In an earlier article in this journal Lloyd Metzler examined the interacting price effects of taxes and subsidies in the context of a Leontief-type input-output model.\(^1\) Such a model eliminates responses to taxes and subsidies in the forms of adjustments in demand, output, wages, or profits and isolates the responses of price changes, considering both the direct price effect of the tax and subsidy and the secondary effects from changing input prices. Metzler demonstrated that the imposition of a tax, \(\tau\), on industry one combined with a subsidy of equal size to industry two raises the price of the first good, \(p_1\), and lowers the price of the second, \(p_2\). The direction of other price movements cannot be determined.\(^2\)

Three points are made here regarding Metzler's conclusions. (1) By modifying the output equations such that the output of each industry is measured as one unit, Metzler calculated changes in total revenues rather than prices. The absolute change in each of Metzler's prices is greater than the change in the actual price, by a factor equal to the respective level of output. (2) Metzler considered only a total tax, \(\tau\), rather than a tax calculated at some rate on a base. His conclusions obtain equally for unit, profits, or wage taxes, all of which are of fixed, exogenous amounts in this type of model. However, when an ad valorem tax (subsidy) is imposed, combined with a subsidy (tax) of equal absolute value, it can only be said that

\[ \sum_{i,j} a_{ij} \leq 1 \text{ for all } i \text{ or } j, \]
\[ \sum_{i,j} a_{ij} < 1 \text{ for some } i \text{ or } j. \]

The cofactor of any diagonal element of such a matrix, \([I - A]\), will be greater than the cofactor of any off-diagonal element in the corresponding row or column. This is somewhat more general than Metzler's conditions in that the diagonal elements of \(A\) may be either positive or negative. See L. A. Metzler, "A Multiple-Country Theorem of Income Transfers," *Journal of Political Economy*, LIX (Feb. 1950) and R. Solow, "On the Structure of Linear Models," *Econometrica*, XX (Jan. 1952).

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2. The mathematical basis for determining the final effects of a tax and subsidy is summarized here. For an indecomposable matrix, \(A\), with nonnegative off-diagonal elements, a sufficient condition for the inverse of \([I - A]\) to exist and to contain only positive elements is that the sum of the elements of any row (column) of \(A\) shall not exceed unity,
\[ \sum_{i,j} a_{ij} \leq 1 \text{ for all } i \text{ or } j, \]
and that at least one such sum shall be less than unity,
\[ \sum_{i,j} a_{ij} < 1 \text{ for some } i \text{ or } j. \]
the price of the taxed (subsidized) good rises (falls); it is impos-
sible to say whether the price of the subsidized (taxed) good will
fall (rise). (3) Metzler considers only the case of an equal tax and
subsidy—a balanced budget. If there is a budget surplus, where
the tax exceeds the subsidy, with an exogenous type of tax all prices
will be higher than under a balanced budget; in the case of an ad
valorem tax (subsidy), the price of the taxed (subsidized) good
will be higher. Conversely, a budget deficit will lower prices.

First, Metzler redefines the units of output in each industry
to equal one. The modified input-coefficients matrix for the new
units of output becomes \( |a_{ij}| \), where \( a_{ij} = a_{ij} x'_i / x'_i \), \( a_{ij} \) are the ori-
ginal input coefficients, and \( x'_i \) is the output level of the \( i^\text{th} \) good at
the point where the units are redefined. His price equations are the
dual of the modified output system, such that the price, \( p_i \), is the
price of the newly defined unit of the \( i^\text{th} \) good. Since the new unit
is \( x'_i \) times the original unit, Metzler's price, \( p_i \), is \( x'_i \) times greater
than the actual price, \( p_i \).

(1) \( p_i = x'_i \pi_i \).

As long as output levels are unchanged, \( p_i \) is the equivalent of total
revenue. Taxes and subsidies are assumed to be imposed as strictly
monetary transfers and have no effect on output levels. Metzler's
price changes, as a result of a tax on industry one and a subsidy to
industry two in amount, \( \tau \), are equivalent, then, to changes in total
revenue, such that

\[
\frac{dp_i}{d\tau} = \frac{d(x'_i \pi_i)}{d\tau} = x'_i \frac{d\pi_i}{d\tau}.
\]

