harmonics of spin weight $s=-2$. We set the normalizations of ${}_{-2}P_{\ell m}$ and ${}_{-2}S^{a\omega}_{\ell m}$ as

$$
\int_0^\pi|{}_{-2}P_{\ell m}|^2\sin\theta d\theta=\int_0^\pi|{}_{-2}S^{a\omega}_{\ell m}|^2\sin\theta d\theta=1. \qquad (\mathrm{A}\cdot3)
$$

Inserting Eq. (A·2) into Eq. (A·1) and collecting the terms of the same order to $(a\omega)^2$, we obtain

$$
[\mathcal{L}_0+\lambda_0]\,{}_{-2}P_{\ell m}=0\,, \qquad (\mathrm{A}\cdot4)
$$
$$
[\mathcal{L}_0+\lambda_0]\,S^{(1)}_{\ell m}=-(4\cos\theta+2m+\lambda_1)\,{}_{-2}P_{\ell m} \qquad (\mathrm{A}\cdot5)
$$
$$
[\mathcal{L}_0+\lambda_0]\,S^{(2)}_{\ell m}=-(4\cos\theta+2m+\lambda_1)S^{(1)}_{\ell m}-(\lambda_2-\sin^2\theta)\,{}_{-2}P_{\ell m}\,, \qquad (\mathrm{A}\cdot6)
$$

where

$$
\mathcal{L}_0\equiv\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)-\frac{(m-2\cos\theta)^2}{\sin^2\theta}-2\,. \qquad (\mathrm{A}\cdot7)
$$

The lowest order equation (A·4) says we have $\lambda_0=(\ell-1)(\ell+2)$.

The first order correction to the eigenvalue, λ_1, is obtained by multiplying Eq. (A·5) by ${}_{-2}P_{\ell m}$ from the left-hand side and integrating it over θ. The result is

$$\lambda_1 = -2m\frac{\ell(\ell+1)+4}{\ell(\ell+1)}\,. \tag{A·8}$$

To obtain $S^{(1)}_{\ell m}$, we set

$$S^{(1)}_{\ell m} = \sum_{\ell'} c^{\ell'}_{\ell m}\ {}_{-2}P_{\ell' m}\,. \tag{A·9}$$

We insert this into Eq. (A·5), multiply it by ${}_{-2}P_{\ell' m}$ and integrate it over θ. Then noting the normalization of the spheroidal harmonics, we have

$$c^{\ell'}_{\ell m} = \begin{cases} \dfrac{4}{(\ell'-1)(\ell'+2)-(\ell-1)(\ell+2)}\displaystyle\int d(\cos\theta)\ {}_{-2}P_{\ell' m}\cos\theta\ {}_{-2}P_{\ell m}\,, & \ell' \neq \ell, \\ 0, & \ell' = \ell. \end{cases}$$

Hence $c^{\ell'}_{\ell m}$ is non-zero only for $\ell' = \ell \pm 1$, and we obtain

$$c^{\ell+1}_{\ell m} = \frac{2}{(\ell+1)^2}\left[\frac{(\ell+3)(\ell-1)(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right]^{1/2},$$
$$c^{\ell-1}_{\ell m} = -\frac{2}{\ell^2}\left[\frac{(\ell+2)(\ell-2)(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2}.$$

The next order equation can be solved similarly. The second order correction to the eigenvalue, λ_2, is obtained by multiplying Eq. (A·6) by ${}_{-2}P_{\ell m}$ from the left-hand side and integrating it over θ. We find

$$\begin{aligned}\lambda_2 &= -4\int d(\cos\theta)_{-2}P_{\ell m}\cos\theta\, S^{(1)}_{\ell m} + \int d(\cos\theta)_{-2}P_{\ell m}\sin^2\theta\ {}_{-2}P_{\ell m} \\ &= -2(\ell+1)(c^{\ell+1}_{\ell m})^2 + 2\ell(c^{\ell-1}_{\ell m})^2 + 1 - \int d(\cos\theta)_{-2}P_{\ell m}\cos^2\theta\ {}_{-2}P_{\ell m}\,,\end{aligned} \tag{A·10}$$

where the last integral becomes

$$\int d(\cos\theta)_{-2}P_{\ell m}\cos^2\theta\ {}_{-2}P_{\ell m} = \frac{1}{3} + \frac{2}{3}\frac{(\ell+4)(\ell-3)(\ell^2+\ell-3m^2)}{\ell(\ell+1)(2\ell+3)(2\ell-1)}\,.$$

As before, to obtain $S^{(2)}_{\ell m}$, we set

$$S^{(2)}_{\ell m} = \sum_{\ell'} d^{\ell'}_{\ell m}\ {}_{-2}P_{\ell' m}\,. \tag{A·11}$$

Inserting Eqs. (A·9) and (A·11) into Eq. (A·6), multiplying it by ${}_{-2}P_{\ell' m}$ and integrate it over θ, we obtain

$$\begin{aligned} d^{\ell'}_{\ell m} = \frac{1}{\lambda_0(\ell)-\lambda_0(\ell')}\Big[&-(2m+\lambda_1(\ell))\left(c^{\ell+1}_{\ell m}\delta_{\ell',\ell+1} + c^{\ell-1}_{\ell m}\delta_{\ell',\ell-1}\right) \\ &-4c^{\ell+1}_{\ell m}\int d(\cos\theta)_{-2}P_{\ell' m}\cos\theta\ {}_{-2}P_{\ell+1 m} - 4c^{\ell-1}_{\ell m}\int d(\cos\theta)_{-2}P_{\ell' m}\cos\theta\ {}_{-2}P_{\ell-1 m} \\ &+\int d(\cos\theta)_{-2}P_{\ell' m}\sin^2\theta\ {}_{-2}P_{\ell m}\Big] \end{aligned} \tag{A·12}$$

for $\ell' \neq \ell$. The integrals in this equation are given by[59),60)]

$$\int d(\cos\theta)_{-2}P_{\ell' m}\cos\theta\,_{-2}P_{\ell m} = \sqrt{\frac{2\ell+1}{2\ell'+1}}\langle \ell,1,m,0|\ell',m\rangle\langle \ell,1,2,0|\ell',2\rangle,$$

$$\int d(\cos\theta)_{-2}P_{\ell' m}\sin^2\theta\,_{-2}P_{\ell m} = \frac{2}{3}\delta_{\ell',\ell} - \frac{2}{3}\sqrt{\frac{2\ell+1}{2\ell'+1}}\langle \ell,2,m,0|\ell',m\rangle\langle \ell,2,2,0|\ell',2\rangle,$$

where $\langle j_1, j_2, m_1, m_2|J, M\rangle$ is a Clebsch-Gordan coefficient. For $\ell = 2$ and 3, the non-vanishing $d_{\ell m}^{\ell'}$ ($\ell' \neq \ell$) are given explicitly as

$$d_{\ell m}^{\ell+1} = \frac{m}{324\sqrt{7}}(3-m)^{1/2}(3+m)^{1/2},$$
$$d_{\ell m}^{\ell+2} = \frac{11}{1764\sqrt{3}}(3-m)^{1/2}(3+m)^{1/2}(4-m)^{1/2}(4+m)^{1/2}$$

for $\ell = 2$, and

$$d_{\ell m}^{\ell+1} = \frac{m}{120\sqrt{21}}(4-m)^{1/2}(4+m)^{1/2},$$
$$d_{\ell m}^{\ell+2} = \frac{1}{180\sqrt{11}}(4-m)^{1/2}(4+m)^{1/2}(5-m)^{1/2}(5+m)^{1/2},$$
$$d_{\ell m}^{\ell-1} = -\frac{m}{324\sqrt{7}}(3-m)^{1/2}(3+m)^{1/2}$$

for $\ell = 3$. As for $d_{\ell m}^{\ell}$, it is determined by the normalization of $_{-2}S_{\ell m}^{a\omega}$, i.e.,

$$\begin{aligned}
1 &= \int d(\cos\theta)|_{-2}S_{\ell m}|^2 \\
&= \int d(\cos\theta)\Big\{(_{-2}P_{\ell m})^2 + 2a\omega\sum_{\ell'} c_{\ell m}^{\ell'}\,_{-2}P_{\ell' m}\,_{-2}P_{\ell m} \\
&\quad +(a\omega)^2\sum_{\ell'\ell''} c_{\ell m}^{\ell'} c_{\ell m}^{\ell''}\,_{-2}P_{\ell' m}\,_{-2}P_{\ell'' m} \\
&\quad + 2(a\omega)^2\sum_{\ell'} d_{\ell m}^{\ell'}\,_{-2}P_{\ell' m}\,_{-2}P_{\ell m} + O((a\omega)^3)\Big\} \\
&= 1 + (a\omega)^2\sum_{\ell'}\left(c_{\ell m}^{\ell'}\right)^2 + 2(a\omega)^2 d_{\ell m}^{\ell} + O((a\omega)^3).
\end{aligned}$$

Then we have

$$d_{\ell m}^{\ell} = -\frac{1}{2}\left\{\left(c_{\ell m}^{\ell+1}\right)^2 + \left(c_{\ell m}^{\ell-1}\right)^2\right\}. \qquad \text{(A·13)}$$

Appendix B
—— The Operators $Q^{(2)}$, $Q^{(3)}$ and $Q^{(4)}$ ——

In this appendix, we show the operators $Q^{(n)}$ for $n = 2$, 3 and 4 which appear in Eq. (3·19).

$$\begin{aligned}
Q^{(2)} = \Bigg[\Bigg(& -28\,i\,m\,q - \frac{32\,i\,m\,q}{\ell} + 8\,i\,\ell\,m\,q + 4\,i\,\ell^2\,m\,q - 13\,q^2 - \frac{6\,q^2}{\ell} - 12\,\ell\,q^2 \\
& -\ell^2\,q^2 + 6\,\ell^3\,q^2 + 2\,\ell^4\,q^2 + 8\,m^2\,q^2 + \frac{32\,m^2\,q^2}{\ell^2} + \frac{8\,m^2\,q^2}{\ell}\Bigg)\frac{1}{z^4} \\
& + \Bigg(16\,m\,q + \frac{24\,m\,q}{\ell^2} + \frac{20\,m\,q}{\ell} - 8\,\ell\,m\,q - 4\,\ell^2\,m\,q - 14\,i\,q^2 - \frac{16\,i\,q^2}{\ell} \\
& +4\,i\,\ell\,q^2 + 2\,i\,\ell^2\,q^2 + 2\,i\,\lambda_1\,m\,q^2 - \frac{4\,i\,\lambda_1\,m\,q^2}{\ell^2} \\
& + \frac{2\,i\,\lambda_1\,m\,q^2}{\ell} - 4\,i\,m^2\,q^2 + \frac{56\,i\,m^2\,q^2}{\ell^2} - \frac{4\,i\,m^2\,q^2}{\ell}\Bigg)\frac{1}{z^3} \\
& + \Bigg(\frac{24\,i\,m\,q}{\ell^2} + \frac{17\,q^2}{2} + \frac{10\,q^2}{\ell} - \frac{13\,\ell\,q^2}{4} - \frac{9\,\ell^2\,q^2}{4} - \frac{3\,\ell^3\,q^2}{4} - \frac{\ell^4\,q^2}{4} - \frac{\lambda_2\,q^2}{2} \\
& -\frac{3\,\ell\,\lambda_2\,q^2}{4} + \frac{\ell^2\,\lambda_2\,q^2}{4} + \frac{3\,\ell^3\,\lambda_2\,q^2}{4} + \frac{\ell^4\,\lambda_2\,q^2}{4} - 2\,\lambda_1\,m\,q^2 \\
& + \frac{4\,\lambda_1\,m\,q^2}{\ell^2} - \frac{2\,\lambda_1\,m\,q^2}{\ell} - \frac{24\,m^2\,q^2}{\ell^2}\Bigg)\frac{1}{z^2}\Bigg]\frac{1}{(\ell+1)^2(\ell^2+\ell-2)} \\
& +\Bigg[\Big(-24\,i\,\lambda_0\,m\,q - 4\,i\,{\lambda_0}^2\,m\,q + 4\,i\,{\lambda_0}^3\,m\,q - 12\,\lambda_0\,q^2 - 6\,{\lambda_0}^2\,q^2 \\
& + 24\,\lambda_0\,m^2\,q^2 - 4\,{\lambda_0}^2\,m^2\,q^2\Big)\frac{1}{z^3} + \Big(24\,\lambda_0\,m\,q - 12\,i\,\lambda_0\,q^2 - 2\,i\,{\lambda_0}^2\,q^2 \\
& +2\,i\,{\lambda_0}^3\,q^2 + 2\,i\,{\lambda_0}^2\,\lambda_1\,m\,q^2 + 24\,i\,\lambda_0\,m^2\,q^2\Big)\frac{1}{z^2}\Bigg]\frac{1}{(\lambda_0+2)^2\lambda_0^2}\frac{d}{dz} \\
& - \frac{q^2}{4z^2}\frac{d^2}{dz^2},
\end{aligned} \tag{B·1}$$

$$\begin{aligned}
Q^{(3)}_{\ell=2} = \Bigg[&\Bigg(\frac{i}{2}\,m\,q + \frac{5\,q^2}{8} - \frac{5\,m^2\,q^2}{9} + \frac{11\,i}{24}\,m\,q^3 - \frac{11\,i}{54}\,m^3\,q^3\Bigg)\frac{1}{z^4} \\
& + \Bigg(\frac{-(m\,q)}{24} + \frac{5\,i}{48}\,q^2 - \frac{65\,i}{216}\,m^2\,q^2 - \frac{m\,q^3}{2} + \frac{16\,m^3\,q^3}{81}\Bigg)\frac{1}{z^3} \\
& + \Bigg(\frac{i}{24}\,m\,q - \frac{q^2}{48} + \frac{17\,m^2\,q^2}{216} - \frac{65\,i}{378}\,m\,q^3 + \frac{17\,i}{252}\,m^3\,q^3\Bigg)\frac{1}{z^2}\Bigg]\frac{d}{dz} \\
& + \Bigg[\Bigg(i\,m\,q + \frac{7\,q^2}{4} - \frac{25\,m^2\,q^2}{18} + \frac{4\,i}{3}\,m\,q^3 - \frac{29\,i}{54}\,m^3\,q^3\Bigg)\frac{1}{z^5}
\end{aligned}$$

$$+\left(\frac{-5\,m\,q}{12}+\frac{19\,i}{24}\,q^2-\frac{103\,i}{108}\,m^2\,q^2-\frac{19\,m\,q^3}{12}+\frac{101\,m^3\,q^3}{162}\right)\frac{1}{z^4}$$
$$+\left(\frac{-q^2}{8}+\frac{41\,m^2\,q^2}{108}-\frac{127\,i}{189}\,m\,q^3+\frac{601\,i}{2268}\,m^3\,q^3\right)\frac{1}{z^3}$$
$$+\left(\frac{-\,(m\,q)}{24}-\frac{i}{48}\,q^2+\frac{17\,i}{216}\,m^2\,q^2+\frac{\lambda_3\,q^3}{8}+\frac{65\,m\,q^3}{378}-\frac{17\,m^3\,q^3}{252}\right)\frac{1}{z^2}\Bigg], \tag{B·2}$$

$$Q^{(3)}_{\ell=3}=\Bigg[\left(\frac{i}{10}\,m\,q+\frac{q^2}{8}-\frac{7\,m^2\,q^2}{180}+\frac{i}{60}\,m\,q^3-\frac{7\,i}{540}\,m^3\,q^3\right)\frac{1}{z^4}$$
$$+\left(\frac{-7\,m\,q}{600}-\frac{83\,i}{1200}\,q^2-\frac{7\,i}{1350}\,m^2\,q^2-\frac{23\,m\,q^3}{720}+\frac{19\,m^3\,q^3}{3240}\right)\frac{1}{z^3}$$
$$+\left(\frac{i}{1200}\,m\,q-\frac{7\,q^2}{1200}+\frac{23\,m^2\,q^2}{5400}-\frac{7\,i}{2160}\,m\,q^3+\frac{11\,i}{12960}\,m^3\,q^3\right)\frac{1}{z^2}\Bigg]\frac{d}{dz}$$
$$+\Bigg[\left(\frac{i}{5}\,m\,q+\frac{q^2}{2}-\frac{31\,m^2\,q^2}{180}+\frac{23\,i}{120}\,m\,q^3-\frac{5\,i}{108}\,m^3\,q^3\right)\frac{1}{z^5}$$
$$+\left(\frac{-8\,m\,q}{75}+\frac{109\,i}{1200}\,q^2-\frac{79\,i}{1350}\,m^2\,q^2-\frac{25\,m\,q^3}{144}+\frac{31\,m^3\,q^3}{810}\right)\frac{1}{z^4}$$
$$+\left(\frac{-13\,i}{1200}\,m\,q+\frac{19\,q^2}{300}+\frac{17\,m^2\,q^2}{1800}-\frac{19\,i}{540}\,m\,q^3+\frac{29\,i}{4320}\,m^3\,q^3\right)\frac{1}{z^3}$$
$$+\left(\frac{-\,(m\,q)}{1200}-\frac{7\,i}{1200}\,q^2+\frac{23\,i}{5400}\,m^2\,q^2+\frac{\lambda_3\,q^3}{8}+\frac{7\,m\,q^3}{2160}-\frac{11\,m^3\,q^3}{12960}\right)\frac{1}{z^2}\Bigg], \tag{B·3}$$

$$Q^{(4)}_{\ell=2}=\Bigg[\Bigg(\frac{-3\,q^2}{8}+\frac{3\,m^2\,q^2}{8}-\frac{23\,i}{24}\,m\,q^3+\frac{43\,i}{108}\,m^3\,q^3-\frac{5\,q^4}{32}+\frac{4\,m^2\,q^4}{9}$$
$$-\frac{79\,m^4\,q^4}{648}\Bigg)\frac{1}{z^5}+\Bigg(\frac{-\,(m\,q)}{8}-\frac{3\,i}{32}\,q^2+\frac{5\,i}{72}\,m^2\,q^2+\frac{77\,m\,q^3}{96}-\frac{263\,m^3\,q^3}{648}$$
$$-\frac{19\,i}{96}\,q^4+\frac{79\,i}{108}\,m^2\,q^4-\frac{137\,i}{648}\,m^4\,q^4\Bigg)\frac{1}{z^4}$$
$$+\Bigg(\frac{-i}{96}\,m\,q-\frac{5\,q^2}{192}+\frac{73\,m^2\,q^2}{864}+\frac{13\,i}{108}\,m\,q^3-\frac{i}{9}\,m^3\,q^3$$
$$+\frac{25\,q^4}{252}-\frac{109\,m^2\,q^4}{252}+\frac{680\,m^4\,q^4}{5103}\Bigg)\frac{1}{z^3}$$

$$+\left(\frac{-(m\,q)}{96}-\frac{i}{192}\,q^2+\frac{11\,i}{288}\,m^2\,q^2+\frac{5\,m\,q^3}{108}-\frac{m^3\,q^3}{3888}+\frac{7\,i}{432}\,q^4\right.$$
$$\left.+\frac{i}{72}\,\lambda_3\,m\,q^4-\frac{1349\,i}{13608}\,m^2\,q^4+\frac{509\,i}{15309}\,m^4\,q^4\right)\frac{1}{z^2}\Bigg]\frac{d}{dz}$$
$$+\Bigg[\left(\frac{-3\,q^2}{4}+\frac{7\,m^2\,q^2}{12}-\frac{47\,i}{24}\,m\,q^3+\frac{19\,i}{27}\,m^3\,q^3-\frac{7\,q^4}{16}+\frac{65\,m^2\,q^4}{72}\right.$$
$$\left.-\frac{71\,m^4\,q^4}{324}\right)\frac{1}{z^6}+\left(\frac{-(m\,q)}{4}-\frac{7\,i}{16}\,q^2+\frac{3\,i}{8}\,m^2\,q^2+\frac{29\,m\,q^3}{12}-\frac{659\,m^3\,q^3}{648}\right.$$
$$\left.-\frac{29\,i}{48}\,q^4+\frac{49\,i}{27}\,m^2\,q^4-\frac{53\,i}{108}\,m^4\,q^4\right)\frac{1}{z^5}$$
$$+\left(\frac{-5\,i}{48}\,m\,q+\frac{5\,q^2}{96}+\frac{41\,m^2\,q^2}{432}+\frac{2287\,i}{3024}\,m\,q^3-\frac{2039\,i}{4536}\,m^3\,q^3\right.$$
$$\left.+\frac{101\,q^4}{288}-\frac{3991\,m^2\,q^4}{3024}+\frac{15853\,m^4\,q^4}{40824}\right)\frac{1}{z^4}$$
$$+\left(\frac{-i}{32}\,q^2+\frac{53\,i}{432}\,m^2\,q^2-\frac{2\,m\,q^3}{27}+\frac{431\,m^3\,q^3}{3888}+\frac{349\,i}{3024}\,q^4\right.$$
$$\left.+\frac{i}{72}\,\lambda_3\,m\,q^4-\frac{7235\,i}{13608}\,m^2\,q^4+\frac{2549\,i}{15309}\,m^4\,q^4\right)\frac{1}{z^3}$$
$$+\left(\frac{-i}{96}\,m\,q+\frac{q^2}{192}-\frac{11\,m^2\,q^2}{288}+\frac{5\,i}{108}\,m\,q^3-\frac{i}{3888}\,m^3\,q^3-\frac{7\,q^4}{432}+\frac{\lambda_4\,q^4}{16}\right.$$
$$\left.-\frac{\lambda_3\,m\,q^4}{72}+\frac{1349\,m^2\,q^4}{13608}-\frac{509\,m^4\,q^4}{15309}\right)\frac{1}{z^2}\Bigg]. \qquad (B\cdot4)$$

Appendix C
—— 4PN Formulas for $R^{\rm in}_{\ell m}$ ——

In this appendix, we show the post-Newtonian expansion of $R^{\rm in}$ in the near zone, where $z=\omega r\ll 1$, for a Kerr black hole which is needed to evaluate gravitational waves at infinity to $O(v^8)$. For convenience, we recover the indices ℓm on $R^{\rm in}$ and give the formulas for $\omega R^{\rm in}_{\ell m}$.