If the price effects of the total tax and subsidy are calculated
from the original price equations of the system, the change in ac-
tual price, \( \pi_i \), would be

\[
\frac{d\pi_i}{d\tau} = \frac{\delta_{i1}/x'_1 - \delta_{i2}/x'_2}{\delta},
\]

where \( \delta \) is the determinant of the unmodified input-coefficients ma-
trix, \( |a_{ij}| \), and the \( \delta_{ij} \) are its signed cofactors. In this form it is
impossible to say anything about the direction of any price change
without knowing output levels, \( x'_1 \) and \( x'_2 \). Expressed in terms of
Metzler's modified equations and total revenue changes, the price
change becomes

\[
\frac{d\pi_i}{d\tau} = \frac{1}{x'_i} \cdot \frac{dp_i}{d\tau} = \frac{1}{x'_i} \frac{\Delta_{i1} - \Delta_{i2}}{\Delta} > 0, \quad i=1,
\]

\[
< 0, \quad i=2.
\]
In this form it can be seen that the price of the taxed good rises and the price of the subsidized good falls, a conclusion that is impossible to draw from examination of the unmodified system, equation (3). Metzler's conclusions about the directions of price changes in \( p_i \) are correct, but the absolute amount of the change in \( p_i \) is \( x'_i \) times greater than the change in actual price, \( \pi_i \).

Second, taxes are usually levied as a proportion of some base rather than as an absolute amount. Common types of taxes are a unit tax, a tax on wages or profits, and an ad valorem tax. Since output, profits, and wages are unaffected by taxes and subsidies in this model, the effect of a unit, wage, or profits tax is the same as a tax in some arbitrary amount, equation (4). We will refer to such taxes as exogenous taxes.

An ad valorem tax, however, depends upon prices and affects price interactions in a more complex manner than do exogenous taxes. Consider a tax on industry one proportional to the value of its output, \( \tau = tp_1 \), combined with a subsidy to industry two of equal absolute value, \( \tau \). In the solution for the new price levels, such a tax must be made a part of the coefficients matrix, creating a new input-and-tax-coefficients matrix. The determinant of the new price equations becomes

\[
\Delta'^* = \begin{vmatrix} 1-a_{11} - t & -a_{21} & \cdots & -a_{n1} \\ -a_{12} + t & 1-a_{22} & \cdots & -a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & 1-a_{nn} \end{vmatrix}
\]

Since the tax rate appears in the determinant, \( \Delta'^* \), and the co-

3. The equivalence of the two expressions, (3) and (4), can easily be illustrated for \( \pi_1 \) in the \( 2 \times 2 \) case and applies equally to other prices and to the general \( n \)-industry case. Since output is unchanged by the tax, \( x'_i = x_i \).

\[
\frac{d\pi_1}{d\tau} = \frac{\delta_{11}/x_1 - \delta_{12}/x_1}{1 - (1-a_{22})/x_2} = \left( 1-a_{22} \right) / x_2
\]

\[
\frac{d\pi_1}{d\tau} = \frac{\Delta_{11} - \Delta_{12}}{x_i \Delta} \left( \frac{1-a_{11}}{(1-a_{22})/x_2} \right) = \left( 1-a_{22} \right) / x_2
\]

4. For reasons of simplicity we shall continue the analysis in terms of the prices, \( p_i \), of the modified units of output, using Metzler's modified equations. Although we refer to a change in \( p_i \) as a change in price, it should be remembered that \( p_i \) is equivalent to total revenue.

5. A constraint must be placed on \( t \) before any conclusions can be drawn, such that \( t \leq a_{12} = a_{22}'/x'_1 \), in order not to violate the assumption of nonnegative off-diagonal elements of the coefficients matrix.
factors, $\Delta^*_{ij}$, of the matrix, the price changes resulting from the tax and subsidy must include the derivatives of $\Delta^*$ and $\Delta^*_{ij}$,

$$\frac{dp_i}{dt} = \left(\sum_{j=1}^{n} \frac{\partial \Delta^*_{ij}}{\partial t} + \frac{\partial \Delta^*_{i2}}{\partial t} + \cdots + \frac{\partial \Delta^*_{in}}{\partial t}\right) / \Delta^* - V_i \frac{\partial \Delta^*}{\partial t} / \Delta^{*2},$$

where $V_i = \lambda_1 \Delta^*_{i1} + \lambda_2 \Delta^*_{i2} + \cdots + \lambda_n \Delta^*_{in} > 0$. It can easily be seen that the derivative of the determinant is negative,

$$\frac{\partial \Delta^*}{\partial t} = -\Delta^*_{11} + \Delta^*_{12} < 0.$$

For the change in the price of the first good, $p_1$, the derivatives of the cofactors, $\Delta^*_{1j}$, are all zero, so that

$$\frac{dp_1}{dt} = V_1 \left(\Delta^*_{11} - \Delta^*_{12}\right) / \Delta^{*2} > 0.$$

However, for all other price changes the derivatives of the cofactors are nonzero and of different signs, and the price change may be in either direction. Thus, we know that when an ad valorem tax is imposed on one good along with an equal subsidy for a second good, the price of the taxed good will rise. It is impossible to say anything about the direction of price change of the subsidized good or any other good without knowledge of the actual input-output structures and tax rates.