$$\omega R^{\rm in}_{2m}=\frac{z^4}{30}+\frac{i}{45}\,z^5-\frac{11\,z^6}{1260}-\frac{i}{420}\,z^7+\frac{23\,z^8}{45360}+\frac{i}{11340}\,z^9$$
$$-\frac{13\,z^{10}}{997920}-\frac{i}{598752}\,z^{11}+\frac{59\,z^{12}}{311351040}$$
$$+\epsilon\left(\frac{-z^3}{15}-\frac{i}{60}\,m\,q\,z^3-\frac{i}{60}\,z^4+\frac{m\,q\,z^4}{45}-\frac{41\,z^5}{3780}+\frac{277\,i}{22680}\,m\,q\,z^5\right.$$
$$-\frac{31\,i}{3780}\,z^6-\frac{7\,m\,q\,z^6}{1620}+\frac{17\,z^7}{5670}-\frac{61\,i}{54432}\,m\,q\,z^7$$

$$+\frac{41\,i}{54432}\,z^8+\frac{47\,m\,q\,z^8}{204120}-\frac{1579\,z^9}{10692000}+\frac{703\,i}{17962560}\,m\,q\,z^9\Bigg)$$
$$+\epsilon^2\Bigg(\frac{z^2}{30}+\frac{i}{40}\,m\,q\,z^2+\frac{q^2\,z^2}{60}-\frac{m^2\,q^2\,z^2}{240}-\frac{i}{60}\,z^3-\frac{m\,q\,z^3}{30}+\frac{i}{90}\,q^2\,z^3$$
$$-\frac{i}{120}\,m^2\,q^2\,z^3+\frac{7937\,z^4}{55125}-\frac{53\,i}{9072}\,m\,q\,z^4-\frac{101\,q^2\,z^4}{35280}+\frac{4213\,m^2\,q^2\,z^4}{635040}$$
$$+\frac{4673\,i}{55125}\,z^5-\frac{13\,m\,q\,z^5}{2835}-\frac{5\,i}{63504}\,q^2\,z^5+\frac{3503\,i}{1143072}\,m^2\,q^2\,z^5-\frac{1665983\,z^6}{55566000}$$
$$-\frac{1777\,i}{544320}\,m\,q\,z^6-\frac{q^2\,z^6}{5040}-\frac{643\,m^2\,q^2\,z^6}{653184}-\frac{107\,z^4\,\ln z}{6300}$$
$$-\frac{107\,i}{9450}\,z^5\,\ln z+\frac{1177\,z^6\,\ln z}{264600}\Bigg)$$
$$+\epsilon^3\Bigg(\Bigg(\frac{-i}{180}\,m\,q-\frac{q^2}{60}+\frac{m^2\,q^2}{240}-\frac{i}{144}\,m\,q^3+\frac{i}{1440}\,m^3\,q^3\Bigg)\,z$$
$$+\Bigg(\frac{i}{120}+\frac{2\,m\,q}{135}-\frac{i}{360}\,q^2+\frac{19\,i}{1440}\,m^2\,q^2+\frac{11\,m\,q^3}{1080}-\frac{m^3\,q^3}{540}\Bigg)\,z^2$$
$$+z^3\Bigg(-\frac{10933}{49000}-\frac{578569\,i}{7938000}\,m\,q-\frac{677\,q^2}{52920}-\frac{529\,m^2\,q^2}{63504}$$
$$+\frac{317\,i}{63504}\,m\,q^3-\frac{167\,i}{84672}\,m^3\,q^3+\frac{107\,\ln z}{3150}+\frac{107\,i}{12600}\,m\,q\,\ln z\Bigg)\Bigg)$$
$$+\epsilon^4\Bigg(\frac{-i}{720}\,m\,q+\frac{m^2\,q^2}{2880}+\frac{i}{288}\,m\,q^3-\frac{i}{2880}\,m^3\,q^3$$
$$+\frac{q^4}{480}-\frac{m^2\,q^4}{720}+\frac{m^4\,q^4}{11520}\Bigg), \qquad \text{(C·1)}$$

$$\omega R^{\rm in}_{3m}=\frac{z^5}{630}+\frac{i}{1260}\,z^6-\frac{z^7}{3780}-\frac{i}{16200}\,z^8+\frac{29\,z^9}{2494800}$$
$$+\frac{i}{554400}\,z^{10}-\frac{47\,z^{11}}{194594400}$$
$$+\epsilon\Bigg(\frac{-z^4}{252}-\frac{i}{1890}\,m\,q\,z^4-\frac{i}{756}\,z^5+\frac{11\,m\,q\,z^5}{22680}+\frac{19\,i}{90720}\,m\,q\,z^6$$
$$-\frac{i}{9450}\,z^7-\frac{m\,q\,z^7}{16200}+\frac{647\,z^8}{14968800}-\frac{247\,i}{17962560}\,m\,q\,z^8\Bigg)$$
$$+\epsilon^2\Bigg(\frac{z^3}{315}+\frac{i}{945}\,m\,q\,z^3+\frac{q^2\,z^3}{1260}-\frac{m^2\,q^2\,z^3}{15120}+\frac{i}{2520}\,z^4-\frac{17\,m\,q\,z^4}{15120}$$
$$+\frac{i}{2160}\,q^2\,z^4-\frac{31\,i}{272160}\,m^2\,q^2\,z^4+\frac{81409\,z^5}{11113200}$$

$$-\frac{313\,i}{907200}\,m\,q\,z^5-\frac{41\,q^2\,z^5}{226800}+\frac{617\,m^2\,q^2\,z^5}{8164800}-\frac{13\,z^5\,\ln z}{26460}\Bigg)$$

$$+\epsilon^3\left(\frac{-z^2}{1260}-\frac{i}{1680}\,m\,q\,z^2-\frac{q^2\,z^2}{840}+\frac{m^2\,q^2\,z^2}{10080}-\frac{i}{5040}\,m\,q^3\,z^2\right), \quad \text{(C·2)}$$

$$\omega R^{\rm in}_{4m}=\frac{z^6}{11340}+\frac{i}{28350}\,z^7-\frac{13\,z^8}{1247400}-\frac{i}{467775}\,z^9+\frac{71\,z^{10}}{194594400}$$

$$+\epsilon\left(\frac{-z^5}{3780}-\frac{i}{45360}\,m\,q\,z^5-\frac{11\,i}{136080}\,z^6+\frac{m\,q\,z^6}{64800}\right.$$

$$\left.+\frac{131\,z^7}{18711000}+\frac{697\,i}{124740000}\,m\,q\,z^7\right)$$

$$+\epsilon^2\left(\frac{z^4}{3528}+\frac{i}{18144}\,m\,q\,z^4+\frac{q^2\,z^4}{21168}-\frac{m^2\,q^2\,z^4}{635040}\right), \quad \text{(C·3)}$$

$$\omega R^{\rm in}_{5m}=\frac{z^7}{207900}+\frac{i}{623700}\,z^8-\frac{z^9}{2316600}$$

$$+\epsilon\left(\frac{-z^6}{59400}-\frac{i}{1039500}\,m\,q\,z^6\right), \quad \text{(C·4)}$$

$$\omega R^{\rm in}_{6m}=\frac{z^8}{4054050}. \quad \text{(C·5)}$$

Appendix D

—— *The Ingoing Regge-Wheeler Functions to $O(\epsilon^3)$* ——

In this appendix, we present a method to calculate the ingoing Regge-Wheeler functions to $O(\epsilon^3)$ which are needed to calculate the luminosity to $O(v^{11})$ beyond Newtonian in the Schwarzschild case. In the Schwarzschild limit, $q \to 0$, and solving Eq. (3·24) recursively is in principle straightforward. Since the general homogeneous solution to the left-hand side of it is given by a linear combination of the spherical Bessel functions j_ℓ and n_ℓ, one can immediately write the integral expression for $\xi^{(n)}_\ell$. Noting that $j_\ell n'_\ell - n_\ell j'_\ell = 1/z^2$, we have

$$\xi^{(n)}_\ell = n_\ell \int dz\, j_\ell W^{(n)} - j_\ell \int dz\, n_\ell W^{(n)}, \quad \text{(D·1)}$$

where the source term $W^{(n)}$ in the Schwarzschild case is given by

$$W^{(n)} = z^2 L^{(1)}[\xi^{(n-1)}] = z^2 e^{-iz}\frac{d}{dz}\left[\frac{1}{z^3}\frac{d}{dz}\left(e^{iz}z^2\xi^{(n-1)}_\ell(z)\right)\right]. \quad \text{(D·2)}$$

We perform the above indefinite integral and set the appropriate boundary condition by examining the asymptotic behavior at $z \to 0$ order by order.

D.1. *General remarks*

Let us first consider the boundary conditions of $\xi_\ell^{(n)}$. In the Schwarzschild case, the original Regge-Wheeler ingoing wave function $X_\ell^{\rm in}$ is related to $\xi_\ell^{\rm in}$ as

$$X_\ell^{\rm in} = z e^{-i\epsilon \ln(z-\epsilon)} \xi_\ell^{\rm in}, \tag{D·3}$$

and it has the asymptotic behavior given by Eq. (3·12), which in the present case reduces to

$$X_\ell^{\rm in} \to \begin{cases} A_\ell^{\rm ref} e^{i\omega r^*} + A_\ell^{\rm inc} e^{-i\omega r^*} & \text{for } z^* \to \infty, \\ A_\ell^{\rm trans} e^{-i\omega r^*} & \text{for } z^* \to -\infty. \end{cases} \tag{D·4}$$

Thus, noting that $z^* = z + \epsilon \ln(z-\epsilon)$, the boundary condition of $\xi_\ell^{\rm in}$ is that it is regular at $z^* \to -\infty$ $(z \to \epsilon)$. To implement this boundary condition to $\xi_\ell^{\rm in}$, we need a different series expansion of it; a series in terms of the variable $x := r/2M = z/\epsilon$ around $x = 1$. We write this expansion as

$$\xi_\ell^{\rm in} = \xi_\ell^{\{0\}}(x) + \epsilon \xi_\ell^{\{1\}}(x) + \epsilon^2 \xi_\ell^{\{2\}}(x) + \cdots. \tag{D·5}$$

On the other hand, there are two independent solutions expanded by functions written in terms of z. We denote the solution whose zeroth order is given by j_ℓ (n_ℓ) as $\xi_{j\ell}$ $(\xi_{n\ell})$. Then the general solution is given by $c_j \xi_{j\ell} + c_n \xi_{n\ell}$. As can be shown by using the result of Poisson and Sasaki,[21)] $\xi_\ell^{\{0\}}$ does not have terms matched with $\xi_{n\ell}$. A term matched with $\xi_{n\ell}$ first appears from $\xi_\ell^{\{1\}}$. For $x \to \infty$ $(\leftrightarrow \epsilon \ll z)$, $\xi_\ell^{\{0\}}(x)$ behaves as $x^\ell = z^\ell \epsilon^{-\ell} \sim \epsilon^{-\ell} \xi_{j\ell}$. On the other hand, $\epsilon \xi_\ell^{\{1\}}$ contains the term that behaves as $\epsilon x^{-\ell-1} = \epsilon^{\ell+2} z^{-\ell-1} \sim \epsilon^{\ell+2} \xi_{n\ell}$. Therefore, in the sense of the post-Minkowskian expansion, the inner boundary condition affects the ingoing wave solution at and beyond $O(\epsilon^{2\ell+2})$. However, in the post-Newtonian sense, since we evaluate ξ_ℓ in the near zone, i.e., for $z = O(v)$, the contribution from $\xi_{n\ell}$ becomes $O(v^{4\ell+5})$ relative to $\xi_{j\ell}$. Since $\ell \geq 2$, we find that the inner boundary condition affects the homogeneous solution at and beyond $O(v^{13})$ in the near zone.*)

Since $j_\ell = O(z^\ell)$ as $z \to 0$, we have $X_\ell^{\rm in} \to O(\epsilon^{\ell+1}) e^{-iz^*}$, or $A_\ell^{\rm trans} = O(\epsilon^{\ell+1})$. On the other hand, from the asymptotic behavior of j_ℓ at $z = \infty$, we find $A_\ell^{\rm inc}$ and $A_\ell^{\rm ref}$ are of order unity. Then using the Wronskian argument, we obtain

$$|A_\ell^{\rm inc}| - |A_\ell^{\rm ref}| = \frac{|A_\ell^{\rm trans}|^2}{|A_\ell^{\rm inc}| + |A_\ell^{\rm ref}|} = O(\epsilon^{2\ell+2}). \tag{D·6}$$

Thus $|A_\ell^{\rm inc}| = |A_\ell^{\rm ref}|$ until we go to $O(\epsilon^{2\ell+2})$ or more. This fact implies that we can make $A_\ell^{\rm inc}$ and $A_\ell^{\rm ref}$ to be complex conjugate to each other to $O(\epsilon^{2\ell+1})$. Hence the imaginary part of $X_\ell^{\rm in}$, which reflects the boundary condition at horizon, appears at $O(\epsilon^{2\ell+2})$ because the Regge-Wheeler equation is real. This is consistent with the argument given in the above paragraph. Provided we choose the phase of $X_\ell^{\rm in}$ in this

*) In Ref. 11), it was erroneously argued that the outgoing gravitational waves are unaffected by the inner boundary condition until we reach $O(\epsilon^6) = O(v^{18})$. As shown here, this is true only in the post-Minkowskian sense.

way, $\mathrm{Im}\left(\xi_\ell^{(n)}\right)$ for a given $n \le 2\ell+1$ is completely determined in terms of $\mathrm{Re}\left(\xi_\ell^{(r)}\right)$ for $r \le n-1$.

To see this explicitly, let us decompose $\xi_\ell^{(n)}$ into the real and imaginary parts:

$$\xi_\ell^{(n)} = f_\ell^{(n)} + i g_\ell^{(n)} . \tag{D$\cdot$7}$$

Inserting this expression into Eq. (D·3) and expanding the result with respect to ϵ by assuming $z \gg \epsilon$, we find

$$\begin{aligned} X_\ell^{\rm in} &= e^{-i\epsilon \ln(z-\epsilon)} z \left(j_\ell + \epsilon(f_\ell^{(1)} + i g_\ell^{(1)}) + \epsilon^2 (f_\ell^{(2)} + i g_\ell^{(2)}) + \epsilon^3 (f_\ell^{(3)} + i g_\ell^{(3)}) + \cdots \right) \\ &= z \Bigg(j_\ell + \epsilon f_\ell^{(1)} + \epsilon^2 \left(f_\ell^{(2)} + g_\ell^{(1)} \ln z - \frac{1}{2} j_\ell (\ln z)^2 \right) \\ &\quad + \epsilon^3 \left(f^{(3)} - \frac{1}{2} (\ln z)^2 f_\ell^{(1)} + \frac{\ln z}{z} j_\ell + \ln z\, g_\ell^{(2)} - \frac{1}{z} g_\ell^{(1)} \right) + \cdots \Bigg) \\ &\quad + iz \Bigg(\epsilon (g_\ell^{(1)} - j_\ell \ln z) + \epsilon^2 \left(g_\ell^{(2)} + \frac{1}{z} j_\ell - f_\ell^{(1)} \ln z \right) \\ &\quad + \epsilon^3 \Bigg(g_\ell^{(3)} - \frac{1}{2} (\ln z)^2 g_\ell^{(1)} - \ln z\, f_\ell^{(2)} + \frac{1}{z} f^{(1)} \\ &\quad + \left(\frac{1}{2z^2} + \frac{1}{6} (\ln z)^3 \right) j_\ell \Bigg) + \cdots \Bigg) . \end{aligned} \tag{D$\cdot$8}$$

Hence we must have

$$\begin{aligned} & g_\ell^{(1)} = j_\ell \ln z , \quad g_\ell^{(2)} = -\frac{1}{z} j_\ell + f_\ell^{(1)} \ln z , \\ & g_\ell^{(3)} = \left(\frac{1}{3} (\ln z)^3 - \frac{1}{2z^2} \right) j_\ell - \frac{1}{z} f_\ell^{(1)} + \ln z\, f_\ell^{(2)} , \quad \cdots . \end{aligned} \tag{D$\cdot$9}$$

For completeness, we also give the relation between the functions $f_\ell^{(n)}$ and the conventional post-Newtonian expansion of $X_\ell^{\rm in}$:

$$\begin{aligned} & X_\ell^{\rm in} = \sum_{n=0}^{\infty} \epsilon^n X_\ell^{(n)} ; \\ & X_\ell^{(0)} = z f_\ell^{(0)} = z j_\ell , \quad X_\ell^{(1)} = z f_\ell^{(1)} , \quad X_\ell^{(2)} = z \left(f_\ell^{(2)} + \frac{1}{2} j_\ell (\ln z)^2 \right) , \\ & X_\ell^{(3)} = z \left(f_\ell^{(3)} + \frac{1}{2} f_\ell^{(1)} (\ln z)^2 - \frac{\ln z}{z} j_\ell \right) , \quad \cdots . \end{aligned} \tag{D$\cdot$10}$$

Now we turn to the asymptotic behavior at $z=\infty$. Let the asymptotic form of $f_\ell^{(n)}$ be

$$f_\ell^{(n)} \to P_\ell^{(n)} j_\ell + Q_\ell^{(n)} n_\ell \quad \text{as } z \to \infty . \quad (n = 1, 2, 3) \tag{D$\cdot$11}$$

Then noting Eq. (D·9) and the equality $e^{-i\epsilon\ln(z-\epsilon)}=e^{-iz^*}e^{iz}$, the asymptotic form of $X_\ell^{\rm in}$ is expressed as

$$\begin{aligned}X_\ell^{\rm in}\to&\frac{1}{2}e^{-iz^*}\left(zh_\ell^{(2)}e^{iz}\right)\Big[1+\epsilon\left\{P_\ell^{(1)}+i\left(Q_\ell^{(1)}+\ln z\right)\right\}\\&+\epsilon^2\left\{\left(P_\ell^{(2)}-Q_\ell^{(1)}\ln z\right)+i\left(Q_\ell^{(2)}+P_\ell^{(1)}\ln z\right)\right\}\\&+\epsilon^3\left\{\left(f_\ell^{(3)}-Q_\ell^{(2)}\ln z\right)+i\left(Q_\ell^{(3)}+P_\ell^{(2)}\ln z+\frac{1}{3}(\ln z)^3\right)\right\}+\cdots\Big]\\&+\frac{1}{2}e^{iz^*}\left(zh_\ell^{(1)}e^{-iz}\right)e^{-2i\epsilon\ln(z-\epsilon)}\Big[1+\epsilon\left\{P_\ell^{(1)}-i\left(Q_\ell^{(1)}-\ln z\right)\right\}\\&+\epsilon^2\left\{\left(P_\ell^{(2)}+Q_\ell^{(1)}\ln z\right)-i\left(Q_\ell^{(2)}-P_\ell^{(1)}\ln z\right)\right\}\\&+\epsilon^3\left\{\left(P_\ell^{(3)}+Q_\ell^{(2)}\ln z\right)-i\left(Q_\ell^{(3)}-P_\ell^{(2)}\ln z-\frac{1}{3}(\ln z)^3\right)\right\}+\cdots\Big].\end{aligned}\tag{D·12}$$

Using the asymptotic behavior of $h_\ell^{(1)}$ and $h_\ell^{(2)}$ given in Eq. (3·57), the incident amplitude $A_\ell^{\rm inc}$ can be readily extracted out as

$$\begin{aligned}A_\ell^{\rm inc}=&\frac{1}{2}i^{\ell+1}e^{-i\epsilon\ln\epsilon}\Big[1+\epsilon\left\{P_\ell^{(1)}+i\left(Q_\ell^{(1)}+\ln z\right)\right\}\\&+\epsilon^2\left\{\left(P_\ell^{(2)}-Q_\ell^{(1)}\ln z\right)+i\left(Q_\ell^{(2)}+P_\ell^{(1)}\ln z\right)\right\}\\&+\epsilon^3\left\{\left(P_\ell^{(3)}-Q_\ell^{(2)}\ln z\right)+i\left(Q_\ell^{(3)}+P_\ell^{(2)}\ln z+\frac{1}{3}(\ln z)^3\right)\right\}+\cdots\Big],\end{aligned}\tag{D·13}$$

where note that the definition of r^*, Eq. (2·9), in the limit $q\to 0$ is

$$\omega r^*=\omega\left(r+2M\ln\frac{r-2M}{2M}\right)=z^*-\epsilon\ln\epsilon,\tag{D·14}$$

which gives rise to the phase $-i\epsilon\ln\epsilon$ of $A_\ell^{\rm inc}$.

An important point to be noted in the above expression for $A_\ell^{\rm inc}$ is that it contains $\ln z$-dependent terms. Since $A_\ell^{\rm inc}$ should be constant, $P_\ell^{(n)}$ and $Q_\ell^{(n)}$ should contain appropriate $\ln z$-dependent terms which exactly cancel the $\ln z$-dependent terms in the formula (D·13).

D.2. *Basic formalism for iteration*

Here we derive the formulas necessary to perform the iteration scheme.

D.2.1. Definitions

We introduce the following functions,

$$\begin{aligned}B_{jj}:=&\int_0^z zj_0j_0dz=-\frac{1}{2}C,\\B_{nj}:=&\int_0^z zn_0j_0dz=-\frac{1}{2}S,\end{aligned}$$

$$B_{jn} := \int_0^z z j_0 n_0 dz = -\frac{1}{2} S,$$
$$B_{nn} := \int_{z_*}^z z n_0 n_0 dz = -B_{jj} + \ln z, \tag{D·15}$$

where

$$S = \int_0^{2z} dy \frac{\sin y}{y} = -\int_x^\infty dy \frac{\sin y}{y} + \frac{\pi}{2},$$
$$C = \int_0^{2z} dy \frac{\cos y - 1}{y} = -\int_{2z}^\infty dy \frac{\cos y}{y} - \gamma - \ln 2z, \tag{D·16}$$

and the lower bound z_* of the integral for the definition of B_{nn} is adjusted so as to make B_{nn} equal to the last expression of the line.

As an extension of these integral sinusoidal functions, we further introduce the following functions:

$$B_{jJ} := \int_{z_*}^z dz z j_0 D_0^J, \tag{D·17}$$
$$B_{nJ} := \int_{z_*}^z dz z n_0 D_0^J, \tag{D·18}$$

where J stands for a sequence of j and n, say, $J = jnnj$, and we have also introduced an extension of the spherical Bessel functions by

$$D_\ell^j := j_\ell, \quad D_\ell^n := n_\ell, \tag{D·19}$$

and

$$D_\ell^{nJ} := n_\ell B_{jJ} - j_\ell B_{nJ},$$
$$D_\ell^{jJ} := j_\ell B_{jJ} + n_\ell B_{nJ}. \tag{D·20}$$

We adopt the following rule to determine the lower bound of the integrals in Eq. (D·18). Whenever we can put $z_* = 0$, we do so, which is always possible when the sequence J ends with j. On the other hand, in the case when J ends with n, there may appear in the integrand the square of n_0 which causes logarithmic divergence if we set $z_* = 0$. In such cases, we use the relation,

$$n_0^2 = \frac{1}{z^2} - j_0^2, \tag{D·21}$$

to replace n_0^2 with the right-hand side and extract out the logarithmically divergent term due to $1/z^2$. Then we set $z_* = 0$ for the j_0^2 term, while we set $z_* = 1$ for the $1/z^2$ term so as to make the resulting logarithmic term zero at $z = 1$. For J of two indices, this is how we have defined B_{nn} in Eq. (D·15). For J of three indices, this applies to B_{njn}. Specifically it is given by

$$B_{njn} = B_{jjj} - B_{jj} \ln z + \frac{1}{2} (\ln z)^2. \tag{D·22}$$

For convenience, in what follows we call B_J the generalized integral sinusoidal functions and D_ℓ^J the generalized spherical Bessel functions.

Note that all the B_J whose J end with n can be expressed in terms of those whose J end with j. For example, for J of three indices, we have

$$\begin{aligned} B_{jjn} &= -B_{nnj} + \ln z B_{nj}\,, \\ B_{jnn} &= 2B_{jjj} + B_{nnj} - \ln z B_{jj}\,, \\ B_{nnn} &= 2B_{njj} - B_{jnj} - \ln z B_{nj}\,, \end{aligned} \qquad (\text{D}\cdot 23)$$

together with Eq. (D·22). Using these relations, we can express all the D_ℓ^J whose J end with n in terms of those whose J end with j.

Further we introduce the following indefinite integral operator,

$$F_{k,\ell}[X] := n_k \int dz j_\ell X - j_k \int dz n_\ell X \qquad (\text{D}\cdot 24)$$

for a function X. Note that Eq. (D·1) is expressed in terms of this operator as

$$\xi_\ell^{(n)} = F_{\ell,\ell}[W^{(n)}]. \qquad (\text{D}\cdot 25)$$

We also introduce the following operator,

$$H_k^J[Y] := n_k \int dz D_0^J Y - j_k \int dz D_0^J \tilde{Y}, \qquad (\text{D}\cdot 26)$$

where Y stands for a linear combination of the generalized Bessel functions with the coefficients given by linear combinations of $z^m(\ln z)^n$ $(m \le 1,\, n \ge 0)$ and $\tilde{Y}$ denotes the quantity which is obtained by replacing j, n, D^{jJ} and D^{nJ} with n, j, D^{nJ} and D^{jJ}, respectively, in the expression of Y. By definition we see that

$$F_{\ell,0}[z^m(\ln z)^n j_0] = H_\ell^j[z^m(\ln z)^n j_0]. \qquad (\text{D}\cdot 27)$$

D.2.2. Basic formulas

The spherical Bessel functions satisfy the recursion relation,

$$\zeta_{m-1} + \zeta_{m+1} = \frac{2m+1}{z}\zeta_m\,, \qquad (\text{D}\cdot 28)$$

where $\zeta_m = j_m$ or n_m. Note that

$$n_m = (-1)^{m+1} j_{-m-1}\,, \quad j_m = (-1)^m n_{-m-1}\,. \qquad (\text{D}\cdot 29)$$

The same recursion relation holds for the generalized spherical Bessel functions,

$$D_{m-1}^{\zeta J} + D_{m+1}^{\zeta J} = \frac{2m+1}{z} D_m^{\zeta J}\,, \qquad (\text{D}\cdot 30)$$

where $\zeta = j$ or n. Further the relations the same as Eqs. (D·29) hold for $D_m^{\zeta J}$,

$$D_m^{nJ} = (-)^{m+1} D_{-m-1}^{jJ}, \quad D_m^{jJ} = (-)^m D_{-m-1}^{nJ}. \qquad (\text{D}\cdot 31)$$

The derivative recursion relation for the spherical Bessel functions is

$$\frac{d}{dz}\zeta_\ell = \frac{1}{2\ell+1}\left\{\ell\zeta_{\ell-1} - (\ell+1)\zeta_{\ell+1}\right\}, \tag{D·32}$$

and this extends to the generalized spherical Bessel functions as

$$\begin{aligned}\frac{d}{dz}D_\ell^{n\zeta J} &= \frac{1}{2\ell+1}\left\{\ell D_{\ell-1}^{n\zeta J} - (\ell+1)D_{\ell+1}^{n\zeta J}\right\} + \frac{R_{\ell,0}}{z}D_0^{\zeta J},\\ \frac{d}{dz}D_\ell^{j\zeta J} &= \frac{1}{2\ell+1}\left\{\ell D_{\ell-1}^{j\zeta J} - (\ell+1)D_{\ell+1}^{j\zeta J}\right\} - \frac{R_{\ell,-1}}{z}D_0^{\zeta J}.\end{aligned} \tag{D·33}$$

Useful integral formulas for the spherical Bessel functions are

$$\begin{aligned}
\int dz\ \zeta_m\zeta_n^* &= \frac{z^2}{(m-n)(m+n+1)}\left(\zeta_m\zeta_{n+1}^* + \zeta_{m-1}\zeta_n^*\right) - \frac{z}{m-n}\zeta_m\zeta_n^*,\\
&\qquad (m \neq n,\ -n-1)\\
\int dz\ \zeta_\ell\zeta_\ell^* &= \frac{1}{2\ell+1}\left\{\int dz\ \zeta_0\zeta_0^* - z\left(\zeta_0\zeta_0^* + 2\sum_{m=1}^{\ell-1}\zeta_m\zeta_m^* + \zeta_\ell\zeta_\ell^*\right)\right\},\\
\int dz\ z\zeta_m\zeta_n^* &= \int dz\ z\zeta_{m-1}\zeta_{n-1}^* - \frac{z^2}{m+n}\left(\zeta_{m-1}\zeta_{n-1}^* + \zeta_m\zeta_n^*\right),\\
\int dz\ z\zeta_\ell\zeta_\ell^* &= \int dz\ z\zeta_0\zeta_0^* - \frac{z^2}{2}\left\{\zeta_0\zeta_0^* + \sum_{m=1}^{\ell-1}\left(\frac{1}{m} + \frac{1}{m+1}\right)\zeta_m\zeta_m^* + \frac{1}{\ell}\zeta_\ell\zeta_\ell^*\right\},\\
\int dz\ z\zeta_\ell\zeta_{\ell-1}^* &= \int dz\ z\zeta_0\zeta_{-1}^* - z^2\left\{\zeta_0\zeta_{-1}^* + \sum_{m=1}^{\ell-1}\frac{4m}{4m^2-1}\zeta_m\zeta_{m-1}^* + \frac{1}{2\ell-1}\zeta_\ell\zeta_{\ell-1}^*\right\},\\
\int dz\ z\zeta_\ell\zeta_{\ell+1}^* &= \int dz\ z\zeta_{-1}\zeta_0^* - z^2\left\{\zeta_{-1}\zeta_0^* + \sum_{m=1}^{\ell}\frac{4m}{4m^2-1}\zeta_{m-1}\zeta_m^* + \frac{1}{2\ell+1}\zeta_\ell\zeta_{\ell+1}^*\right\},
\end{aligned} \tag{D·34}$$

where ζ_m or ζ_m^* stands for j_m or n_m.

The following polynomial of $1/z$ plays an important role in the calculations:

$$\begin{aligned}R_{m,k} &= z^2(n_m j_k - j_m n_k)\\ &= -\sum_{r=0}^{[(m-k-1)/2]}(-1)^r\frac{(m-k-1-r)!\ \Gamma\left(m+\frac{1}{2}-r\right)}{r!\ (m-k-1-2r)!\ \Gamma\left(k+\frac{3}{2}+r\right)}\left(\frac{2}{z}\right)^{m-k-1-2r}\end{aligned} \tag{D·35}$$

for $m > k$ and

$$R_{m,k} = -R_{k,m} \tag{D·36}$$

for $m < k$. By construction, a recursion formula similar to that satisfied by the spherical Bessel functions holds:

$$R_{m,k-1} + R_{m,k+1} = \frac{2k+1}{z}R_{m,k}. \tag{D·37}$$

Note that the indices m and k can be negative as well. As examples, we write down the explicit forms for some special cases:

$$R_{k,k}=0,\quad R_{k,k+1}=1,\quad R_{k,k-1}=-1,\quad R_{k,k-2}=-\frac{2k-1}{z}.\qquad (\mathrm{D}\cdot 38)$$

D.2.3. Source terms

The source term can be rewritten as

$$W_\ell^{(n)}=z^2\left[\frac{d^2}{dz^2}+\frac{2}{z}\frac{d}{dz}+\left(1-\frac{\ell(\ell+1)}{z^2}\right)\right]\frac{\xi_\ell^{(n-1)}}{z} + \left[\frac{d}{dz}-2z+\frac{\ell(\ell+1)-4}{z}\right]\xi_\ell^{(n-1)}+i\left[2z\frac{d}{dz}+1\right]\xi_\ell^{(n-1)}\qquad (\mathrm{D}\cdot 39)$$

The contribution to $\xi_\ell^{(n)}$ from the first term is given by $z^{-1}\xi_\ell^{(n-1)}$. So we focus on the second and third terms. Note that the operators of the second and third terms have opposite parities; the second term is odd while the third term is even under the transformation $z\to -z$. To perform the integration of these terms, we introduce the concept of the standard form of source terms as follows.

First consider the first iteration, $n=1$. Since $\xi_\ell^{(0)}=j_\ell$ and since we only need to calculate the real part of $\xi_\ell^{(1)}$, we only need to consider the second term. Using the recursion relations (D·28) and (D·32), it may be rewritten in the form,

$$\alpha_0 z j_\ell+\beta_0 j_{\ell-1}+\beta_1 j_{\ell+1}\,.\qquad (\mathrm{D}\cdot 40)$$

Then using the integral formulas (D·34), the integrals $F_{\ell,\ell}[zj_\ell]$ and $F_{e\ll,\ell}[j_{\ell\pm1}]$ are readily evaluated to give

$$F_{\ell,\ell}\left[zj_\ell\right]=D_\ell^{nj}-\frac{1}{2}\left\{R_{\ell,0}j_0+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{\ell,m}j_m\right\},$$
$$F_{\ell,\ell}\left[j_k\right]=-\frac{1}{(\ell-k)(\ell+k+1)}j_k.\quad (k=\ell\pm 1)\qquad (\mathrm{D}\cdot 41)$$

The real part of $\xi_\ell^{(1)}$ is expressed in terms of these functions, while the imaginary part is $(\ln z)j_\ell$ as given by Eq. (D·9). The result is Eq. (3·34) with $q=0$.

At the second iteration, $n=2$, we insert the real part of $\xi_\ell^{(1)}$ to the second term in Eq. (D·39) and the imaginary part $i(\ln z)j_\ell$ to the third term, to evaluate the real part of $\xi_\ell^{(2)}$. Let us focus on the contribution of the terms of the form $R_{\ell,m}j_m$ in $\xi_\ell^{(1)}$ for the moment. Since $R_{\ell,m}$ are polynomials in $1/z$, we cannot apply the integral formulas (D·34) directly. So, by using the recursion relation (D·28) we get rid of the inverse powers of z. Then we find the corresponding source term may be expressed in the form,

$$z(\tilde\alpha_- j_{\ell-1}+\tilde\alpha_+ j_{\ell+1})+\sum_{m\neq 0}\tilde\beta_m j_{\ell+2m}\,.\qquad (\mathrm{D}\cdot 42)$$

Similarly, at the third iteration, $n = 3$, the terms in $\xi_\ell^{(2)}$ having the form $z^k j_n$ ($k \leq 0$) will give rise to the source term which can be written in the form,

$$z(\alpha_0 j_\ell + \alpha_- j_{-\ell-2} + \alpha_+ j_{-\ell}) + \sum_{m \neq -\ell} \beta_m j_{\ell+2m-1}\,, \tag{D·43}$$

which is a generalization of Eq. (D·40). Because the operator of the second term in Eq. (D·39) has the odd parity, the source terms for $n = 2$ and 3 take different forms. We call Eqs. (D·42) and (D·43) the standard forms. For convenience we call the former the even standard form and the latter the odd standard form. Now turning to the terms with D_m^J or $(\ln z) j_m$, since they satisfy the same recursion relation as j_m do, the same idea can be extended to them in a natural sense. The standard form for them is then defined by Eqs. (D·42) and (D·43) with j_m replaced by D_m^J or $(\ln z) j_m$. Note that D_ℓ^{nj} at $n = 2$ plays an analogous role of j_ℓ at $n = 1$. Hence the odd standard form of D_m^{nj} appears at $n = 2$. On the other hand, since the second and third terms in Eq. (D·39) have opposite parities, the parity of the standard form of $(\ln z) j_m$ is equal to that of j_m.

To summarize, the source term at the second iteration consists of the standard forms of

$$j_m : \text{ even}, \quad (\ln z) j_m : \text{ even}, \quad D_m^{nj} : \text{ odd}. \tag{D·44}$$

The integration of these terms can be done by using the formulas given in §D.3 below. The resulting $\xi_\ell^{(2)}$ are given by Eqs. (3·39) for $\ell = 2$, 3 and Eq. (4·16) for $\ell = 4$. Then we find there appear new types of the source term at the third iteration, which are

$$z D_\ell^{nnj}\,, \quad D_{\ell\pm1}^{nnj}\,, \quad z(\ln z)^2 j_\ell\,, \quad (\ln z)^2 j_{\ell\pm1}\,, \quad z(\ln z) D_{\ell\pm1}^{nj}\,, \tag{D·45}$$

in addition to the opposite parity terms of Eq. (D·44),

$$j_m : \text{ odd}, \quad (\ln z) j_m : \text{ odd}, \quad D_m^{nj} : \text{ even}. \tag{D·46}$$

D.3. *Reduction of integrals*

In this subsection, we reduce the expressions $F_{\ell,\ell}$[source terms] to those written in terms of D_m^J. For this purpose we need to evaluate integrals such as

$$F_{k,\ell}[z D_\ell^{nJ}] := n_k \int dz\, z j_\ell D_\ell^{nJ} - j_k \int dz\, z n_\ell D_\ell^{nJ}. \tag{D·47}$$

As an example let us show how this is evaluated. Using the basic integral formulas (D·34), we integrate it by part as

$$\begin{aligned} F_{k,\ell}[z D_\ell^{nJ}] &= n_k \int dz\ z j_0 D_0^{nJ} - j_k \int dz\ z n_0 D_0^{nJ} \\ &+ \Bigg[j_k \left(-\frac{z^2}{2}\right) \left\{ n_0 j_0 + \sum_{m=1}^{\ell-1} \left(\frac{1}{m} + \frac{1}{m+1}\right) n_m j_m + \frac{1}{\ell} n_\ell j_\ell \right\} \\ &- n_k \left(-\frac{z^2}{2}\right) \left\{ j_0 j_0 + \sum_{m=1}^{\ell-1} \left(\frac{1}{m} + \frac{1}{m+1}\right) j_m j_m + \frac{1}{\ell} j_\ell j_\ell \right\} \Bigg] B_{nJ} \end{aligned}$$

$$-\left[j_k\left(-\frac{z^2}{2}\right)\left\{n_0n_0+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)n_mn_m+\frac{1}{\ell}n_\ell n_\ell\right\}\right.$$
$$\left.-n_k\left(-\frac{z^2}{2}\right)\left\{j_0n_0+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)j_mn_m+\frac{1}{\ell}j_\ell n_\ell\right\}\right]B_{jJ}$$
$$-n_k\left[\int dz\left(-\frac{z^2}{2}\right)\left\{j_0n_0+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)j_mn_m+\frac{1}{\ell}j_\ell n_\ell\right\}zj_0D_0^J\right.$$
$$\left.-\int dz\left(-\frac{z^2}{2}\right)\left\{j_0j_0+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)j_mj_m+\frac{1}{\ell}j_\ell j_\ell\right\}zn_0D_0^J\right]$$
$$+j_k\left[\int dz\left(-\frac{z^2}{2}\right)\left\{n_0n_0+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)n_mn_m+\frac{1}{\ell}n_\ell n_\ell\right\}zj_0D_0^J\right.$$
$$\left.-\int dz\left(-\frac{z^2}{2}\right)\left\{n_0j_0+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)n_mj_m+\frac{1}{\ell}n_\ell j_\ell\right\}zn_0D_0^J\right]$$
$$=D_k^{nnJ}+\frac{1}{2}\left\{R_{0k}D_0^{nJ}+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{mk}D_m^{nJ}+\frac{1}{\ell}R_{\ell k}D_\ell^{nJ}\right\}$$
$$+\frac{1}{2}n_k\int zdz\left(\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{m0}j_m+\frac{1}{\ell}R_{\ell 0}j_\ell\right)D_0^J$$
$$-\frac{1}{2}j_k\int zdz\left(\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{m0}n_m+\frac{1}{\ell}R_{\ell 0}n_\ell\right)D_0^J$$
$$=D_k^{nnJ}+\frac{1}{2}\left\{R_{0k}D_0^{nJ}+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{mk}D_m^{nJ}+\frac{1}{\ell}R_{\ell k}D_\ell^{nJ}\right\}$$
$$+\frac{1}{2}H_k^J\left[\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{m0}j_m+\frac{1}{\ell}R_{\ell 0}j_\ell\right]. \qquad (\text{D}\cdot 48)$$

The reduction of the H_k^J term in the last expression is done similarly. In the following, we give formulas for each type of the source terms.

D.3.1. j_m-terms

The source terms have the standard form (D·42) or (D·43).

Using Eqs. (D·34), their integrals are evaluated as

$$F_{\ell,\ell}[\zeta_m]=-\frac{1}{(\ell-m)(\ell+m+1)}\zeta_m, \quad (m\neq\ell,\ -\ell-1) \qquad (\text{D}\cdot 49)$$

$$F_{\ell,\ell}[z\zeta_\ell]=D_\ell^{n\zeta}-\frac{1}{2}\left\{R_{\ell,0}\zeta_0+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{\ell,m}\zeta_m\right\}, \qquad (\text{D}\cdot 50)$$

$$F_{\ell,\ell}[zj_{\ell-1}]=-D_\ell^{nn}-\left\{-R_{\ell,0}n_0+\sum_{m=1}^{\ell-1}\frac{4m}{4m^2-1}R_{\ell,m}j_{m-1}\right\}, \qquad (\text{D}\cdot 51)$$

$$F_{\ell,\ell}\left[zn_{\ell-1}\right] = D_\ell^{nj} - \left\{R_{\ell,0}j_0 + \sum_{m=1}^{\ell-1} \frac{4m}{4m^2-1} R_{\ell,m} n_{m-1}\right\}, \tag{D·52}$$

$$F_{\ell,\ell}\left[z\zeta_{\ell+1}\right] = -D_\ell^{j\zeta} - \left\{R_{\ell,-1}\zeta_0 + \sum_{m=1}^{\ell} \frac{4m}{4m^2-1} R_{\ell,m-1}\zeta_m\right\}, \tag{D·53}$$

where ζ represents j or n. Note also that rather general formulas,

$$F_{k,\ell}[\zeta_m] = \frac{1}{(\ell-m)(\ell+m+1)}\left\{R_{k,\ell}\zeta_{m+1} + R_{k,\ell-1}\zeta_m\right\}$$
$$-\frac{1}{\ell-m}\frac{R_{k,\ell}}{z}\zeta_m, \quad (m \neq \ell, \ -\ell-1) \tag{D·54}$$

$$F_{k,\ell}[z\zeta_\ell] = D_k^{n\zeta} - \frac{1}{2}\left\{R_{k,0}\zeta_0 + \sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right) R_{k,m}\zeta_m + \frac{1}{\ell}R_{k,\ell}\zeta_\ell\right\}, \tag{D·55}$$

hold.

D.3.2. $(\ln z)j_m$-terms

The source terms are in the form (D·40) or (D·42) with j_m replaced by $(\ln z)j_m$.

First we give the general formula for $F_{\ell,\ell}[(\ln z)j_m]$. Using the first formula in Eqs. (D·34), we obtain

$$F_{\ell,\ell}[(\ln z)j_m] = \frac{1}{(\ell-m)(\ell+m+1)}\left(-\ln z + \frac{1}{(\ell+m+1)}\right) j_m$$
$$+\frac{1}{\ell-m}F_{\ell,\ell}\left[-\frac{2z}{\ell+m+1}j_{m+1} + j_m\right] \tag{D·56}$$

for $m \neq \ell, \ -\ell-1$.

Next we consider the remaining term of the odd parity. With the aid of the second formula in Eqs. (D·34), after the integration by part, we have

$$F_{\ell,\ell}[z(\ln z)j_\ell] = F_{\ell,0}[z(\ln z)j_0] - \frac{\ln z}{2}\left\{R_{\ell,0}j_0 + \sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{\ell,m}j_m\right\}$$
$$+\left(\sum_{m=1}^{\ell}\frac{1}{m}\right)D_\ell^{nj} - \frac{1}{4}\left\{2\left(\sum_{m=1}^{\ell}\frac{1}{m}\right) - 1\right\}R_{\ell,0}j_0$$
$$-\frac{1}{4}\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)\left\{2\left(\sum_{k=m+1}^{\ell}\frac{1}{k}\right) + \frac{1}{m(m+1)}\right\}R_{\ell,m}j_m. \tag{D·57}$$

To evaluate the first term, we use the following trick. Note that

$$B_{jnJ} = \int_0^z dz\ zj_0\left(n_0B_{jJ} - j_0B_{nJ}\right)$$
$$= \int_0^z dz\ zn_0\left(j_0B_{jJ} + n_0B_{nJ}\right) - \int_0^z \frac{dz}{z}B_{nJ}$$

$$= B_{njJ} - (\ln z)B_{nJ} + \int_0^z dz (\ln z)\frac{dB_{nJ}}{dz}, \tag{D·58}$$

where we have assumed J do not end with n, i.e., $J \neq n, jn, \cdots$. In the same way,

$$B_{jjJ} = -B_{nnJ} + (\ln z)B_{jJ} - \int_0^z dz (\ln z)\frac{dB_{jJ}}{dz}. \tag{D·59}$$

Thus we obtain

$$\begin{aligned} \int_0^z dz (\ln z)\frac{dB_{nJ}}{dz} &= B_{jnJ} - B_{njJ} + (\ln z)B_{nJ}, \\ \int_0^z dz (\ln z)\frac{dB_{jJ}}{dz} &= -B_{jjJ} - B_{nnJ} + (\ln z)B_{jJ}. \end{aligned} \tag{D·60}$$

Using Eqs. (D·60), we find

$$F_{\ell,0}[z(\ln z)j_0] = -D_\ell^{njj} - D_\ell^{jnj} + (\ln z)D_\ell^{nj}. \tag{D·61}$$

As for the remaining terms of the even parity, we have

$$\begin{aligned} F_{\ell,\ell}[z(\ln z)j_{\ell+1}] &= -D_\ell^{nnj} + D_\ell^{jjj} - \ln z D_\ell^{jj} \\ &\quad -R_{\ell,-1}(\ln z)j_0 - \ln z \sum_{m=1}^{\ell} \frac{4m}{4m^2-1} R_{\ell,m-1} j_m \\ &\quad +F_{\ell,-1}[zj_0] + \sum_{m=1}^{\ell} \frac{4m}{4m^2-1} F_{\ell,m-1}[zj_m] + \frac{1}{2\ell+1} F_{\ell,\ell}[zj_{\ell+1}], \\ F_{\ell,\ell}[z(\ln z)j_{\ell-1}] &= D_\ell^{jjj} - D_\ell^{nnj} - (\ln z)D_\ell^{jj} + \frac{1}{2}(\ln z)^2 j_\ell \\ &\quad -R_{\ell,-1}(\ln z)j_0 + \ln z \sum_{m=1}^{\ell-1} \frac{4m}{4m^2-1} R_{\ell,m} j_{m-1} \\ &\quad -F_{\ell,0}[zn_0] + \sum_{m=1}^{\ell-1} \frac{4m}{4m^2-1} F_{\ell,m}[zj_{m-1}] + \frac{1}{2\ell+1} F_{\ell,\ell}[zj_{\ell-1}]. \end{aligned} \tag{D·62}$$

D.3.3. $(\ln z)^2 j_m$-terms

The source terms we have to evaluate are $z(\ln z)^2 j_\ell$ and $(\ln z)^2 j_{\ell\pm1}$.