On the other hand, if a subsidy were awarded to industry two proportional to the value of its output, $\tau = sp_2$, with the necessary taxation imposed on industry one, the price of the subsidized good would definitely fall, but it is impossible in the general case to say anything about the price of the taxed good.

$$\frac{dp_2}{ds} = V_2 \left(\Delta^{**}_{21} - \Delta^{**}_{22}\right) / \Delta^{**2} < 0,$$

6. The parameter, $\lambda_i$, represents value added for the $i$th good. The price of a newly defined unit of the $i$th good is

$$p_i = (\lambda_1 \Delta^*_{i1} + \lambda_2 \Delta^*_{i2} + \cdots + \lambda_n \Delta^*_{in}) / \Delta^* = V_i / \Delta^*.$$

7. The determinant resulting from the ad valorem subsidy would be

$$\Delta^{***} = \Delta^{**} \equiv \begin{vmatrix} 1-\alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1n} \\ -\alpha_{21} & 1-\alpha_{22} & \cdots & -\alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\alpha_{n1} & -\alpha_{n2} & \cdots & 1-\alpha_{nn} \end{vmatrix}$$

No constraint need be placed on $t$ in the case of the ad valorem subsidy, because the subsidy changes neither the signs of the off-diagonal elements nor the column sums of the coefficients matrix.
where $\Delta^{**}$ and $\Delta^{**}_{ij}$ are the determinant and cofactors for the input-and-subsidy-coefficients matrix.\(^8\)

Metzler's point was to emphasize the interacting effects of other prices on the input costs and on the price of any good and to show that these secondary price interactions would not be sufficient to dominate the direct upward pressure on the price of the taxed good or the downward pressure on the price of the subsidized good. We show here that in cases of an ad valorem tax (subsidy) the direct pressures are known to dominate only for the price of the taxed (subsidized) good, but the secondary effects may dominate the price change in the industry receiving the equivalent subsidy (tax).

Third, we consider the price effects of an unbalanced budget, in which the tax and subsidy are not equal. Consider an exogenous tax on industry one, $c_t\tau$, and a subsidy to industry two, $c_s\tau$, with the budget initially balanced, $c_t = c_s$. Either an increase in $c_t$, raising the tax, or a decrease in $c_s$, lowering the subsidy, will produce a budget surplus. The directional effect on prices of incurring a budget surplus by either method is the same — all prices rise.

\[
\begin{align*}
\frac{dp_t}{dc_t} &= \frac{\Delta_{11}}{\Delta} > 0, \\
-\frac{dp_t}{dc_s} &= \frac{\Delta_{12}}{\Delta} > 0.
\end{align*}
\]

For a given budget surplus, the absolute amount of price increase depends upon the method of incurring the surplus. An increase in taxes will affect the price of the taxed good, $p_1$, more than would a decline in the subsidy, $\Delta_{11} > \Delta_{12}$, but the opposite is true for the price, $p_2$, of the subsidized good, $\Delta_{21} < \Delta_{22}$. A move to a budget deficit would be the negative of (9) and all prices would fall.

In the case of the ad valorem tax we are limited to speaking about the price change of the good on which an ad valorem tax (subsidy) is imposed. When an ad valorem tax on industry one, $c_tp_1$, is combined with a subsidy to the second industry, $c_sp_1$, both proportional to $p_1$, the budget is balanced when $c_t = c_s$ and a budget surplus is incurred by raising $c_t$ or lowering $c_s$. Either will raise the price of the taxed good,

\[
\frac{dp_1}{dc_t} = V_1\Delta^{*}_{11}/\Delta^{*2} > 0,
\]

8. If more than one price level is involved in the tax (subsidy), such as an ad valorem tax (subsidy) on more than one industry or a tax (subsidy) on variable costs of one industry, it will be impossible to say anything even about the direction of price change of the taxed (subsidized) good.
\[
\frac{dp_1}{-dc_s} = \frac{V_1 \Delta^*_{12}}{\Delta^*_{22}} > 0,
\]
but it is impossible to predict the effect of the budget surplus on other prices. When an ad valorem subsidy to the second industry, \( c_s p_2 \), is combined with a tax to industry one, \( c_t p_2 \), both proportional to the price of the subsidized good, we know that a budget surplus will raise the price of the subsidized good,

\[
\frac{dp_2}{dc_t} = \frac{V_2 \Delta^{**}_{21}}{\Delta^{**}_{22}} > 0
\]
but nothing can be said about the other price changes. A budget deficit would be the negative of (11) or (12); the price to which the tax or subsidy is proportional will fall as a result of a budget deficit.

The conclusions that surplus financing leads to higher prices and deficit financing to lower prices may appear to be strangely in contrast to the usual generalizations about the price effects of deficit financing. It must be remembered that this model isolates the effects of interacting prices, ignoring the effects of changes in demand. In the case of deficit or surplus financing, these price effects that tend to offset demand effects are frequently ignored.