Using the first formula in Eqs. (D·34), we obtain

$$\begin{aligned} F_{\ell,\ell}[(\ln z)^2 j_m] = \frac{1}{(\ell-m)(\ell+m+1)} \Bigg\{ &\left[-(\ln z)^2 + \frac{2\ln z}{\ell+m+1} - \frac{2}{(\ell+m+1)^2} \right] j_m \\ &+2F_{\ell,\ell}\left[(\ln z)\left(-2zj_{m+1} + (\ell+m+1)j_m\right) + \frac{2zj_{m+1}}{\ell+m+1} \right] \Bigg\} \end{aligned} \tag{D·63}$$

for $m \neq \ell, -\ell-1$.

Also, with the aid of the second formula in Eqs. (D·34), we find

$$F_{\ell,\ell}[z(\ln z)^2 j_\ell] = F_{\ell,0}[z(\ln z)^2 j_0] - \frac{(\ln z)^2}{2}\left\{R_{\ell,0}j_0 + \sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{\ell,m}j_m\right\}$$
$$+F_{\ell,0}\left[z(\ln z)j_0\right] + \sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)F_{\ell,m}\left[z(\ln z)j_m\right]$$
$$+\frac{1}{\ell}F_{\ell,\ell}\left[z(\ln z)j_\ell\right]. \qquad \text{(D·64)}$$

The evaluation of the first term in the above equation is done as follows. Using a technique similar to the one used to derive Eqs. (D·60), we obtain

$$\int_0^z dz\,(\ln z)^2\frac{dB_{nJ}}{dz} = 2\left[-B_{nnnJ} - B_{jjnJ} + B_{njjJ} - B_{jnjJ}\right]$$
$$+2\,(\ln z)\left[B_{jnJ} - B_{njJ}\right] + (\ln z)^2 B_{nJ},$$
$$\int_0^z dz\,(\ln z)^2\frac{dB_{jJ}}{dz} = 2\left[-B_{jnnJ} + B_{njnJ} + B_{jjjJ} + B_{nnjJ}\right]$$
$$-2\,(\ln z)\left[B_{nnJ} + B_{jjJ}\right] + (\ln z)^2 B_{jJ}. \qquad \text{(D·65)}$$

Using these, we can rewrite the first term as

$$F_{\ell,0}[z(\ln z)^2 j_0] = 2\left(-D_\ell^{nnnj} + D_\ell^{jjnj} + D_\ell^{njjj} + D_\ell^{jnjj}\right)$$
$$-2(\ln z)\left(D_\ell^{jnj} + D_\ell^{njj}\right) + (\ln z)^2 D_\ell^{nj}. \qquad \text{(D·66)}$$

We need one more formula to evaluate (D·64):

$$F_{\ell,m}[z(\ln z)j_m] = F_{\ell,0}[z(\ln z)j_0] - \frac{1}{4m^2}R_{\ell,m}j_m$$
$$-\frac{\ln z}{2}\left\{R_{\ell,0}j_0 + \sum_{k=1}^{m-1}\left(\frac{1}{k}+\frac{1}{k+1}\right)R_{\ell,k}j_k + \frac{1}{m}R_{\ell,m}j_m\right\}$$
$$+\left(\sum_{k=1}^{m}\frac{1}{k}\right)D_\ell^{nj} - \frac{1}{4}\left\{2\left(\sum_{k=1}^{m}\frac{1}{k}\right) - 1\right\}R_{\ell,0}j_0$$
$$-\frac{1}{4}\sum_{k=1}^{m-1}\left(\frac{1}{k}+\frac{1}{k+1}\right)\left\{2\left(\sum_{p=k+1}^{m}\frac{1}{p}\right) + \frac{1}{k(k+1)}\right\}R_{\ell,k}j_k.$$
$$\text{(D·67)}$$

D.3.4. D_k^{nJ}-terms

The source terms are the even and odd standard forms of D_m^{nj}, and zD_ℓ^{nnj} and $D_{\ell\pm1}^{nnj}$.

Necessary formulas are

$$F_{\ell,\ell}[zD_{\ell+1}^{nJ}] = -D_\ell^{jnJ} - R_{\ell,-1}D_0^{nJ} - \sum_{m=1}^{\ell}\frac{4m}{4m^2-1}R_{\ell,m-1}D_m^{nJ}$$

$$
\begin{aligned}
&+H_\ell^J\left[z\left\{\sum_{m=1}^{\ell}\frac{4m}{4m^2-1}R_{m,0}j_{m-1}+\frac{R_{\ell+1,0}}{2\ell+1}j_\ell\right\}\right],\\
F_{\ell,\ell}[zD_{\ell-1}^{nJ}] &= D_\ell^{njJ}-R_{\ell,0}D_0^{jJ}-\sum_{m=1}^{\ell-1}\frac{4m}{4m^2-1}R_{\ell,m}D_{m-1}^{nJ}+D_\ell^{nJ}\\
&+H_\ell^J\left[z\left\{\sum_{m=2}^{\ell-1}\frac{4m}{4m^2-1}R_{m-1,0}j_m+\frac{R_{\ell-1,0}}{2\ell-1}j_\ell\right\}\right],\\
F_{\ell,\ell}[D_m^{nJ}] &= -\frac{1}{(\ell-m)(\ell+m+1)}D_m^{nJ}\\
&-H_\ell^J\left[\frac{z}{(\ell-m)(\ell+m+1)}\left(R_{m+1,0}j_\ell+R_{m,0}j_{\ell-1}\right)-\frac{R_{m,0}}{\ell-m}j_\ell\right],\\
F_{\ell,\ell}[zD_\ell^{nJ}] &= D_\ell^{nnJ}+\frac{1}{2}\left\{R_{0,\ell}D_0^{nJ}+\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{m,k}D_m^{nJ}\right\}\\
&+H_\ell^J\left[\frac{z}{2}\left(\sum_{m=1}^{\ell-1}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{m,0}j_m+\frac{1}{\ell}R_{\ell,0}j_\ell\right)\right],\\
F_{\ell,\ell}[D_{\ell+1}^{nJ}] &= \frac{1}{2(\ell+1)}D_{\ell+1}^{nJ}+H_\ell^J\left[-\frac{z}{2(\ell+1)}\left(R_{0,\ell+2}j_\ell+R_{0,\ell+1}j_{\ell-1}\right)+R_{0,\ell+1}j_\ell\right],\\
F_{\ell,\ell}[D_{\ell-1}^{nJ}] &= -\frac{1}{2\ell}D_{\ell-1}^{nJ}+H_\ell^J\left[\frac{z}{2\ell}\left(R_{0,\ell}j_\ell+R_{0,\ell-1}j_{\ell-1}\right)-R_{0,\ell-1}j_\ell\right]. \qquad \text{(D·68)}
\end{aligned}
$$

In order to evaluate the terms involving the integral operator H_ℓ^J, we recast its argument into the form,

$$
z(\alpha_{-2}j_{-2}+\alpha_{-1}j_{-1}+\alpha_0 j_0+\alpha_1 j_1)+\sum_{n\neq 0,-1}\beta_n j_n. \qquad \text{(D·69)}
$$

Then all the necessary terms can be easily evaluated as

$$
\begin{aligned}
H_\ell^j[j_m] &= \frac{1}{m(m+1)}\left(-R_{\ell,m+1}j_0+R_{\ell,m}n_0\right)+\frac{R_{\ell,m}}{mz}j_0, \quad (m\neq 0,\,-1)\\
H_\ell^j[zj_1] &= -D_\ell^{nn}+R_{\ell,0}n_0-R_{\ell,1}j_0,\\
H_\ell^j[zj_0] &= D_\ell^{nj},\\
H_\ell^j[zj_{-1}] &= -D_\ell^{jj},\\
H_\ell^j[zj_{-2}] &= -D_\ell^{jn}-j_0R_{\ell,-2}+j_{-1}R_{\ell,-1}, \qquad \text{(D·70)}
\end{aligned}
$$

and

$$
\begin{aligned}
H_\ell^{nj}[zj_1] &= D_\ell^{njj}-\frac{R_{\ell,0}}{z}D_0^{nj},\\
H_\ell^{nj}[zj_0] &= D_\ell^{nnj},\\
H_\ell^{nj}[zj_{-1}] &= -D_\ell^{jnj},\\
H_\ell^{nj}[zj_{-2}] &= \text{(unnecessary for our present calculation)},
\end{aligned}
$$

$$H_k^{nj}[j_m] = \frac{1}{m(m+1)}\left(R_{k,m}D_1^{nj} + R_{k,m-1}D_0^{nj}\right)$$
$$-\frac{1}{mz}R_{k,m}D_0^{nj} + \frac{1}{m(m+1)}H_k^j[zj_m]. \qquad \text{(D·71)}$$

The last term $H_k^j[zj_m]$ can be reduced recursively to those given in Eq. (D·70).

D.3.5. $z(\ln z)D_m^{nj}$-terms

What we need to evaluate are $F_{\ell,\ell}[z(\ln z)D_{\ell\pm 1}^{nj}]$.

First we consider $F_{\ell,\ell}\left[z(\ln z)D_{\ell+1}^{nj}\right]$. It is evaluated as

$$F_{\ell,\ell}\left[z(\ln z)D_{\ell+1}^{nj}\right]$$
$$= F_{\ell,-1}\left[z(\ln z)D_0^{nj}\right] - (\ln z)\left\{R_{\ell,-1}D_0^{nj} + \sum_{m=1}^{\ell}\frac{4m}{4m^2-1}R_{\ell,m-1}D_m^{nj}\right\}$$
$$+F_{\ell,-1}\left[zD_0^{nj}\right] + \sum_{m=1}^{\ell}\frac{4m}{4m^2-1}F_{\ell,m-1}\left[zD_m^{nj}\right] + \frac{1}{2\ell+1}F_{\ell,\ell}\left[zD_{\ell+1}^{nj}\right]$$
$$+H_\ell^j\left[z(\ln z)\left\{\sum_{m=1}^{\ell}\frac{4m}{4m^2-1}R_{m,0}j_{m-1} + \frac{1}{2\ell+1}R_{\ell+1,0}j_\ell\right\}\right], \qquad \text{(D·72)}$$

where the first term is further expressed in terms of D_ℓ^J as

$$F_{\ell,-1}\left[z(\ln z)D_0^{nj}\right] = -D_\ell^{nnnj} + D_\ell^{jjnj} - (\ln z)D_\ell^{jnj}. \qquad \text{(D·73)}$$

The other unknown terms in Eq. (D·73) are also evaluated as

$$F_{\ell,m-1}\left[zD_m^{nj}\right] = -D_\ell^{jnj} - \left\{R_{\ell,-1}D_0^{nj} + \sum_{k=1}^{m-1}\frac{4k}{4k^2-1}R_{\ell,k-1}D_k^{nj} + \frac{R_{\ell,m-1}}{2m-1}D_m^{nj}\right\}$$
$$+H_\ell^j\left[z\left\{\sum_{k=1}^{m-1}\frac{4k}{4k^2-1}R_{k,0}j_{k-1} + \frac{R_{m,0}}{2m-1}j_{m-1}\right\}\right], \quad (m \geq 1)$$
$$F_{\ell,-1}\left[zD_0^{nj}\right] = -D_\ell^{jnj},$$
$$H_\ell^j\left[z(\ln z)j_0\right] = F_{\ell,0}[z(\ln z)j_0] = -D_\ell^{njj} - D_\ell^{jnj} + (\ln z)D_\ell^{nj},$$
$$H_\ell^j\left[(\ln z)j_m\right] = (\ln z)\left\{\frac{1}{m(m+1)}\left(-R_{\ell,m+1}j_0 + R_{\ell,m}n_0\right) + \frac{R_{\ell,m}}{mz}j_0\right\}$$
$$+H_\ell^j\left[\frac{2z}{m(m+1)}j_{m+1}\right] + \frac{2m+1}{m^2(m+1)^2}\left(R_{\ell,m+1}j_0 - R_{\ell,m}n_0\right)$$
$$-\frac{R_{\ell,m}}{m^2 z}j_0. \qquad \text{(D·74)}$$

In the same way, we can evaluate $F_{\ell,\ell}\left[z(\ln z)D_{\ell-1}^{nj}\right]$ as

$$F_{\ell,\ell}\left[z(\ln z)D_{\ell-1}^{nj}\right]$$

$$=F_{\ell,0}\left[z(\ln z)D_0^{jj}\right]-(\ln z)\left\{R_{\ell,0}D_{-1}^{nj}+\sum_{m=1}^{\ell-1}\frac{4m}{4m^2-1}R_{\ell,m}D_{m-1}^{nj}\right\}$$
$$+F_{\ell,0}\left[zD_{-1}^{nj}\right]+\sum_{m=1}^{\ell-1}\frac{4m}{4m^2-1}F_{\ell,m}\left[zD_{m-1}^{nj}\right]+\frac{1}{2\ell-1}F_{\ell,\ell}\left[zD_{\ell-1}^{nj}\right]$$
$$+H_\ell^j\left[z(\ln z)\left(j_0+\sum_{m=2}^{\ell-1}\frac{4m}{4m^2-1}R_{m-1,0}j_m+\frac{1}{2\ell-1}R_{\ell-1,0}j_\ell\right)\right], \quad (D\cdot 75)$$

where the unknown terms on the right-hand side are evaluated as

$$F_{\ell,0}\left[z(\ln z)D_0^{jj}\right]=-D_\ell^{njjj}-D_\ell^{jnjj}+(\ln z)D_\ell^{njj},$$
$$F_{\ell,m}\left[zD_{m-1}^{nj}\right]=D_\ell^{njj}-\left\{R_{\ell,0}D_{-1}^{nj}+\sum_{k=1}^{m-1}\frac{4k}{4k^2-1}R_{\ell,k}D_{k-1}^{nj}+\frac{R_{\ell,m}}{2m-1}D_{m-1}^{nj}\right\}$$
$$+H_\ell^j\left[z\left(j_0+\sum_{k=1}^{m-1}\frac{4k}{4k^2-1}R_{k-1,0}j_k+\frac{R_{m-1,0}}{2m-1}j_m\right)\right], \quad (m\ge 1)$$
$$F_{\ell,0}\left[zD_{-1}^{nj}\right]=D_\ell^{njj}. \quad (D\cdot 76)$$

D.4. *The asymptotic behavior*

Using the results of the preceding subsection, we obtain the analytic expression for $\xi_\ell^{(3)}$. The real part of it is given in Eq. (4·14) for $\ell=2$ and in Eq. (4·15) for $\ell=3$, while the imaginary part is determined by Eq. (D·9). To obtain $A_\ell^{\rm inc}$ to $O(\epsilon^3)$, we then see that all what we need to evaluate are the asymptotic behaviors of D_ℓ^{nj}, D_ℓ^{nnj}, D_ℓ^{nnnj} and $F_{\ell,0}[z(\ln z)j_0]$. Although the last of these can be expressed in terms of D_ℓ^J as given by Eq. (D·61), we find it is easier to evaluate the integral directly as it is.

Here in order to evaluate the asymptotic behavior of these functions, we first give necessary basic formulas. Then we evaluate the asymptotic behavior of all the necessary B_J and $F_{\ell,0}[z(\ln z)j_0]$. In this subsection $x=2z$.

D.4.1. Basic asymptotic formulas

First, we give the most basic formulas:

$$S=\int_0^x dy\frac{\sin y}{y}\to\frac{\pi}{2},$$
$$C=\int_0^x dy\frac{\cos y-1}{y}\to-\ln x-\gamma,$$
$$\int_0^x dy\frac{\sin y}{y}\ln y\to-\frac{\pi}{2}\gamma,$$
$$\int_0^x dy\frac{\cos y-1}{y}\ln y\to-\frac{1}{2}(\ln x)^2-\frac{\pi^2}{24}+\frac{\gamma^2}{2},$$
$$\int_0^x dy\frac{\sin y}{y}(\ln y)^2\to\frac{\pi}{2}\gamma^2+\frac{\pi^3}{24},$$

$$\int_0^x dy \frac{\cos y - 1}{y} (\ln y)^2 \to -\frac{1}{3}(\ln x)^3 - \frac{\gamma^3}{3} + \frac{\gamma\pi^2}{12} - \frac{2}{3}\zeta(3), \tag{D·77}$$

where $\zeta(z)$ is the Riemann zeta function and $\zeta(3) = 1.202\cdots$.

There appear several expressions to be estimated that diverge if we evaluate them term by term. But they give finite results when combined together. They are

$$\begin{aligned}
&\int_0^x \frac{dy}{y+1}\{\ln(y+1) - \ln y\} \to \frac{\pi^2}{6},\\
&\int_0^x \frac{dy}{y+1}\left\{(\ln(y+1))^2 - (\ln y)^2\right\} \to 0,\\
&\int_0^x \frac{dy}{y+1}\{\ln(y+2) - \ln(y+1)\} \to \frac{\pi^2}{12},\\
&\int_0^x \frac{dy}{y+1}\left\{(\ln(y+2))^2 - (\ln(y+1))^2\right\} \to 2\phi(3,1/2) + \frac{\pi^2}{6}\ln 2 - \frac{1}{3}(\ln 2)^3,\\
&\int_0^x \frac{dy}{y+2}\ln(y+1)\{\ln(y+2) - \ln(y+1)\}\\
&\to -\phi(3,1/2) - \frac{\pi^2}{12}\ln 2 + \frac{1}{6}(\ln 2)^3 + \frac{3}{2}\zeta(3),
\end{aligned} \tag{D·78}$$

where $\phi(a,b)$ represents the modified zeta function. We mention that the $\phi(3,1/2)$ terms are found to cancel out in the final expression for $A_\ell^{\rm inc}$.

There are several formulas which require multiple integrations. They are evaluated as

$$\int_0^x \frac{dy}{y}\left[\int_y^\infty \frac{du}{u}(\cos(u-y) - \cos u)\right] \to \frac{\pi^2}{6}, \tag{D·79}$$

$$\int_0^x \frac{dy}{y}\ln y\left[\int_y^\infty \frac{du}{u}(\cos(u-y) - \cos u)\right] \to -\frac{\pi^2\gamma}{6}, \tag{D·80}$$

$$\begin{aligned}
&\int_0^x \frac{dy}{y}\ln y\left[\int_y^\infty \frac{du}{u}(\cos(u+y) - \cos u)\right]\\
&\qquad \to \phi(3,1/2) + \frac{\pi^2}{12}\ln 2 - \frac{1}{6}(\ln 2)^3 + \frac{\pi^2\gamma}{12},
\end{aligned} \tag{D·81}$$

$$\int_0^x \frac{dy}{y}\ln y\left[\int_y^\infty \frac{du}{u}(\sin(u-y) - \sin u)\right] \to \frac{\pi^3}{12}, \tag{D·82}$$

$$\int_0^x \frac{dy}{y}\ln y\left[\int_y^\infty \frac{du}{u}(\sin(u+y) - \sin u)\right] \to -\frac{\pi^3}{24}, \tag{D·83}$$

$$\int_0^x \frac{dy}{y}\left[\int_y^\infty \frac{du}{u}(\sin(u-y) - \sin u)\right] \to 0. \tag{D·84}$$

These are obtained by changing the variable from u to $u' = (u/y) - 1$, performing the dy-integration first and using the formulas (D·78).

Next we give several formulas that contain S and C. Let us recall the definition of S and C:

$$S = \int_0^x \frac{dy}{y}\sin y = -\int_x^\infty \frac{dy}{y}\sin y + \frac{\pi}{2}, \tag{D·85}$$

$$C = \int_0^x \frac{dy}{y}(\cos y - 1) = -\int_x^\infty \frac{dy}{y}\cos y - \ln x - \gamma. \tag{D·86}$$

By using the integration by part, we have

$$\begin{aligned}\int_0^x dy \ln y\left(\frac{\sin y}{y}S + \frac{\cos y - 1}{y}C\right) &= \frac{\pi}{2}\int_0^x dy\frac{\sin y}{y}\ln y - \int_0^x dy\frac{\cos y - 1}{y}\left(\gamma\ln y + (\ln y)^2\right) \\ &\quad - \int_0^x dy\frac{\ln y}{y}\left[\int_y^\infty \frac{du}{u}(\cos(u-y) - \cos u)\right],\end{aligned} \tag{D·87}$$

whose asymptotic behavior is determined by Eqs. (D·77) and (D·80). Similarly we have

$$\begin{aligned}\int_0^x dy \ln y\left(\frac{\sin y}{y}S - \frac{\cos y - 1}{y}C\right) &= \frac{\pi}{2}\int_0^x dy\frac{\sin y}{y}\ln y + \int_0^x dy\frac{\cos y - 1}{y}\left(\gamma\ln y + (\ln y)^2\right) \\ &\quad + \int_0^x dy\frac{\ln y}{y}\left[\int_y^\infty \frac{du}{u}(\cos(u+y) - \cos u)\right],\end{aligned} \tag{D·88}$$

and

$$\begin{aligned}\int_0^x dy \ln y\left(\frac{\sin y}{y}C \pm \frac{\cos y - 1}{y}S\right) &= -\int_0^x dy\frac{\sin y}{y}\ln y(\ln y + \gamma) \pm \frac{\pi}{2}\int_0^x dy\frac{\cos y - 1}{y}\ln y \\ &\quad \mp \int_0^x dy\frac{\ln y}{y}\left[\int_y^\infty \frac{du}{u}(\sin(u \pm y) - \sin u)\right],\end{aligned} \tag{D·89}$$

whose asymptotic behaviors are determined by Eqs. (D·77), (D·81), (D·82) and (D·83). Then we obtain

$$\int_0^x \frac{dy}{y}\left(S^2 + C^2\right) \to \frac{\pi^2}{4}(\ln x + \gamma) + \frac{1}{3}(\ln x + \gamma)^3 - \frac{4}{3}\zeta(3), \tag{D·90}$$

$$\begin{aligned}\int_0^x \frac{dy}{y}\left(S^2 - C^2\right) \to &\frac{\pi^2}{4}\ln x - \frac{1}{3}(\ln x + \gamma)^3 + \frac{\pi^2\gamma}{4} + \frac{4}{3}\zeta(3) \\ &-2\left(\phi(3, 1/2) + \frac{\pi^2}{12}\ln 2 - \frac{1}{6}(\ln 2)^3\right)\end{aligned} \tag{D·91}$$

These two formulas are obtained by reducing them to the forms to which Eqs. (D·87) and (D·88) can be applied, respectively.

Finally, we present two complicated formulas. The first is

$$\begin{aligned} I_1 :=& \int_0^x \frac{dy}{y}\left(-2\sin y SC + (\cos y - 1)(S^2 - C^2)\right) \\ \to& 2\left(\phi(3,1/2) + \frac{\pi^2}{12}\ln 2 - \frac{1}{6}(\ln 2)^3\right) \\ &-\frac{7}{3}\zeta(3) - \frac{1}{4}\pi^2(\ln x + \gamma) + \frac{1}{3}(\ln x + \gamma)^3, \end{aligned} \tag{D·92}$$

where we have used the equalities,

$$SC = \int_0^\infty \frac{\sin(t+2)x}{t+2}\ln(t+1)dt + \frac{\pi}{2}C - (\ln x + \gamma)S + \frac{\pi}{2}(\ln x + \gamma), \tag{D·93}$$

$$S^2 - C^2 = -2\int_0^\infty \frac{\cos(t+2)x}{t+2}\ln(t+1)dt + \pi S - \frac{\pi^2}{4} + 2(\ln x + \gamma)C + (\ln x + \gamma)^2, \tag{D·94}$$

and applied the formulas (D·77), (D·87) and the last one of Eqs. (D·78). The second is

$$\begin{aligned} I_2 :=& \int_0^x \frac{dy}{y}\left(2(\cos y - 1)SC + \sin y(S^2 - C^2)\right) \\ \to& \frac{\pi}{2}\left((\ln x + \gamma)^2 + \frac{\pi^2}{4}\right), \end{aligned} \tag{D·95}$$

which is obtained in the same way by applying the formulas (D·77), (D·84) and (D·89).

D.4.2. The asymptotic behavior of B_J

As we have mentioned, what we have to evaluate are the asymptotic behaviors of D_ℓ^{nj}, D_ℓ^{nnj}, D_ℓ^{nnnj} and $F_{\ell,0}[z(\ln z)j_0]$. Hence, recalling the definition of D_ℓ^J, we need to evaluate $B_{nj}, B_{jj}, B_{nnj}, B_{jnj}, B_{nnnj}$ and B_{jnnj} in addition to $F_{\ell,0}[z(\ln z)j_0]$.

The formulas for B_J with two indices are given by the first two equations of (D·77):

$$B_{nj} = -\frac{1}{2}S \to -\frac{\pi}{4}, \quad B_{jj} = -\frac{1}{2}C \to \frac{1}{2}(\ln x + \gamma). \tag{D·96}$$

As for $F_{\ell,0}[z(\ln z)j_0]$, its asymptotic behavior is directly evaluated as

$$\begin{aligned} F_{\ell,0}[z(\ln z)j_0] =& n_\ell \int_0^z dz z(\ln z)j_0^2 - j_\ell \int_0^z dz z(\ln z)j_0 n_0 \\ \to& \frac{1}{4}\left((\ln z)^2 + \frac{\pi^2}{12} - (\gamma + \ln 2)^2\right)n_\ell - \frac{\pi}{4}(\gamma + \ln 2)j_\ell . \end{aligned} \tag{D·97}$$

The formulas for B_J with three indices are given by

$$\begin{aligned} B_{jnj} =& \frac{1}{4}\int_0^x dy\left(\frac{\sin y}{y}C - \frac{\cos y - 1}{y}S\right) \\ \to& \frac{\pi}{8}(\ln x + \gamma), \end{aligned} \tag{D·98}$$

where we have used Eqs. (D·77) and (D·84), and

$$B_{nnj} = -\frac{1}{4}\int_0^x dy\left(\frac{\cos y+1}{y}C + \frac{\sin y}{y}S\right)$$
$$\to -\frac{1}{8}\left(\frac{5}{12}\pi^2 - (\ln x+\gamma)^2\right), \qquad (D·99)$$

where we have used Eqs. (D·77).

For B_J with four indices, we have

$$\begin{aligned}B_{jnnj} &= -\frac{1}{2}\int_0^x dy\frac{\sin y}{y}B_{jnj} + \frac{1}{2}\int_0^x dy\frac{\cos y - 1}{y}B_{nnj} \\ &= -\frac{1}{2}\left[SB_{jnj} - CB_{nnj}\right] + \frac{1}{8}\int_0^x dy\left(\frac{\sin y}{y}C - \frac{\cos y-1}{y}S\right)S \\ &\quad +\frac{1}{8}\int_0^x dy\left(\frac{\sin y}{y}S + \frac{\cos y+1}{y}C\right)C \\ &= -\frac{1}{2}\left[SB_{jnj} - CB_{nnj}\right] - \frac{1}{8}I_1 + \frac{1}{4}\int_0^x dy\frac{C^2}{y} \\ &\to \frac{1}{24}\left[\frac{5}{8}\pi^2(\ln x+\gamma) - \frac{1}{2}(\ln x+\gamma)^3 - \zeta(3)\right],\end{aligned} \qquad (D·100)$$

where we have used Eqs. (D·90), (D·91) and (D·92), and

$$\begin{aligned}B_{nnnj} &= \frac{1}{2}\int_0^x dy\frac{\sin y}{y}B_{nnj} + \frac{1}{2}\int_0^x dy\frac{\cos y + 1}{y}B_{jnj} \\ &= \int_0^x dyB_{jnj} + \frac{1}{2}\left[CB_{jnj} + SB_{nnj}\right] \\ &\quad -\frac{1}{8}\int_0^x dy\left(\frac{\sin y}{y}C - \frac{\cos y-1}{y}S\right)C \\ &\quad +\frac{1}{8}\int_0^x dy\left(\frac{\sin y}{y}S + \frac{\cos y+1}{y}C\right)S \\ &= \ln x B_{jnj} - \frac{1}{4}\int_0^x dy\ln y\left(\frac{\sin y}{y}C - \frac{\cos y-1}{y}S\right) \\ &\quad +\frac{1}{2}\left[CB_{jnj} + SB_{nnj}\right] + \frac{1}{8}I_2 + \frac{1}{4}\int_0^x dy\frac{SC}{y} \\ &\to \frac{\pi}{32}\left[-\frac{1}{4}\pi^2 + (\ln x+\gamma)^2\right],\end{aligned} \qquad (D·101)$$

where we have used Eqs. (D·89) and (D·95). We should note that Eq. (D·89) has been also used in evaluating the term $(1/4)\int_0^x dy(SC/y)$.

With these results, the asymptotic incoming amplitudes $A_\ell^{\rm inc}$ to the required order are obtained, which are given in Eqs. (4·17) in the text.

Appendix E

—— 5.5PN Formulas for $R_\ell^{\rm in}$ and $(dE/dt)_{\ell m}$ in the Schwarzschild Case ——

In this appendix, we show the post-Newtonian expansion of $R^{\rm in}$ which are necessary to evaluate the 5.5PN gravitational wave luminosity in the case of a Schwarzschild black hole. Then we show each (ℓ, m)-mode contribution to the total luminosity from a particle in circular orbit.

For convenience, we give the formulas for $c_0\omega R^{\rm in}_\ell$, where we recover the index ℓ on $R^{\rm in}$ and $c_0 = (\ell-1)\ell(\ell+1)(\ell+2) - 6i\epsilon$

$$
\begin{aligned}
c_0\omega R^{\rm in}_2 = &\left(\frac{4\,z^4}{5} + \frac{8\,i}{15}\,z^5 - \frac{22\,z^6}{105} - \frac{2\,i}{35}\,z^7 + \frac{23\,z^8}{1890} + \frac{2\,i}{945}\,z^9 - \frac{13\,z^{10}}{41580}\right.\\
&\left. - \frac{i}{24948}\,z^{11} + \frac{59\,z^{12}}{12972960} + \frac{i}{2162160}\,z^{13} - \frac{83\,z^{14}}{1945944000} - \frac{i}{277992000}\,z^{15}\right)\\
&+\left(\frac{-8\,z^3}{5} - \frac{3\,i}{5}\,z^4 - \frac{8\,z^5}{63} - \frac{13\,i}{90}\,z^6 + \frac{109\,z^7}{1890} + \frac{341\,i}{22680}\,z^8\right.\\
&\left. - \frac{9403\,z^9}{3118500} - \frac{293\,i}{594000}\,z^{10} + \frac{38963\,z^{11}}{567567000} + \frac{75529\,i}{9081072000}\,z^{12}\right)\epsilon\\
&+\left(\frac{4\,z^2}{5} + \frac{123317\,z^4}{36750} + \frac{231479\,i}{110250}\,z^5 - \frac{889954\,z^6}{1157625} - \frac{454499\,i}{2315250}\,z^7\right.\\
&+\frac{215321483\,z^8}{5501034000} + \frac{35106811\,i}{5501034000}\,z^9 - \frac{214\,z^4\,\ln z}{525} - \frac{428\,i}{1575}\,z^5\,\ln z\\
&\left.+\frac{1177\,z^6\,\ln z}{11025} + \frac{107\,i}{3675}\,z^7\,\ln z - \frac{2461\,z^8\,\ln z}{396900} - \frac{107\,i}{99225}\,z^9\,\ln z\right)\epsilon^2\\
&+\left(\frac{-66823\,z^3}{12250} - \frac{99851\,i}{55125}\,z^4 - \frac{504569\,z^5}{694575} - \frac{2488639\,i}{3969000}\,z^6 + \frac{428\,z^3\,\ln z}{525}\right.\\
&\left.+\frac{107\,i}{350}\,z^4\,\ln z + \frac{428\,z^5\,\ln z}{6615} + \frac{1391\,i}{18900}\,z^6\,\ln z\right)\epsilon^3\\
&+\left(\frac{471487\,z^2}{220500} - \frac{263\,i}{1260}\,z^3 - \frac{214\,z^2\,\ln z}{525}\right)\epsilon^4,
\end{aligned}
\tag{E·1}
$$

$$
\begin{aligned}
c_0\omega R^{\rm in}_3 = &\left(\frac{4\,z^5}{21} + \frac{2\,i}{21}\,z^6 - \frac{2\,z^7}{63} - \frac{i}{135}\,z^8 + \frac{29\,z^9}{20790} + \frac{i}{4620}\,z^{10} - \frac{47\,z^{11}}{1621620}\right.\\
&\left. - \frac{i}{294840}\,z^{12} + \frac{23\,z^{13}}{64864800} + \frac{i}{29937600}\,z^{14}\right)\\
&+\left(\frac{-10\,z^4}{21} - \frac{53\,i}{315}\,z^5 + \frac{z^6}{210} - \frac{i}{90}\,z^7 + \frac{751\,z^8}{155925} + \frac{1483\,i}{1247400}\,z^9\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{23\,z^{10}}{102960} - \frac{367\,i}{10810800}\,z^{11}\Bigg)\,\epsilon \\
&+ \left(\frac{8\,z^3}{21} + \frac{i}{14}\,z^4 + \frac{40337\,z^5}{46305} + \frac{79099\,i}{185220}\,z^6 - \frac{12562147\,z^7}{91683900} - \frac{4840537\,i}{157172400}\,z^8\right. \\
&\quad \left. - \frac{26\,z^5\,\ln z}{441} - \frac{13\,i}{441}\,z^6\,\ln z + \frac{13\,z^7\,\ln z}{1323} + \frac{13\,i}{5670}\,z^8\,\ln z\right)\epsilon^2 \\
&+ \left(\frac{-2\,z^2}{21} - \frac{182981\,z^4}{92610} - \frac{3753697\,i}{5556600}\,z^5\right. \\
&\quad \left. + \frac{65\,z^4\,\ln z}{441} + \frac{689\,i}{13230}\,z^5\,\ln z\right)\epsilon^3, \qquad \text{(E·2)}
\end{aligned}
$$

$$
\begin{aligned}
c_0\omega R_4^{\rm in} &= \left(\frac{2\,z^6}{63} + \frac{4\,i}{315}\,z^7 - \frac{13\,z^8}{3465} - \frac{8\,i}{10395}\,z^9\right. \\
&\quad \left. + \frac{71\,z^{10}}{540540} + \frac{i}{54054}\,z^{11} - \frac{37\,z^{12}}{16216200} - \frac{i}{4054050}\,z^{13}\right) \\
&+ \left(\frac{-2\,z^5}{21} - \frac{4\,i}{135}\,z^6 + \frac{142\,z^7}{51975} - \frac{31\,i}{51975}\,z^8 + \frac{929\,z^9}{2702700} + \frac{8\,i}{96525}\,z^{10}\right)\epsilon \\
&+ \left(\frac{5\,z^4}{49} + \frac{97\,i}{4410}\,z^5 + \frac{958223891\,z^6}{6051137400} + \frac{239560304\,i}{3781960875}\,z^7\right. \\
&\quad \left. - \frac{1571\,z^6\,\ln z}{218295} - \frac{3142\,i}{1091475}\,z^7\,\ln z\right)\epsilon^2 \\
&+ \left(\frac{-20\,z^3}{441} - \frac{i}{196}\,z^4\right)\epsilon^3, \qquad \text{(E·3)}
\end{aligned}
$$

$$
\begin{aligned}
c_0\omega R_5^{\rm in} &= \left(\frac{2\,z^7}{495} + \frac{2\,i}{1485}\,z^8 - \frac{7\,z^9}{19305} - \frac{i}{15015}\,z^{10} + \frac{17\,z^{11}}{1621620} + \frac{i}{737100}\,z^{12}\right) \\
&+ \left(\frac{-7\,z^6}{495} - \frac{67\,i}{17325}\,z^7 + \frac{1831\,z^8}{4054050} - \frac{43\,i}{4054050}\,z^9\right)\epsilon \\
&+ \left(\frac{28\,z^5}{1485} + \frac{59\,i}{14850}\,z^6\right)\epsilon^2, \qquad \text{(E·4)}
\end{aligned}
$$

$$
c_0\omega R_6^{\rm in} = \left(\frac{8\,z^8}{19305} + \frac{16\,i}{135135}\,z^9 - \frac{4\,z^{10}}{135135} - \frac{2\,i}{405405}\,z^{11}\right)
$$

$$+\left(\frac{-32\,z^7}{19305}-\frac{163\,i}{405405}\,z^8\right)\epsilon, \tag{E·5}$$

$$c_0\omega R_7^{\rm in} = \frac{8\,z^9}{225225}+\frac{2\,i}{225225}\,z^{10}. \tag{E·6}$$

Next we show the contribution of each (ℓ, m)-mode to the gravitational wave luminosity to $O(v^{11})$ in the case of a circular orbit around a Schwarzschild black hole. We set

$$\left\langle \frac{dE}{dt}\right\rangle = \frac{1}{2}\left(\frac{dE}{dt}\right)_{\rm N}\sum_{\ell,m}\eta_{\ell,m}\,, \tag{E·7}$$

where $(dE/dt)_{\rm N}$ is the Newtonian quadrupole luminosity, Eq. (4·19). In the present case we have $\eta_{\ell,m}=\eta_{\ell,-m}$. Hence we show only the modes $m>0$.

$$\begin{aligned}
\eta_{2,2} = {} & 1-\frac{107\,v^2}{21}+4\,\pi\,v^3+\frac{4784\,v^4}{1323}-\frac{428\,\pi\,v^5}{21} \\
& +v^6\left(\frac{99210071}{1091475}-\frac{1712\,\gamma}{105}+\frac{16\,\pi^2}{3}-\frac{3424\,\ln 2}{105}-\frac{1712\,\ln v}{105}\right) \\
& +\frac{19136\,\pi\,v^7}{1323} \\
& +v^8\left(-\frac{27956920577}{81265275}+\frac{183184\,\gamma}{2205}-\frac{1712\,\pi^2}{63}+\frac{366368\,\ln 2}{2205}+\frac{183184\,\ln v}{2205}\right) \\
& +v^9\left(\frac{396840284\,\pi}{1091475}-\frac{6848\,\gamma\,\pi}{105}-\frac{13696\,\pi\,\ln 2}{105}-\frac{6848\,\pi\,\ln v}{105}\right) \\
& +v^{10}\left(\frac{187037845924}{6257426175}-\frac{8190208\,\gamma}{138915}+\frac{76544\,\pi^2}{3969}\right. \\
& \left.-\frac{16380416\,\ln 2}{138915}-\frac{8190208\,\ln v}{138915}\right) \\
& +v^{11}\left(\frac{-111827682308\,\pi}{81265275}+\frac{732736\,\gamma\,\pi}{2205}\right. \\
& \left.+\frac{1465472\,\pi\,\ln 2}{2205}+\frac{732736\,\pi\,\ln v}{2205}\right),
\end{aligned} \tag{E·8}$$

$$\begin{aligned}
\eta_{2,1} = {} & \frac{v^2}{36}-\frac{17\,v^4}{504}+\frac{\pi\,v^5}{18}-\frac{2215\,v^6}{254016}-\frac{17\,\pi\,v^7}{252} \\
& +v^8\left(\frac{15707221}{26195400}-\frac{107\,\gamma}{945}+\frac{\pi^2}{27}-\frac{107\,\ln 2}{945}-\frac{107\,\ln v}{945}\right) \\
& -\frac{2215\,\pi\,v^9}{127008} \\
& +v^{10}\left(-\frac{6435768121}{57210753600}+\frac{1819\,\gamma}{13230}-\frac{17\,\pi^2}{378}+\frac{1819\,\ln 2}{13230}+\frac{1819\,\ln v}{13230}\right) \\
& +v^{11}\left(\frac{15707221\,\pi}{13097700}-\frac{214\,\gamma\,\pi}{945}-\frac{214\,\pi\,\ln 2}{945}-\frac{214\,\pi\,\ln v}{945}\right),
\end{aligned} \tag{E·9}$$

$$\eta_{3,3} = \frac{1215\,v^2}{896} - \frac{1215\,v^4}{112} + \frac{3645\,\pi\,v^5}{448} + \frac{243729\,v^6}{9856} - \frac{3645\,\pi\,v^7}{56}$$
$$+v^8\left(\frac{25037019729}{125565440} - \frac{47385\,\gamma}{1568} + \frac{3645\,\pi^2}{224} - \frac{47385\,\ln 2}{1568}\right.$$
$$\left.-\frac{47385\,\ln 3}{1568} - \frac{47385\,\ln v}{1568}\right)$$
$$+\frac{731187\,\pi\,v^9}{4928}$$
$$+v^{10}\left(-\frac{2074855555161}{1381219840} + \frac{47385\,\gamma}{196} - \frac{3645\,\pi^2}{28} + \frac{47385\,\ln 2}{196}\right.$$
$$\left.+\frac{47385\,\ln 3}{196} + \frac{47385\,\ln v}{196}\right)$$
$$+v^{11}\left(\frac{75111059187\,\pi}{62782720} - \frac{142155\,\gamma\,\pi}{784} - \frac{142155\,\pi\,\ln 2}{784}\right.$$
$$\left.-\frac{142155\,\pi\,\ln 3}{784} - \frac{142155\,\pi\,\ln v}{784}\right), \qquad \text{(E·10)}$$

$$\eta_{3,2} = \frac{5\,v^4}{63} - \frac{193\,v^6}{567} + \frac{20\,\pi\,v^7}{63} + \frac{86111\,v^8}{280665} - \frac{772\,\pi\,v^9}{567}$$
$$+v^{10}\left(\frac{960188809}{178783605} - \frac{1040\,\gamma}{1323} + \frac{80\,\pi^2}{189} - \frac{2080\,\ln 2}{1323} - \frac{1040\,\ln v}{1323}\right)$$
$$+\frac{344444\,\pi\,v^{11}}{280665}, \qquad \text{(E·11)}$$

$$\eta_{3,1} = \frac{v^2}{8064} - \frac{v^4}{1512} + \frac{\pi\,v^5}{4032} + \frac{437\,v^6}{266112} - \frac{\pi\,v^7}{756}$$
$$+v^8\left(-\frac{1137077}{50854003200} - \frac{13\,\gamma}{42336} + \frac{\pi^2}{6048} - \frac{13\,\ln 2}{42336} - \frac{13\,\ln v}{42336}\right)$$
$$+\frac{437\,\pi\,v^9}{133056}$$
$$+v^{10}\left(-\frac{38943317051}{5034546316800} + \frac{13\,\gamma}{7938} - \frac{\pi^2}{1134} + \frac{13\,\ln 2}{7938} + \frac{13\,\ln v}{7938}\right)$$
$$+v^{11}\left(\frac{-1137077\,\pi}{25427001600} - \frac{13\,\gamma\,\pi}{21168} - \frac{13\,\pi\,\ln 2}{21168} - \frac{13\,\pi\,\ln v}{21168}\right), \qquad \text{(E·12)}$$

$$\eta_{4,4} = \frac{1280\,v^4}{567} - \frac{151808\,v^6}{6237} + \frac{10240\,\pi\,v^7}{567} + \frac{560069632\,v^8}{6243237} - \frac{1214464\,\pi\,v^9}{6237}$$
$$+v^{10}\left(\frac{36825600631808}{88497884475} - \frac{25739264\,\gamma}{392931} + \frac{81920\,\pi^2}{1701} - \frac{25739264\,\ln 2}{130977}\right.$$
$$\left.-\frac{25739264\,\ln v}{392931}\right) + \frac{4480557056\,\pi\,v^{11}}{6243237}, \qquad \text{(E·13)}$$

$$\eta_{4,3} = \frac{729\,v^6}{4480} - \frac{28431\,v^8}{24640} + \frac{2187\,\pi\,v^9}{2240} + \frac{620077923\,v^{10}}{246646400} - \frac{85293\,\pi\,v^{11}}{12320}, \tag{E·14}$$

$$\eta_{4,2} = \frac{5\,v^4}{3969} - \frac{437\,v^6}{43659} + \frac{20\,\pi\,v^7}{3969} + \frac{7199152\,v^8}{218513295} - \frac{1748\,\pi\,v^9}{43659} + v^{10}\left(\frac{9729776708}{619485191325} - \frac{25136\,\gamma}{2750517} + \frac{80\,\pi^2}{11907} - \frac{50272\,\ln 2}{2750517} - \frac{25136\,\ln v}{2750517}\right) + \frac{28796608\,\pi\,v^{11}}{218513295}, \tag{E·15}$$

$$\eta_{4,1} = \frac{v^6}{282240} - \frac{101\,v^8}{4656960} + \frac{\pi\,v^9}{141120} + \frac{7478267\,v^{10}}{139848508800} - \frac{101\,\pi\,v^{11}}{2328480}, \tag{E·16}$$

$$\eta_{5,5} = \frac{9765625\,v^6}{2433024} - \frac{2568359375\,v^8}{47443968} + \frac{48828125\,\pi\,v^9}{1216512} + \frac{7060478515625\,v^{10}}{25904406528} - \frac{12841796875\,\pi\,v^{11}}{23721984}, \tag{E·17}$$

$$\eta_{5,4} = \frac{4096\,v^8}{13365} - \frac{18231296\,v^{10}}{6081075} + \frac{32768\,\pi\,v^{11}}{13365}, \tag{E·18}$$

$$\eta_{5,3} = \frac{2187\,v^6}{450560} - \frac{150903\,v^8}{2928640} + \frac{6561\,\pi\,v^9}{225280} + \frac{600654447\,v^{10}}{2665062400} - \frac{452709\,\pi\,v^{11}}{1464320}, \tag{E·19}$$

$$\eta_{5,2} = \frac{4\,v^8}{40095} - \frac{15644\,v^{10}}{18243225} + \frac{16\,\pi\,v^{11}}{40095}, \tag{E·20}$$

$$\eta_{5,1} = \frac{v^6}{127733760} - \frac{179\,v^8}{2490808320} + \frac{\pi\,v^9}{63866880} + \frac{290803\,v^{10}}{971415244800} - \frac{179\,\pi\,v^{11}}{1245404160}, \tag{E·21}$$

$$\eta_{6,6} = \frac{26244\,v^8}{3575} - \frac{2965572\,v^{10}}{25025} + \frac{314928\,\pi\,v^{11}}{3575}, \tag{E·22}$$

$$\eta_{6,5} = \frac{244140625\,v^{10}}{435891456}, \tag{E·23}$$

$$\eta_{6,4} = \frac{131072\,v^8}{9555975} - \frac{4063232\,v^{10}}{22297275} + \frac{1048576\,\pi\,v^{11}}{9555975}, \tag{E·24}$$

$$\eta_{6,3} = \frac{59049\,v^{10}}{98658560}, \tag{E·25}$$

$$\eta_{6,2} = \frac{4\,v^8}{5733585} - \frac{4\,v^{10}}{495495} + \frac{16\,\pi\,v^{11}}{5733585}, \tag{E·26}$$

$$\eta_{6,1} = \frac{v^{10}}{7192209024}, \tag{E·27}$$

$$\eta_{7,7} = \frac{96889010407\,v^{10}}{7116595200}, \tag{E·28}$$

$$\eta_{7,5} = \frac{6103515625\,v^{10}}{181330845696}, \tag{E·29}$$

$$\eta_{7,3} = \frac{1594323\,v^{10}}{205209804800}, \tag{E·30}$$

$$\eta_{7,1} = \frac{v^{10}}{5983917907968}. \tag{E·31}$$

Appendix F
—— 4PN Luminosity in Terms of the Orbital Frequency ——

Here we present the (ℓ, m)-mode contributions to the gravitational wave luminosity for a circular orbit on the equatorial plane around a Kerr black hole. In stead of $v \equiv (M/r_0)^{1/2}$, the formulas are expressed in terms of the parameter $x \equiv (M\Omega_\varphi)^{1/3}$, where Ω_φ is the orbital angular frequency, which is more relevant in the actual analysis of observed gravitational wave signals. We express the partial mode contributions as

$$\left\langle \frac{dE}{dt} \right\rangle \equiv \frac{16}{5}\left(\frac{\mu}{M}\right)^2 x^{10} \sum_{\ell,m} \eta_{\ell,m}\,. \tag{F·1}$$

Since $\eta_{\ell,m} = \eta_{\ell,-m}$, we show only the modes $m > 0$ below.

$$\begin{aligned}
\eta_{2,2} = {} & 1 - \frac{107\,x^2}{21} + \left(4\,\pi - \frac{8\,q}{3}\right) x^3 + \left(\frac{4784}{1323} + 2\,q^2\right) x^4 \\
& + \left(\frac{-428\,\pi}{21} + \frac{52\,q}{27}\right) x^5 \\
& + \left(\frac{99210071}{1091475} - \frac{1712\,\gamma}{105} + \frac{16\,\pi^2}{3} - \frac{32\,\pi\,q}{3} - \frac{1817\,q^2}{567}\right. \\
& \left. - \frac{3424\,\ln 2}{105} - \frac{1712\,\ln x}{105}\right) x^6 + \left(\frac{19136\,\pi}{1323} + \frac{364856\,q}{11907} + 8\,\pi\,q^2 - \frac{8\,q^3}{3}\right) x^7 \\
& + \left(-\frac{27956920577}{81265275} + \frac{183184\,\gamma}{2205} - \frac{1712\,\pi^2}{63} + \frac{208\,\pi\,q}{27}\right. \\
& \left. + \frac{105022\,q^2}{9261} + q^4 + \frac{366368\,\ln 2}{2205} + \frac{183184\,\ln x}{2205}\right) x^8 ,
\end{aligned} \tag{F·2}$$

$$\begin{aligned}
\eta_{2,1} = {} & \frac{x^2}{36} - \frac{q\,x^3}{12} + \left(-\frac{17}{504} + \frac{q^2}{16}\right) x^4 \\
& + \left(\frac{\pi}{18} + \frac{215\,q}{9072}\right) x^5 \\
& + \left(-\frac{2215}{254016} - \frac{\pi\,q}{6} + \frac{313\,q^2}{1512}\right) x^6 \\
& + \left(\frac{-17\,\pi}{252} - \frac{18127\,q}{190512} + \frac{\pi\,q^2}{8} - \frac{7\,q^3}{24}\right) x^7 \\
& + \left(\frac{15707221}{26195400} - \frac{107\,\gamma}{945} + \frac{\pi^2}{27} + \frac{215\,\pi\,q}{4536}\right. \\
& \left. + \frac{44299\,q^2}{95256} + \frac{q^4}{16} - \frac{107\,\ln 2}{945} - \frac{107\,\ln x}{945}\right) x^8 ,
\end{aligned} \tag{F·3}$$

$$\eta_{3,3} = \frac{1215\,x^2}{896} - \frac{1215\,x^4}{112} + \left(\frac{3645\,\pi}{448} - \frac{1215\,q}{224}\right) x^5 + \left(\frac{243729}{9856} + \frac{3645\,q^2}{896}\right) x^6 + \left(\frac{-3645\,\pi}{56} + \frac{41229\,q}{1792}\right) x^7 + \left(\frac{25037019729}{125565440} - \frac{47385\,\gamma}{1568} + \frac{3645\,\pi^2}{224} - \frac{3645\,\pi\,q}{112} - \frac{236925\,q^2}{14336} - \frac{47385\,\ln 2}{1568} - \frac{47385\,\ln 3}{1568} - \frac{47385\,\ln x}{1568}\right) x^8, \qquad (F\cdot4)$$

$$\eta_{3,2} = \frac{5\,x^4}{63} - \frac{40\,q\,x^5}{189} + \left(-\frac{193}{567} + \frac{80\,q^2}{567}\right) x^6 + \left(\frac{20\,\pi}{63} + \frac{982\,q}{1701}\right) x^7 + \left(\frac{86111}{280665} - \frac{160\,\pi\,q}{189} + \frac{80\,q^2}{189}\right) x^8, \qquad (F\cdot5)$$

$$\eta_{3,1} = \frac{x^2}{8064} - \frac{x^4}{1512} + \left(\frac{\pi}{4032} - \frac{25\,q}{18144}\right) x^5 + \left(\frac{437}{266112} + \frac{17\,q^2}{24192}\right) x^6 + \left(\frac{-\pi}{756} + \frac{2257\,q}{435456}\right) x^7 + \left(-\frac{1137077}{50854003200} - \frac{13\,\gamma}{42336} + \frac{\pi^2}{6048} - \frac{25\,\pi\,q}{9072} + \frac{12863\,q^2}{3483648} - \frac{13\,\ln 2}{42336} - \frac{13\,\ln x}{42336}\right) x^8, \qquad (F\cdot6)$$

$$\eta_{4,4} = \frac{1280\,x^4}{567} - \frac{151808\,x^6}{6237} + \left(\frac{10240\,\pi}{567} - \frac{20480\,q}{1701}\right) x^7 + \left(\frac{560069632}{6243237} + \frac{5120\,q^2}{567}\right) x^8, \qquad (F\cdot7)$$

$$\eta_{4,3} = \frac{729\,x^6}{4480} - \frac{729\,q\,x^7}{1792} + \left(-\frac{28431}{24640} + \frac{3645\,q^2}{14336}\right) x^8, \qquad (F\cdot8)$$

$$\eta_{4,2} = \frac{5\,x^4}{3969} - \frac{437\,x^6}{43659} + \left(\frac{20\,\pi}{3969} - \frac{170\,q}{11907}\right) x^7 + \left(\frac{7199152}{218513295} + \frac{200\,q^2}{27783}\right) x^8, \qquad (F\cdot9)$$

$$\eta_{4,1} = \frac{x^6}{282240} - \frac{q\,x^7}{112896} + \left(-\frac{101}{4656960} + \frac{5\,q^2}{903168}\right) x^8, \qquad (F\cdot10)$$

$$\eta_{5,5} = \frac{9765625\,x^6}{2433024} - \frac{2568359375\,x^8}{47443968}, \qquad (F\cdot11)$$

$$\eta_{5,4} = \frac{4096\,x^8}{13365}, \qquad (F\cdot12)$$

$$\eta_{5,3} = \frac{2187\,x^6}{450560} - \frac{150903\,x^8}{2928640}, \tag{F·13}$$

$$\eta_{5,2} = \frac{4\,x^8}{40095}, \tag{F·14}$$

$$\eta_{5,1} = \frac{x^6}{127733760} - \frac{179\,x^8}{2490808320}, \tag{F·15}$$

$$\eta_{6,6} = \frac{26244\,x^8}{3575}, \tag{F·16}$$

$$\eta_{6,4} = \frac{131072\,x^8}{9555975}, \tag{F·17}$$

$$\eta_{6,2} = \frac{4\,x^8}{5733585}. \tag{F·18}$$

Appendix G

—— 4PN Luminosity for a Slightly Eccentric Orbit in the Schwarzschild Case ——

We define the partial mode contribution $\eta_{\ell,m,n}$ as

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2}\left(\frac{dE}{dt}\right)_{\rm N} \sum_{\ell,m,n} \eta_{\ell,m,n}, \tag{G·1}$$

where $(dE/dt)_{\rm N}$ is the Newtonian quadrupole luminosity, Eq. (4·19), and $\eta_{\ell,-m,-n} = \eta_{\ell,m,n}$ because of the symmetry in the Teukolsky equation. Up to $O(v^8)$, $\eta_{\ell,1,-1}$ do not appear irrespective of ℓ, and other modes are as follows:
For $\ell = 2$,

$$\begin{aligned}
\eta_{2,2,0} = {}& 1 - \frac{107\,v^2}{21} + 4\,\pi\,v^3 + \frac{4784\,v^4}{1323} - \frac{428\,\pi\,v^5}{21} + \frac{99210071\,v^6}{1091475} - \frac{1712\,\gamma\,v^6}{105} \\
&+ \frac{16\,\pi^2\,v^6}{3} + \frac{19136\,\pi\,v^7}{1323} - \frac{27956920577\,v^8}{81265275} + \frac{183184\,\gamma\,v^8}{2205} - \frac{1712\,\pi^2\,v^8}{63} \\
&- \frac{3424\,v^6\,\ln(2)}{105} + \frac{366368\,v^8\,\ln(2)}{2205} - \frac{1712\,v^6\,\ln(v)}{105} + \frac{183184\,v^8\,\ln(v)}{2205} \\
&+ e^2\Bigg(-10 + \frac{932\,v^2}{21} - 46\,\pi\,v^3 - \frac{14270\,v^4}{147} + \frac{4748\,\pi\,v^5}{21} - \frac{516582901\,v^6}{363825} \\
&+ \frac{22256\,\gamma\,v^6}{105} - \frac{208\,\pi^2\,v^6}{3} - \frac{189502\,\pi\,v^7}{441} + \frac{405725734982\,v^8}{127702575} - \frac{352672\,\gamma\,v^8}{315} \\
&+ \frac{3296\,\pi^2\,v^8}{9} + \frac{44512\,v^6\,\ln(2)}{105} - \frac{705344\,v^8\,\ln(2)}{315} \\
&+ \frac{22256\,v^6\,\ln(v)}{105} - \frac{352672\,v^8\,\ln(v)}{315}\Bigg),
\end{aligned} \tag{G·2}$$

$$\begin{aligned}
\eta_{2,1,0} = {}& \frac{v^2}{36} - \frac{17\,v^4}{504} + \frac{\pi\,v^5}{18} - \frac{2215\,v^6}{254016} - \frac{17\,\pi\,v^7}{252} + \frac{15707221\,v^8}{26195400} \\
&- \frac{107\,\gamma\,v^8}{945} + \frac{\pi^2\,v^8}{27} - \frac{107\,v^8\,\ln(2)}{945} - \frac{107\,v^8\,\ln(v)}{945}
\end{aligned}$$

$$+e^2\left(\frac{-2\,v^2}{9}+\frac{93\,v^4}{112}-\frac{19\,\pi\,v^5}{36}+\frac{60667\,v^6}{84672}+\frac{169\,\pi\,v^7}{84}-\frac{1877507981\,v^8}{419126400}\right.$$
$$\left.+\frac{1177\,\gamma\,v^8}{945}-\frac{11\,\pi^2\,v^8}{27}+\frac{1177\,v^8\,\ln(2)}{945}+\frac{1177\,v^8\,\ln(v)}{945}\right), \qquad (\text{G}\cdot3)$$

$$\eta_{2,2,1}=e^2\left(\frac{729}{64}-\frac{3645\,v^2}{64}+\frac{2187\,\pi\,v^3}{32}+\frac{24057\,v^4}{256}-\frac{6561\,\pi\,v^5}{16}+\frac{9067629321\,v^6}{3449600}\right.$$
$$-\frac{234009\,\gamma\,v^6}{560}+\frac{2187\,\pi^2\,v^6}{16}+\frac{102789\,\pi\,v^7}{128}-\frac{545827954239\,v^8}{44844800}$$
$$+\frac{234009\,\gamma\,v^8}{80}-\frac{15309\,\pi^2\,v^8}{16}-\frac{234009\,v^6\,\ln(2)}{560}+\frac{234009\,v^8\,\ln(2)}{80}$$
$$-\frac{234009\,v^6\,\ln(3)}{560}+\frac{234009\,v^8\,\ln(3)}{80}$$
$$\left.-\frac{234009\,v^6\,\ln(v)}{560}+\frac{234009\,v^8\,\ln(v)}{80}\right), \qquad (\text{G}\cdot4)$$

$$\eta_{2,2,-1}=e^2\left(\frac{9}{64}+\frac{1041\,v^2}{448}+\frac{9\,\pi\,v^3}{32}+\frac{2224681\,v^4}{112896}+\frac{615\,\pi\,v^5}{112}+\frac{11918100443\,v^6}{93139200}\right.$$
$$-\frac{321\,\gamma\,v^6}{560}+\frac{3\,\pi^2\,v^6}{16}+\frac{3083119\,\pi\,v^7}{56448}+\frac{181180743580847\,v^8}{228843014400}-\frac{50611\,\gamma\,v^8}{3920}$$
$$+\frac{473\,\pi^2\,v^8}{112}-\frac{321\,v^6\,\ln(2)}{560}-\frac{50611\,v^8\,\ln(2)}{3920}$$
$$\left.-\frac{321\,v^6\,\ln(v)}{560}-\frac{50611\,v^8\,\ln(v)}{3920}\right), \qquad (\text{G}\cdot5)$$

$$\eta_{2,1,1}=e^2\left(\frac{4\,v^2}{9}-\frac{172\,v^4}{63}+\frac{16\,\pi\,v^5}{9}+\frac{10300\,v^6}{1323}-\frac{856\,\pi\,v^7}{63}+\frac{145520063\,v^8}{3274425}\right.$$
$$\left.-\frac{6848\,\gamma\,v^8}{945}+\frac{64\,\pi^2\,v^8}{27}-\frac{13696\,v^8\,\ln(2)}{945}-\frac{6848\,v^8\,\ln(v)}{945}\right), \qquad (\text{G}\cdot6)$$

$$\eta_{2,0,1}=e^2\left(\frac{1}{96}-\frac{145\,v^2}{672}+\frac{\pi\,v^3}{48}+\frac{282521\,v^4}{169344}-\frac{83\,\pi\,v^5}{168}-\frac{776764633\,v^6}{139708800}-\frac{107\,\gamma\,v^6}{2520}\right.$$
$$+\frac{\pi^2\,v^6}{72}+\frac{384203\,\pi\,v^7}{84672}+\frac{6362456713\,v^8}{3467318400}+\frac{20009\,\gamma\,v^8}{17640}-\frac{187\,\pi^2\,v^8}{504}$$
$$\left.-\frac{107\,v^6\,\ln(2)}{2520}+\frac{20009\,v^8\,\ln(2)}{17640}-\frac{107\,v^6\,\ln(v)}{2520}+\frac{20009\,v^8\,\ln(v)}{17640}\right). \qquad (\text{G}\cdot7)$$

For $\ell=3$,

$$\eta_{3,3,0}$$
$$=\frac{1215\,v^2}{896}-\frac{1215\,v^4}{112}+\frac{3645\,\pi\,v^5}{448}+\frac{243729\,v^6}{9856}-\frac{3645\,\pi\,v^7}{56}+\frac{25037019729\,v^8}{125565440}$$
$$-\frac{47385\,\gamma\,v^8}{1568}+\frac{3645\,\pi^2\,v^8}{224}-\frac{47385\,v^8\,\ln(2)}{1568}-\frac{47385\,v^8\,\ln(3)}{1568}-\frac{47385\,v^8\,\ln(v)}{1568}$$

$$+e^2\left(\frac{-10935\,v^2}{448}+\frac{37665\,v^4}{256}-\frac{142155\,\pi\,v^5}{896}-\frac{4428189\,v^6}{9856}+\frac{455625\,\pi\,v^7}{448}\right.$$
$$-\frac{81628707987\,v^8}{17937920}+\frac{142155\,\gamma\,v^8}{224}-\frac{10935\,\pi^2\,v^8}{32}$$
$$\left.+\frac{142155\,v^8\,\ln(2)}{224}+\frac{142155\,v^8\,\ln(3)}{224}+\frac{142155\,v^8\,\ln(v)}{224}\right), \qquad \text{(G·8)}$$

$$\eta_{3,2,0}=\frac{5\,v^4}{63}-\frac{193\,v^6}{567}+\frac{20\,\pi\,v^7}{63}+\frac{86111\,v^8}{280665}$$
$$+e^2\left(\frac{-65\,v^4}{63}+\frac{947\,v^6}{189}-\frac{290\,\pi\,v^7}{63}-\frac{442003\,v^8}{40095}\right), \qquad \text{(G·9)}$$

$$\eta_{3,1,0}=\frac{v^2}{8064}-\frac{v^4}{1512}+\frac{\pi\,v^5}{4032}+\frac{437\,v^6}{266112}-\frac{\pi\,v^7}{756}-\frac{1137077\,v^8}{50854003200}$$
$$-\frac{13\,\gamma\,v^8}{42336}+\frac{\pi^2\,v^8}{6048}-\frac{13\,v^8\,\ln(2)}{42336}-\frac{13\,v^8\,\ln(v)}{42336}$$
$$+e^2\left(\frac{-v^2}{4032}+\frac{65\,v^4}{16128}-\frac{\pi\,v^5}{1152}-\frac{11063\,v^6}{798336}+\frac{5\,\pi\,v^7}{448}+\frac{614545391\,v^8}{30512401920}\right.$$
$$\left.+\frac{65\,\gamma\,v^8}{42336}-\frac{5\,\pi^2\,v^8}{6048}+\frac{65\,v^8\,\ln(2)}{42336}+\frac{65\,v^8\,\ln(v)}{42336}\right), \qquad \text{(G·10)}$$

$$\eta_{3,3,1}=e^2\left(\frac{640\,v^2}{21}-\frac{46720\,v^4}{189}+\frac{5120\,\pi\,v^5}{21}+\frac{2135648\,v^6}{2673}-\frac{408320\,\pi\,v^7}{189}\right.$$
$$+\frac{485145507664\,v^8}{59594535}-\frac{532480\,\gamma\,v^8}{441}+\frac{40960\,\pi^2\,v^8}{63}$$
$$\left.-\frac{532480\,v^8\,\ln(2)}{441}-\frac{532480\,v^8\,\ln(4)}{441}-\frac{532480\,v^8\,\ln(v)}{441}\right), \qquad \text{(G·11)}$$

$$\eta_{3,3,-1}=e^2\left(\frac{15\,v^2}{14}+\frac{1055\,v^4}{126}+\frac{30\,\pi\,v^5}{7}+\frac{1530967\,v^6}{37422}+\frac{2515\,\pi\,v^7}{63}\right.$$
$$\left.+\frac{5409381833\,v^8}{19864845}-\frac{520\,\gamma\,v^8}{49}+\frac{40\,\pi^2\,v^8}{7}-\frac{1040\,v^8\,\ln(2)}{49}-\frac{520\,v^8\,\ln(v)}{49}\right), \qquad \text{(G·12)}$$

$$\eta_{3,2,1}=e^2\left(\frac{3645\,v^4}{1792}-\frac{13851\,v^6}{896}+\frac{10935\,\pi\,v^7}{896}+\frac{2808837\,v^8}{49280}\right), \qquad \text{(G·13)}$$

$$\eta_{3,2,-1}=e^2\left(\frac{5\,v^4}{1792}+\frac{1609\,v^6}{24192}+\frac{5\,\pi\,v^7}{896}+\frac{4153669\,v^8}{5132160}\right), \qquad \text{(G·14)}$$

$$\eta_{3,1,1}=e^2\left(\frac{v^2}{126}-\frac{23\,v^4}{126}+\frac{2\,\pi\,v^5}{63}+\frac{6437\,v^6}{4158}-\frac{7\,\pi\,v^7}{9}-\frac{1688449328\,v^8}{297972675}\right.$$
$$\left.-\frac{104\,\gamma\,v^8}{1323}+\frac{8\,\pi^2\,v^8}{189}-\frac{208\,v^8\,\ln(2)}{1323}-\frac{104\,v^8\,\ln(v)}{1323}\right), \qquad \text{(G·15)}$$

$$\eta_{3,0,1} = e^2\left(\frac{v^4}{2688} - \frac{55\,v^6}{6048} + \frac{\pi\,v^7}{1344} + \frac{203417\,v^8}{2395008}\right). \tag{G·16}$$

For $\ell = 4$,

$$\begin{aligned}\eta_{4,4,0} = {} & \frac{1280\,v^4}{567} - \frac{151808\,v^6}{6237} + \frac{10240\,\pi\,v^7}{567} + \frac{560069632\,v^8}{6243237} \\ & + e^2\left(\frac{-37120\,v^4}{567} + \frac{3284224\,v^6}{6237} - \frac{312320\,\pi\,v^7}{567} - \frac{60497401856\,v^8}{31216185}\right),\end{aligned} \tag{G·17}$$

$$\eta_{4,3,0} = \frac{729\,v^6}{4480} - \frac{28431\,v^8}{24640} + e^2\left(\frac{-2187\,v^6}{640} + \frac{2062341\,v^8}{98560}\right), \tag{G·18}$$

$$\begin{aligned}\eta_{4,2,0} = {} & \frac{5\,v^4}{3969} - \frac{437\,v^6}{43659} + \frac{20\,\pi\,v^7}{3969} + \frac{7199152\,v^8}{218513295} \\ & + e^2\left(\frac{-25\,v^4}{3969} + \frac{2743\,v^6}{43659} - \frac{130\,\pi\,v^7}{3969} - \frac{27410\,v^8}{77077}\right),\end{aligned} \tag{G·19}$$

$$\eta_{4,1,0} = \frac{v^6}{282240} - \frac{101\,v^8}{4656960} + e^2\left(\frac{-v^6}{56448} + \frac{4951\,v^8}{18627840}\right), \tag{G·20}$$

$$\begin{aligned}\eta_{4,4,1} = {} & e^2\left(\frac{48828125\,v^4}{580608} - \frac{11767578125\,v^6}{12773376} + \frac{244140625\,\pi\,v^7}{290304}\right. \\ & \left. + \frac{3640732421875\,v^8}{874266624}\right),\end{aligned} \tag{G·21}$$

$$\eta_{4,4,-1} = e^2\left(\frac{32805\,v^4}{7168} + \frac{321489\,v^6}{22528} + \frac{98415\,\pi\,v^7}{3584} + \frac{251783118603\,v^8}{6314147840}\right), \tag{G·22}$$

$$\eta_{4,3,1} = e^2\left(\frac{2048\,v^6}{315} - \frac{216064\,v^8}{3465}\right), \tag{G·23}$$

$$\eta_{4,3,-1} = e^2\left(\frac{2\,v^6}{35} + \frac{24\,v^8}{35}\right), \tag{G·24}$$

$$\eta_{4,2,1} = e^2\left(\frac{295245\,v^8}{495616}\right), \tag{G·25}$$

$$\eta_{4,2,-1} = e^2\left(\frac{5\,v^4}{254016} + \frac{5501\,v^6}{11176704} + \frac{5\,\pi\,v^7}{127008} + \frac{5568858419\,v^8}{895030456320}\right), \tag{G·26}$$

$$\eta_{4,1,1} = e^2\left(\frac{2\,v^6}{2205} - \frac{28\,v^8}{1485}\right), \tag{G·27}$$

$$\eta_{4,0,1} = e^2\left(\frac{v^4}{225792} - \frac{113\,v^6}{709632} + \frac{\pi\,v^7}{112896} + \frac{1469592037\,v^8}{596686970880}\right). \tag{G·28}$$

For $\ell = 5$,

$$\eta_{5,5,0}= \frac{9765625\,v^6}{2433024} - \frac{2568359375\,v^8}{47443968} + e^2\left(\frac{-419921875\,v^6}{2433024} + \frac{336376953125\,v^8}{189775872}\right), \tag{G·29}$$

$$\eta_{5,4,0}= \frac{4096\,v^8}{13365} - \frac{131072\,e^2\,v^8}{13365}, \tag{G·30}$$

$$\eta_{5,3,0}= \frac{2187\,v^6}{450560} - \frac{150903\,v^8}{2928640} + e^2\left(\frac{-2187\,v^6}{40960} + \frac{5736501\,v^8}{11714560}\right), \tag{G·31}$$

$$\eta_{5,2,0}= \frac{4\,v^8}{40095} - \frac{32\,e^2\,v^8}{40095}, \tag{G·32}$$

$$\eta_{5,1,0}= \frac{v^6}{127733760} - \frac{179\,v^8}{2490808320} + e^2\left(\frac{v^6}{25546752} - \frac{43\,v^8}{9963233280}\right), \tag{G·33}$$

$$\eta_{5,5,1}= e^2\left(\frac{19683\,v^6}{88} - \frac{17498187\,v^8}{5720}\right), \tag{G·34}$$

$$\eta_{5,5,-1}= e^2\left(\frac{512\,v^6}{33} - \frac{54272\,v^8}{6435}\right), \tag{G·35}$$

$$\eta_{5,4,1}= e^2\left(\frac{48828125\,v^8}{2737152}\right), \tag{G·36}$$

$$\eta_{5,4,-1}= e^2\left(\frac{19683\,v^8}{56320}\right), \tag{G·37}$$

$$\eta_{5,3,1}= e^2\left(\frac{512\,v^6}{13365} - \frac{16384\,v^8}{173745}\right), \tag{G·38}$$

$$\eta_{5,3,-1}= e^2\left(\frac{11\,v^6}{9720} + \frac{13\,v^8}{1080}\right), \tag{G·39}$$

$$\eta_{5,2,-1}= e^2\left(\frac{v^8}{2566080}\right), \tag{G·40}$$

$$\eta_{5,1,1}= e^2\left(\frac{v^6}{13860} - \frac{1091\,v^8}{540540}\right), \tag{G·41}$$

$$\eta_{5,0,1}= e^2\left(\frac{v^8}{9580032}\right). \tag{G·42}$$

Note that up to $O(v^8)$, $\eta_{5,2,1}$ does not appear.
For $\ell = 6$,

$$\eta_{6,6,0}= \frac{26244\,v^8}{3575} - \frac{314928\,e^2\,v^8}{715}, \tag{G·43}$$

$$\eta_{6,4,0}= \frac{131072\, v^8}{9555975} - \frac{524288\, e^2\, v^8}{1911195}, \tag{G·44}$$

$$\eta_{6,2,0}= \frac{4\, v^8}{5733585} + \frac{16\, e^2\, v^8}{5733585}, \tag{G·45}$$

$$\eta_{6,6,1}= e^2 \frac{678223072849\, v^8}{1186099200}, \tag{G·46}$$

$$\eta_{6,6,-1}= e^2 \frac{244140625\, v^8}{5271552}, \tag{G·47}$$

$$\eta_{6,4,1}= e^2 \frac{244140625\, v^8}{782825472}, \tag{G·48}$$

$$\eta_{6,4,-1}= e^2 \frac{964467\, v^8}{80537600}, \tag{G·49}$$

$$\eta_{6,2,1}= e^2 \frac{6561\, v^8}{32215040}, \tag{G·50}$$

$$\eta_{6,2,-1}= e^2 \frac{5\, v^8}{4696952832}, \tag{G·51}$$

$$\eta_{6,0,1}= e^2 \frac{v^8}{2739889152}. \tag{G·52}$$

Note that up to $O(v^8)$, $\eta_{6,5,k}$, $\eta_{6,3,k}$ and $\eta_{6,1,k}$ do not appear.

Now, we define $\eta_\ell^{(0)}$, $\eta_\ell^{(2)}$, and $\eta_{z\ell}^{(2)}$ as

$$\left\langle \frac{dE}{dt} \right\rangle = \left(\frac{dE}{dt}\right)_{\rm N} \sum_\ell \left(\eta_\ell^{(0)} + \eta_\ell^{(2)} e^2\right), \tag{G·53}$$

$$\left\langle \frac{dJ_z}{dt} \right\rangle = \frac{1}{\Omega} \left(\frac{dE}{dt}\right)_{\rm N} \sum_\ell \left(\eta_\ell^{(0)} + \eta_{z\ell}^{(2)} e^2\right). \tag{G·54}$$

Then,

$$\begin{aligned}\eta_2^{(0)} &= 1 - \frac{1277\, v^2}{252} + 4\,\pi\, v^3 + \frac{37915\, v^4}{10584} - \frac{2561\,\pi\, v^5}{126} + \frac{76187\,\pi\, v^7}{5292} \\ &+ v^6 \left(\frac{2116278473}{23284800} - \frac{1712\,\gamma}{105} + \frac{16\,\pi^2}{3} - \frac{3424\,\ln(2)}{105} - \frac{1712\,\ln(v)}{105}\right) \\ &+ v^8 \left(-\frac{2455920939443}{7151344200} + \frac{548803\,\gamma}{6615} - \frac{5129\,\pi^2}{189} \right. \\ &\left. + \frac{219671\,\ln(2)}{1323} + \frac{548803\,\ln(v)}{6615}\right),\end{aligned} \tag{G·55}$$

$$\begin{aligned}\eta_2^{(2)} &= \frac{37}{24} - \frac{2581\, v^2}{252} + \frac{1087\,\pi\, v^3}{48} + \frac{346561\, v^4}{21168} - \frac{29857\,\pi\, v^5}{168} + \frac{35639309\,\pi\, v^7}{84672} \\ &+ v^6 \left(\frac{93579660049}{69854400} - \frac{65056\,\gamma}{315} + \frac{608\,\pi^2}{9} + \frac{1712\,\ln(2)}{315} \right. \\ &\left. - \frac{234009\,\ln(3)}{560} - \frac{65056\,\ln(v)}{315}\right)\end{aligned}$$

$$+v^8\left(-\frac{3395552510663}{416078208}+\frac{4730363\,\gamma}{2646}-\frac{221045\,\pi^2}{378}\right.$$
$$\left.+\frac{2914573\,\ln(2)}{4410}+\frac{234009\,\ln(3)}{80}+\frac{4730363\,\ln(v)}{2646}\right), \qquad (G\cdot 56)$$

$$\eta_3^{(0)}=\frac{1367\,v^2}{1008}-\frac{32567\,v^4}{3024}+\frac{16403\,\pi\,v^5}{2016}+\frac{152122\,v^6}{6237}-\frac{13991\,\pi\,v^7}{216}$$
$$+v^8\left(\frac{5712521850527}{28605376800}-\frac{79963\,\gamma}{2646}+\frac{6151\,\pi^2}{378}-\frac{79963\,\ln(2)}{2646}\right.$$
$$\left.-\frac{47385\,\ln(3)}{1568}-\frac{79963\,\ln(v)}{2646}\right), \qquad (G\cdot 57)$$

$$\eta_3^{(2)}=\frac{1801\,v^2}{252}-\frac{78509\,v^4}{864}+\frac{40083\,\pi\,v^5}{448}+\frac{8163047\,v^6}{21384}-\frac{1894997\,\pi\,v^7}{1728}$$
$$+v^8\left(\frac{446664927141403}{114421507200}-\frac{771979\,\gamma}{1323}+\frac{59383\,\pi^2}{189}-\frac{87347\,\ln(2)}{147}\right.$$
$$\left.+\frac{142155\,\ln(3)}{224}-\frac{532480\,\ln(4)}{441}-\frac{771979\,\ln(v)}{1323}\right), \qquad (G\cdot 58)$$

$$\eta_4^{(0)}=\frac{8965\,v^4}{3969}-\frac{84479081\,v^6}{3492720}+\frac{23900\,\pi\,v^7}{1323}+\frac{51619996697\,v^8}{582702120}, \qquad (G\cdot 59)$$

$$\eta_4^{(2)}=\frac{2946739\,v^4}{127008}-\frac{58555205\,v^6}{155232}+\frac{107560723\,\pi\,v^7}{338688}+\frac{5187619686371\,v^8}{2330808480}, \qquad (G\cdot 60)$$

$$\eta_5^{(0)}=\frac{1002569\,v^6}{249480}-\frac{3145396841\,v^8}{58378320}, \qquad (G\cdot 61)$$

$$\eta_5^{(2)}=\frac{2491525\,v^6}{37422}-\frac{150181214159\,v^8}{116756640}, \qquad (G\cdot 62)$$

$$\eta_6^{(0)}=\frac{210843872\,v^8}{28667925}, \qquad (G\cdot 63)$$

$$\eta_6^{(2)}=\frac{142651028551\,v^8}{802701900} \qquad (G\cdot 64)$$

and

$$\eta_{z2}^{(2)}=-\frac{5}{8}+\frac{137\,v^2}{24}+\frac{49\,\pi\,v^3}{8}-\frac{235675\,v^4}{14112}-\frac{20437\,\pi\,v^5}{504}+\frac{883609\,\pi\,v^7}{14112}$$
$$+v^6\left(\frac{303218627}{470400}-\frac{19367\,\gamma}{210}+\frac{181\,\pi^2}{6}+\frac{20009\,\ln(2)}{210}\right.$$
$$\left.-\frac{78003\,\ln(3)}{280}-\frac{19367\,\ln(v)}{210}\right)$$
$$+v^8\left(-\frac{1553872210987}{488980800}+\frac{3197053\,\gamma}{4410}-\frac{29879\,\pi^2}{126}\right.$$

$$-\frac{2649641\ln(2)}{13230}+\frac{234009\ln(3)}{140}+\frac{3197053\ln(v)}{4410}\Bigg), \tag{G·65}$$

$$\eta_{z3}^{(2)}=\frac{67\,v^2}{32}-\frac{66497\,v^4}{2016}+\frac{43193\,\pi\,v^5}{1008}+\frac{2711543\,v^6}{18144}-\frac{2203487\,\pi\,v^7}{4032}$$
$$+v^8\left(\frac{321229428757}{133358400}-\frac{586079\,\gamma}{1764}+\frac{45083\,\pi^2}{252}-\frac{1842685\ln(2)}{5292}\right.$$
$$\left.+\frac{1848015\ln(3)}{3136}-\frac{133120\ln(4)}{147}-\frac{586079\ln(v)}{1764}\right), \tag{G·66}$$

$$\eta_{z4}^{(2)}=\frac{478195\,v^4}{42336}-\frac{64132457\,v^6}{317520}+\frac{5239355\,\pi\,v^7}{28224}+\frac{670696042069\,v^8}{537878880}, \tag{G·67}$$

$$\eta_{z5}^{(2)}=\frac{1778041\,v^6}{45360}-\frac{300193429\,v^8}{374220}, \tag{G·68}$$

$$\eta_{z6}^{(2)}=\frac{113949013\,e^2\,v^8}{980100}. \tag{G·69}$$

Appendix H
—— *Asymptotic Amplitudes and $R^{\rm up}$* ——

In this appendix, we show the asymptotic amplitudes $B^{\rm inc}$, $B^{\rm trans}$ and $C^{\rm trans}$ and the post-Newtonian expansion of $R^{\rm up}$ which are used in §12 to evaluate the black hole absorption rate to $O(v^{13})$ relative to the quadrupole energy flux at infinity in section.

(a) $\ell=2$

$$B^{\rm inc}$$
$$=\frac{1}{\omega}\frac{1}{\kappa^4\epsilon^4}e^{\frac{1}{2}\pi i(\nu+3)}e^{i\epsilon\kappa}e^{-i\epsilon\ln\epsilon}$$
$$\times\left\{\frac{15}{4}+\left(-\frac{15}{2}i\gamma-\frac{25}{12}mq-\frac{15}{8}\pi-\frac{15}{4}i\psi^{(0)}(3+\frac{imq}{\kappa})-\frac{15}{4}i\ln(2)+\frac{125}{16}i\right)\epsilon\right.$$
$$+\left(\frac{1089}{56}\gamma+\frac{725}{2352}\kappa^2+\frac{1089}{112}\ln(2)-\frac{15}{2}\gamma^2+\frac{12625}{21168}m^2q^2+\frac{35}{32}\pi^2\right.$$
$$-\frac{535}{144}imq+\frac{15}{4}i\gamma\pi-\frac{125}{32}i\pi+\frac{25}{6}i\gamma mq+\frac{25}{24}m\pi q-\frac{15}{2}\gamma\ln(2)$$
$$-\frac{15}{2}\gamma\psi^{(0)}(3+\frac{imq}{\kappa})-\frac{15}{4}\ln(2)\psi^{(0)}(3+\frac{imq}{\kappa})+\frac{107}{56}\ln(\epsilon)+\frac{107}{56}\ln(\kappa)$$
$$-\frac{20573}{960}-\frac{15}{8}(\ln(2))^2+\frac{1089}{112}\psi^{(0)}(3+\frac{imq}{\kappa})-\frac{15}{8}\left(\psi^{(0)}(3+\frac{imq}{\kappa})\right)^2$$
$$-\frac{15}{8}\psi^{(1)}(3+\frac{imq}{\kappa})+\frac{25}{12}imq\ln(2)+\frac{25}{12}imq\psi^{(0)}(3+\frac{imq}{\kappa})$$
$$\left.\left.+\frac{15}{8}i\pi\ln(2)+\frac{15}{8}i\pi\psi^{(0)}(3+\frac{imq}{\kappa})-\frac{15}{4}\psi^{(1)}(3+\frac{imq}{\kappa})\kappa^{-1}\right)\epsilon^2\right\}, \tag{H·1}$$

$$B^{\rm trans} = \left(\frac{\omega}{\epsilon\kappa}\right)^4 e^{i\epsilon_+ \ln\kappa}\left\{1 + \left(\frac{5}{6}\,i\kappa - \frac{5}{18}\,mq\right)\epsilon \right.$$
$$\left. + \left(\frac{325}{7938}\,m^2q^2 + \frac{5}{18} - \frac{15}{49}\,\kappa^2 - \frac{85}{378}\,i\kappa\, mq\right)\epsilon^2\right\}, \tag{H·2}$$

$$C^{\rm trans} = \omega^3 2^{i\epsilon} e^{-\frac{\pi}{2}i(\nu-1)} e^{i\epsilon\ln\epsilon}\left\{2 + \left(-\frac{1}{3}\,i\kappa - \pi + \frac{1}{9}\,mq\right)\epsilon\right.$$
$$+\left(\frac{1}{4}\,\pi^2 - \frac{1}{189}\,i\kappa\, mq + \frac{2}{49}\,q^2 - \frac{1}{3}\,imq - \frac{11}{3969}\,m^2q^2\right.$$
$$\left.\left. - \frac{1}{18}\,m\pi\, q + \frac{1}{6}\,i\kappa\,\pi - \frac{67}{441}\right)\epsilon^2\right\}, \tag{H·3}$$

$$R^{\rm up} = -3\,\frac{i}{z} - 3 + \frac{3}{2}\,iz + \frac{1}{2}\,z^2 - \frac{1}{8}\,iz^3 + \frac{31}{40}\,z^4 + \frac{43}{80}\,iz^5 - \frac{117}{560}\,z^6 - \frac{769}{13440}\,iz^7$$
$$+\epsilon\left(\left(-\frac{3}{2}\,i + mq\right)\frac{1}{z^2} + \left(-3\,\gamma + \kappa + 3\,i\pi - \frac{1}{6}\,imq\right)\frac{1}{z} - \frac{9}{4}\,i + 3\,i\gamma\right.$$
$$-i\kappa + 3\,\pi + \frac{1}{3}\,mq + \left(-\frac{5}{2} + \frac{3}{2}\,\gamma - \frac{1}{2}\,\kappa - \frac{3}{2}\,i\pi - \frac{1}{4}\,imq\right)z$$
$$+\left(\frac{85}{48}\,i - \frac{1}{2}\,i\gamma + \frac{1}{6}\,i\kappa - \frac{1}{2}\,\pi - \frac{7}{72}\,mq\right)z^2$$
$$+\left(-\frac{13}{60} - \frac{1}{8}\,\gamma + \frac{1}{24}\,\kappa + \frac{1}{8}\,i\pi - \frac{269}{720}\,imq\right)z^3$$
$$\left.+\left(-\frac{1559}{2400}\,i + \frac{1}{40}\,i\gamma + \frac{31}{120}\,i\kappa - \frac{3}{8}\,\pi + \frac{49}{180}\,mq + \frac{4}{5}\,i\ln(2) + \frac{4}{5}\,i\ln(z)\right)z^4\right)$$
$$+\epsilon^2\left(\left(-\frac{3}{4}\,i - \frac{3}{28}\,i\kappa^2 + mq + \frac{11}{56}\,im^2q^2\right)z^{-3} + \left(-1 - \frac{3}{2}\,\gamma + \frac{1}{2}\,\kappa + \frac{1}{14}\,\kappa^2\right.\right.$$
$$\left.+\frac{3}{2}\,i\pi - \frac{7}{12}\,imq - i\gamma\, mq + \frac{1}{3}\,i\kappa\, mq - m\pi\, q - \frac{31}{504}\,m^2q^2\right)z^{-2}$$
$$+\left(\frac{183}{28}\,i - \frac{107}{70}\,i\gamma + \frac{3}{2}\,i\gamma^2 - i\gamma\,\kappa - \frac{4}{49}\,i\kappa^2 + 3\,\gamma\,\pi - \kappa\,\pi - \frac{7}{4}\,i\pi^2 - \frac{1}{4}\,mq\right.$$
$$-\frac{1}{6}\,\gamma\, mq + \frac{19}{252}\,\kappa\, mq + \frac{1}{6}\,im\pi\, q + \frac{781}{21168}\,im^2q^2 - \frac{107}{70}\,i\ln(2)$$
$$\left.-\frac{107}{70}\,i\ln(z)\right)z^{-1} + \frac{1791}{280} - \frac{529}{140}\,\gamma + \frac{3}{2}\,\gamma^2 + \frac{3}{4}\,\kappa - \gamma\,\kappa - \frac{11}{392}\,\kappa^2 + \frac{9}{4}\,i\pi$$
$$-3\,i\gamma\,\pi + i\kappa\,\pi - \frac{7}{4}\,\pi^2 + \frac{13}{24}\,imq - \frac{1}{3}\,i\gamma\, mq + \frac{23}{252}\,i\kappa\, mq - \frac{1}{3}\,m\pi\, q$$
$$-\frac{563}{21168}\,m^2q^2 - \frac{107}{70}\,\ln(2) - \frac{107}{70}\,\ln(z) + \left(-\frac{13751}{3360}\,i\right.$$

$$+\frac{457}{140}i\gamma - \frac{3}{4}i\gamma^2 - \frac{5}{6}i\kappa + \frac{1}{2}i\gamma\kappa + \frac{17}{4704}i\kappa^2 + \frac{5}{2}\pi$$
$$-\frac{3}{2}\gamma\pi + \frac{1}{2}\kappa\pi + \frac{7}{8}i\pi^2 + \frac{2}{3}mq - \frac{1}{4}\gamma mq$$
$$+\frac{37}{504}\kappa mq + \frac{1}{4}im\pi q + \frac{751}{28224}im^2q^2 + \frac{107}{140}i\ln(2) + \frac{107}{140}i\ln(z)\Big)z\Bigg)$$
$$+\epsilon^3\Bigg(\Big(-\frac{3}{8}i - \frac{9}{56}i\kappa^2 + \frac{3}{4}mq + \frac{15}{112}\kappa^2mq + \frac{33}{112}im^2q^2$$
$$-\frac{3}{112}m^3q^3\Big)z^{-4} + \Big(-1 - \frac{3}{4}\gamma + \frac{1}{4}\kappa - \frac{1}{7}\kappa^2 - \frac{3}{28}\gamma\kappa^2 + \frac{1}{28}\kappa^3$$
$$+\frac{3}{4}i\pi + \frac{3}{28}i\kappa^2\pi - \frac{157}{168}imq - i\gamma mq + \frac{1}{3}i\kappa mq - \frac{5}{84}i\kappa^2mq - m\pi q$$
$$+\frac{41}{504}m^2q^2 + \frac{11}{56}\gamma m^2q^2 - \frac{11}{168}\kappa m^2q^2 - \frac{11}{56}im^2\pi q^2$$
$$-\frac{23}{1008}im^3q^3\Big)z^{-3} + \Big(-\frac{1}{3}i\kappa m\pi q + \frac{31}{504}i\gamma m^2q^2$$
$$-\frac{1}{72}i\kappa m^2q^2 + i\gamma m\pi q - \frac{1}{2}\gamma^2mq + \frac{7}{12}m\pi^2q + \frac{31}{504}m^2\pi q^2 + \frac{107}{210}mq\ln(2)$$
$$+\frac{107}{210}mq\ln(z) + \frac{2335}{42336}im^2q^2 - \frac{1}{2}i\gamma\kappa + \frac{1}{3}\gamma\kappa mq + \frac{7}{12}im\pi q - \frac{25}{168}i\kappa q^2$$
$$-\frac{1}{14}i\gamma\kappa^2 - \frac{239}{14112}\kappa^2mq + \pi + \frac{1269}{280}i + \frac{3}{2}\gamma\pi - \frac{1}{2}\kappa\pi - \frac{1}{14}\kappa^2\pi + \frac{25}{504}mq^3$$
$$+\frac{69}{196}i\kappa^2 + \frac{33}{140}i\gamma - \frac{31}{168}i\kappa - \frac{107}{140}i\ln(2) - \frac{107}{140}i\ln(z) + \frac{3}{4}i\gamma^2$$
$$-\frac{7}{8}i\pi^2 - \frac{1}{8}i\kappa^3 - \frac{31}{420}\gamma mq + \frac{103}{504}\kappa mq - \frac{421}{210}mq - \frac{239}{127008}m^3q^3\Big)z^{-2}\Bigg)$$
$$+\epsilon^4\Bigg(\Big(-\frac{3}{16}i - \frac{9}{56}i\kappa^2 - \frac{1}{112}i\kappa^4 + \frac{1}{2}mq + \frac{15}{56}\kappa^2mq + \frac{33}{112}im^2q^2$$
$$+\frac{11}{252}i\kappa^2m^2q^2 - \frac{3}{56}m^3q^3 - \frac{23}{8064}im^4q^4\Big)z^{-5}\Bigg). \qquad \text{(H·4)}$$

(b) $\ell = 3$

$$B^{\rm inc} = \frac{1}{\omega}\frac{1}{\kappa^5\epsilon^5}e^{\frac{1}{2}\pi i(\nu+3)}e^{i\epsilon\kappa}e^{-i\epsilon\ln\epsilon}\Bigg\{\frac{945}{2}\frac{\kappa}{3\kappa + imq}$$
$$-\frac{63}{8}\frac{\kappa}{(3\kappa + imq)^2}\Big(90\kappa\pi + 30imq\pi + 180i\kappa\ln(2) - 60mq\ln(2)$$
$$+45\kappa mq + 15im^2q^2 + 360i\kappa\gamma - 120\gamma mq - 591i\kappa + 197mq$$

$$+180\, i\psi^{(0)}(\frac{3\,\kappa+imq}{\kappa})\kappa - 60\,\psi^{(0)}(\frac{3\,\kappa+imq}{\kappa})mq - 60\,i\Bigg)\epsilon\Bigg\}, \quad \text{(H·5)}$$

$$B^{\text{trans}} = \left(\frac{\omega}{\epsilon\kappa}\right)^4 e^{i\epsilon_+\ln\kappa}\left(1+\left(-\frac{11}{72}\,mq+\frac{2}{3}\,i\kappa\right)\epsilon\right), \quad \text{(H·6)}$$

$$C^{\text{trans}} = \omega^3 2^{i\epsilon} e^{-\frac{\pi}{2}i(\nu-1)} e^{i\epsilon\ln\epsilon}\left(2+\left(-\pi+\frac{1}{36}\,mq-\frac{2}{3}\,i\kappa\right)\epsilon\right), \quad \text{(H·7)}$$

$$\begin{aligned}R^{\text{up}} &= -45\,\frac{i}{z^2} - 30\,z^{-1} + \frac{15}{2}\,i + \frac{5}{8}\,iz^2\\
&+\epsilon\left(\left(\frac{45}{4}\,mq - 45\,i\kappa + 90\,i\left(-\frac{1}{2}+\frac{1}{2}\,\kappa\right)\right)z^{-3}\right.\\
&+\left(45\,i\pi + 30 + \frac{15}{2}\,\kappa - 45\,\gamma - \frac{15}{8}\,imq\right)z^{-2}\\
&\left.+\left(30\,\pi - 45\,i + \frac{35}{24}\,mq + 30\,i\gamma - 5\,i\kappa\right)z^{-1}\right)\\
&+\epsilon^2\left(\frac{135}{8}\,\kappa\,mq - \frac{75}{2}\,i\kappa^2\right.\\
&+135\,i\kappa\left(-\frac{1}{2}+\frac{1}{2}\,\kappa\right) - \frac{135}{4}\left(-\frac{1}{2}+\frac{1}{2}\,\kappa\right)mq + \frac{15}{8}\,im^2q^2\\
&\left.-45\,i\left(-\left(-\frac{1}{2}+\frac{1}{2}\,\kappa\right)^2 + 2\left(-1+\kappa\right)\left(-\frac{1}{2}+\frac{1}{2}\,\kappa\right)\right)\right)z^{-4}.\end{aligned} \quad \text{(H·8)}$$

(c) $\ell = 4$

$$B^{\text{inc}} = \frac{1}{\omega}\frac{1}{\kappa^6\epsilon^6}e^{\frac{1}{2}\pi i(\nu+3)}e^{i\epsilon\kappa}e^{-i\epsilon\ln\epsilon}\left\{79380\,\frac{\kappa^2}{(3\,\kappa+imq)\,(4\,\kappa+imq)}\right\}, \quad \text{(H·9)}$$

$$B^{\text{trans}} = \left(\frac{\omega}{\epsilon\kappa}\right)^4 e^{i\epsilon_+\ln\kappa}\left(1+O(\epsilon)\right), \quad \text{(H·10)}$$

$$C^{\text{trans}} = \omega^3 2^{i\epsilon} e^{-\frac{\pi}{2}i(\nu-1)} e^{i\epsilon\ln\epsilon}\left(2+O(\epsilon)\right), \quad \text{(H·11)}$$

$$R^{\text{up}} = -\frac{630i}{z^3}\,. \quad \text{(H·12)}$$

Appendix I
—— *Energy Absorption by a Kerr Black Hole* ——

Here we give the (ℓ, m)-components of the energy absorption rate to $O(v^8)$ beyond the lowest order for the Kerr black hole, that is $O(v^{13})$ relative to the quadrupole luminosity at infinity.

$$
\begin{aligned}
\eta^H_{2,2} =& -\frac{1}{4}\,q-\frac{3}{4}\,q^3+v^2\left(-\frac{3}{4}\,q-\frac{9}{4}\,q^3\right)\\
&+\left(\frac{27}{4}\,q^2+2\,qB_2+\frac{15}{4}\,q^4+6\,q^3B_2+\frac{13}{2}\,\kappa\,q^2+3\,q^4\kappa+\frac{1}{2}\,\kappa+\frac{1}{2}\right)v^3\\
&+\left(-\frac{199}{42}\,q-\frac{593}{42}\,q^3+\frac{2}{7}\,q^5\right)v^4\\
&+\left(\frac{721}{36}\,q^2+6\,qB_2+\frac{127}{12}\,q^4+18\,q^3B_2+\frac{39}{2}\,\kappa\,q^2+9\,q^4\kappa+\frac{3}{2}\,\kappa+\frac{3}{2}\right)v^5\\
&+\left(-\frac{607076}{11025}\,q-4\,B_2+\frac{428}{105}\,\gamma\,q+\frac{2}{3}\,\pi^2q+\frac{428}{105}\,q\ln 2\right.\\
&-4\,qC_2-12\,q^3C_2-36\,q^4B_2-56\,q^2B_2+\frac{428}{35}\,q^3\gamma+\frac{428}{35}\,q^3\ln 2\\
&+2\,q^3\pi^2+\frac{428}{105}\,q\ln\kappa+\frac{428}{105}\,qA_2+\frac{428}{35}\,q^3\ln\kappa+6\,\frac{q^7}{\kappa}+\frac{428}{35}\,q^3A_2\\
&-8\,qB_2{}^2-24\,q^3B_2{}^2+\frac{856}{105}\,q\ln v+\frac{856}{35}\,q^3\ln v-4\,\frac{B_2}{\kappa}-32\,\frac{q^3}{\kappa}\\
&-31\,\frac{q}{\kappa}+57\,\frac{q^5}{\kappa}-48\,\frac{q^2B_2}{\kappa}+28\,\frac{q^4B_2}{\kappa}-24\,\frac{q^3C_2}{\kappa}-8\,\frac{qC_2}{\kappa}\\
&\left.+24\,\frac{q^6B_2}{\kappa}-\frac{11883052}{99225}\,q^3+\frac{548}{27}\,q^5\right)v^6\\
&+\left(\frac{16747}{126}\,q^2+\frac{15893}{189}\,q^4-\frac{155}{126}\,q^6+123\,\kappa\,q^2+\frac{1142}{21}\,q^4\kappa-\frac{8}{7}\,\kappa\,q^6\right.\\
&\left.+\frac{199}{21}\,\kappa+\frac{199}{21}+\frac{796}{21}\,qB_2+\frac{2372}{21}\,q^3B_2-\frac{16}{7}\,q^5B_2\right)v^7\\
&+\left(-\frac{761349}{3920}\,q-12\,B_2+\frac{3076}{105}\,\gamma\,q+2\,\pi^2q+\frac{4868}{105}\,q\ln 2-12\,qC_2\right.\\
&-36\,q^3C_2-\frac{308}{3}\,q^4B_2-\frac{1496}{9}\,q^2B_2+\frac{3076}{35}\,q^3\gamma+\frac{4868}{35}\,q^3\ln 2+6\,q^3\pi^2\\
&+\frac{428}{35}\,q\ln\kappa+\frac{428}{35}\,qA_2+\frac{1284}{35}\,q^3\ln\kappa+\frac{46}{3}\,\frac{q^7}{\kappa}+\frac{1284}{35}\,q^3A_2-24\,qB_2{}^2\\
&-72\,q^3B_2{}^2+\frac{872}{21}\,q\ln v+\frac{872}{7}\,q^3\ln(v)-12\,\frac{B_2}{\kappa}-\frac{272}{3}\,\frac{q^3}{\kappa}-\frac{833}{9}\,\frac{q}{\kappa}\\
&+\frac{1511}{9}\,\frac{q^5}{\kappa}-144\,\frac{q^2B_2}{\kappa}+84\,\frac{q^4B_2}{\kappa}-72\,\frac{q^3C_2}{\kappa}-24\,\frac{qC_2}{\kappa}\\
&\left.+72\,\frac{q^6B_2}{\kappa}-\frac{140529967}{317520}\,q^3+\frac{46465}{756}\,q^5-\frac{191}{588}\,q^7\right)v^8, \qquad \text{(I·1)}
\end{aligned}
$$

$$
\begin{aligned}
\eta^H_{2,1} =& \, v^2\left(\frac{3}{16}\,q^3-\frac{1}{4}\,q\right)+\left(-\frac{1}{4}\,q^4+\frac{1}{3}\,q^2\right)v^3\\
&+\left(\frac{1}{12}\,q^5+\frac{8}{9}\,q^3-\frac{4}{3}\,q\right)v^4
\end{aligned}
$$

$$+\left(-\frac{137}{48}q^4-\frac{3}{4}q^3B_1+\frac{265}{72}q^2+qB_1-\frac{3}{4}q^4\kappa+\frac{1}{2}\kappa+\frac{7}{8}\kappa q^2+\frac{1}{2}\right)v^5$$
$$+\left(-\frac{382}{63}q+\frac{865}{756}q^3+\frac{2321}{1008}q^5-\frac{2}{3}\kappa q-\frac{7}{6}\kappa q^3+q^5\kappa+q^4B_1-\frac{4}{3}q^2B_1\right)v^6$$
$$+\left(\frac{8}{3}+\frac{4643}{252}q^2-\frac{32}{9}q^3B_1-\frac{1405}{3024}q^6-\frac{65}{18}q^4\kappa-\frac{123269}{9072}q^4\right.$$
$$\left.+\frac{44}{9}\kappa q^2-\frac{1}{3}q^5B_1-\frac{1}{3}\kappa q^6+\frac{16}{3}qB_1+\frac{8}{3}\kappa\right)v^7$$
$$+\left(-\frac{6292409}{176400}q-2B_1+\frac{107}{105}\gamma q+\frac{1}{6}\pi^2q+\frac{107}{105}q\ln 2-\frac{107}{140}q^3\gamma\right.$$
$$-\frac{107}{140}q^3\ln 2-\frac{1}{8}q^3\pi^2+\frac{107}{105}q\ln\kappa-\frac{107}{140}q^3\ln\kappa-\frac{55}{6}\frac{q^7}{\kappa}+\frac{214}{105}q\ln v$$
$$-\frac{107}{70}q^3\ln v+\frac{65}{12}\frac{q^3}{\kappa}-\frac{245}{18}\frac{q}{\kappa}+\frac{625}{36}\frac{q^5}{\kappa}+\frac{73}{6}q^4B_1-qC_1$$
$$-\frac{283}{18}q^2B_1+\frac{3}{2}q^3{B_1}^2+\frac{107}{105}qA_1-\frac{107}{140}q^3A_1-3\frac{q^6B_1}{\kappa}+\frac{13}{2}\frac{q^4B_1}{\kappa}$$
$$-\frac{3}{2}\frac{q^2B_1}{\kappa}-2\frac{qC_1}{\kappa}+\frac{3}{2}\frac{q^3C_1}{\kappa}-2q{B_1}^2+\frac{3}{4}q^3C_1-2\frac{B_1}{\kappa}$$
$$\left.+\frac{381643}{6350400}q^3+\frac{6439}{378}q^5-\frac{11}{168}q^7\right)v^8, \tag{I·2}$$

$$\eta^H_{3,3}=\left(-\frac{75}{112}q^5-\frac{555}{896}q^3-\frac{15}{224}q\right)v^4+\left(-\frac{45}{112}q-\frac{1665}{448}q^3-\frac{225}{56}q^5\right)v^6$$
$$+\left(\frac{225}{28}q^5B_3+\frac{375}{112}q^6+\frac{15}{112}+\frac{1905}{448}\kappa q^2+\frac{2055}{224}q^4\kappa+\frac{15}{112}\kappa\right.$$
$$\left.+\frac{1665}{224}q^3B_3+\frac{45}{56}qB_3+\frac{10995}{896}q^4+\frac{2055}{448}q^2\right)v^7$$
$$+\left(-\frac{17697}{896}q^3-\frac{18315}{896}q^5+\frac{125}{112}q^7-\frac{481}{224}q\right)v^8, \tag{I·3}$$

$$\eta^H_{3,2}=\left(\frac{25}{189}q^5-\frac{5}{63}q-\frac{110}{567}q^3\right)v^6$$
$$+\left(\frac{5}{42}q^2+\frac{55}{189}q^4-\frac{25}{126}q^6\right)v^7$$
$$+\left(\frac{25}{336}q^7-\frac{40}{63}q-\frac{14485}{9072}q^3+\frac{205}{216}q^5\right)v^8, \tag{I·4}$$

$$\eta^H_{3,1}=\left(-\frac{1}{224}q+\frac{59}{8064}q^3-\frac{1}{336}q^5\right)v^4$$
$$+\left(\frac{767}{12096}q^3-\frac{13}{504}q^5-\frac{13}{336}q\right)v^6$$
$$+\left(\frac{1}{84}q^5B_1+\frac{1}{56}qB_1+\frac{31}{4032}\kappa q^2-\frac{55}{2016}q^4\kappa+\frac{1}{84}\kappa q^6+\frac{109}{3024}q^6\right.$$
$$\left.-\frac{6395}{72576}q^4-\frac{59}{2016}q^3B_1+\frac{1}{112}+\frac{185}{4032}q^2+\frac{1}{112}\kappa\right)v^7$$

$$+\left(-\frac{1}{336}q^7-\frac{431}{2016}q+\frac{25105}{72576}q^3-\frac{3271}{24192}q^5\right)v^8, \tag{I·5}$$

$$\eta^H_{4,4}=-\frac{5}{2268}v^8q\left(9+7q^2\right)\left(3q^2+1\right)\left(15q^2+1\right), \tag{I·6}$$

$$\eta^H_{4,2}=-\frac{5}{63504}v^8q\left(5q^2-9\right)\left(3q^2-4\right)\left(3q^2+1\right). \tag{I·7}$$

Appendix J
—— $(dE/dt)_{\rm H}$ *in Terms of the Orbital Frequency* ——

In this appendix, we describe the absorption rate $\langle dE/dt\rangle_{\rm H}$ by a Kerr black hole in terms of $x\equiv(M\Omega_\varphi)^{1/3}$. Using the relation,

$$v=x\Big(1+\frac{1}{3}qx^3+\frac{2}{9}q^2x^6+O(x^9)\Big), \tag{J·1}$$

we have

$$\left(\frac{dE}{dt}\right)_{\rm H}=\frac{32}{5}\left(\frac{\mu}{M}\right)^2x^{10}x^5\Bigg[-\frac{1}{4}q-\frac{3}{4}q^3+\left(-q-\frac{33}{16}q^3\right)x^2$$
$$+\left(2qB_2+\frac{1}{2}+\frac{13}{2}\kappa q^2+\frac{35}{6}q^2-\frac{1}{4}q^4+\frac{1}{2}\kappa+3q^4\kappa+6q^3B_2\right)x^3$$
$$+\left(-\frac{43}{7}q-\frac{17}{56}q^5-\frac{4651}{336}q^3\right)x^4$$
$$+\Bigg(\frac{433}{24}q^2-\frac{95}{24}q^4+2-\frac{3}{4}q^3B_1+2\kappa+\frac{33}{4}q^4\kappa+6qB_2$$
$$+18q^3B_2+\frac{163}{8}\kappa q^2+qB_1\Bigg)x^5$$
$$+\Bigg(-\frac{2586329}{44100}q-4B_2-\frac{1640747}{19600}q^3+19q^5\kappa+\frac{428}{105}\gamma q+\frac{2}{3}\pi^2q$$
$$+\frac{428}{105}q\ln(2)-4qC_2-12q^3C_2-44q^2B_2+\frac{428}{35}q^3\gamma+\frac{428}{35}q^3\ln(2)$$
$$+2q^3\pi^2+\frac{428}{105}q\ln(\kappa)+\frac{428}{105}qA_2+\frac{428}{35}q^3\ln(\kappa)+6\frac{q^7}{\kappa}$$
$$+\frac{428}{35}q^3A_2-8qB_2{}^2-24q^3B_2{}^2-4\frac{B_2}{\kappa}-32\frac{q^3}{\kappa}-31\frac{q}{\kappa}$$
$$+57\frac{q^5}{\kappa}+q^4B_1-\frac{4}{3}q^2B_1+\frac{7}{3}\kappa q+\frac{227}{6}\kappa q^3+\frac{455}{16}q^5$$
$$-48\frac{q^2B_2}{\kappa}+28\frac{q^4B_2}{\kappa}-24\frac{q^3C_2}{\kappa}-8\frac{qC_2}{\kappa}$$
$$+24\frac{q^6B_2}{\kappa}+\frac{856}{105}q\ln(x)+\frac{856}{35}q^3\ln(x)\Bigg)x^6$$
$$+\Bigg(\frac{19687}{168}q^2-\frac{145}{336}q^6-\frac{4729}{1008}q^4+\frac{899}{168}qB_1-\frac{41}{28}\kappa q^6$$

$$
\begin{aligned}
&+\frac{45}{56}\,qB_3-\frac{803}{224}\,q^3B_1+\frac{1665}{224}\,q^3B_3+\frac{86}{7}+\frac{719}{12}\,q^4\kappa\\
&+\frac{796}{21}\,qB_2+\frac{86}{7}\,\kappa-\frac{16}{7}\,q^5B_2+\frac{225}{28}\,q^5B_3\\
&-\frac{9}{28}\,q^5B_1+\frac{22201}{168}\,\kappa\,q^2+\frac{2372}{21}\,q^3B_2\Big)x^7\\
&+\Big(-\frac{19366807}{88200}\,q-12\,B_2-2\,B_1-\frac{2062220497}{6350400}\,q^3-qC_1+55\,q^5\kappa\\
&+\frac{1061}{35}\,\gamma\,q+\frac{13}{6}\,\pi^2q+\frac{995}{21}\,q\ln(2)-12\,qC_2-36\,q^3C_2\\
&+\frac{52}{3}\,q^4B_2-\frac{1136}{9}\,q^2B_2+\frac{12197}{140}\,q^3\gamma+\frac{3873}{28}\,q^3\ln(2)\\
&+\frac{47}{8}\,q^3\pi^2+\frac{1391}{105}\,q\ln(\kappa)+\frac{428}{35}\,qA_2+\frac{5029}{140}\,q^3\ln(\kappa)\\
&+\frac{37}{6}\,\frac{q^7}{\kappa}+\frac{1284}{35}\,q^3A_2-24\,q{B_2}^2-72\,q^3{B_2}^2-12\,\frac{B_2}{\kappa}\\
&-\frac{341}{4}\,\frac{q^3}{\kappa}-\frac{637}{6}\,\frac{q}{\kappa}+\frac{741}{4}\,\frac{q^5}{\kappa}\\
&+\frac{43}{6}\,q^4B_1-\frac{163}{18}\,q^2B_1+\frac{3}{2}\,q^3{B_1}^2+\frac{107}{105}\,qA_1\\
&-\frac{107}{140}\,q^3A_1-2\,q{B_1}^2+\frac{3}{4}\,q^3C_1-2\,\frac{B_1}{\kappa}+\frac{40}{3}\,\kappa\,q+\frac{815}{6}\,\kappa\,q^3\\
&+\frac{1265}{18}\,q^5+\frac{25}{252}\,q^7-144\,\frac{q^2B_2}{\kappa}+84\,\frac{q^4B_2}{\kappa}-72\,\frac{q^3C_2}{\kappa}-24\,\frac{qC_2}{\kappa}\\
&+72\,\frac{q^6B_2}{\kappa}-3\,\frac{q^6B_1}{\kappa}+\frac{13}{2}\,\frac{q^4B_1}{\kappa}-\frac{3}{2}\,\frac{q^2B_1}{\kappa}-2\,\frac{qC_1}{\kappa}+\frac{3}{2}\,\frac{q^3C_1}{\kappa}\\
&+\frac{4574}{105}\,q\ln(x)+\frac{8613}{70}\,q^3\ln(x)\Big)x^8\,. \qquad (\mathrm{J}\cdot 2)
\end{aligned}
$$

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