ON LAMINAR BOUNDARY-LAYER FLOW NEAR A POSITION OF SEPARATION

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SUMMARY

Singularities are considered in the solution of the laminar boundary-layer equations at a position of separation. A singularity of the type here considered occurred in a careful numerical computation by Hartree for a linearly decreasing velocity distribution outside the boundary layer; it may occur generally. Whenever it does occur, the boundary-layer equations cease to be valid at and near separation on the upstream side, and also downstream of separation. The work suggests that singularities may arise in the solution of non-linear parabolic equations due to their non-linearity. The formulae found may help computers of laminar boundary layers, who desire more than a rough solution, to have an end-point at which to aim.

1. Introduction and summary

FOR a flow at a large Reynolds number along an immersed solid surface a boundary layer is formed through which the velocity rises rapidly from zero at the surface to its value in the main stream. The approximate equations for the two-dimensional flow of a fluid of constant density ρ and kinematic viscosity ν in a boundary layer are

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \\ u = \frac{\partial \psi}{\partial y}, \qquad v &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\}, \tag{1}$$

where x is distance measured along the solid boundary in the plane of the flow, y is distance normal to the surface, u, v are the velocity components in the directions of x and y increasing, p the pressure, and ψ the stream function. According to the approximations of boundary-layer theory, p and $\partial p/\partial x$ may be taken independent of y, and if U is the velocity just outside the boundary layer in the main stream,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = U\frac{dU}{dx}. \tag{2}$$

Moreover, according to these approximations v/u and $(\partial u/\partial x)/(\partial u/\partial y)$ are small.

As boundary conditions we have that $u = 0$ and $v = 0$ (or $\psi = 0$) at $y = 0$, u is given as a function of y for some initial value of x, and the velocity passes over smoothly into the velocity of the main stream, i.e. $u \to U$, $\partial u/\partial y \to 0$, $\partial^2 u/\partial y^2 \to 0$, etc., as $y \to \infty$.

If $dU/dx < 0$ and $\partial p/\partial x > 0$, then $\partial u/\partial y$ at the wall $y = 0$ decreases as x increases until it vanishes; beyond the section at which it is zero, a slow back flow sets in along the wall, and the boundary layer separates from the surface.

Let d be any representative length of the system, U_0 a representative velocity, such as the undisturbed stream velocity, and R the Reynolds number $U_0 d/\nu$. The equations may be made non-dimensional by writing

$$x' = x/d, \quad y' = R^{\frac{1}{2}}y/d, \quad u' = u/U_0, \quad v' = R^{\frac{1}{2}}v/U_0, \quad p' = p/\rho U_0^2. \tag{3}$$

In the non-dimensional form, let $x' = 0$ be the initial section, at which u' is given as or approximated by a polynomial or power-series

$$u' = a_1 y' + a_2 y'^2 + \dots, \tag{4}$$

and let

$$-\frac{\partial p'}{\partial x'} = p_0 + p_1 x' + p_2 x'^2 + \dots. \tag{5}$$

Then it is known that if there is to be a solution without singularities, certain equations must be satisfied:

$$2a_2 + p_0 = 0, \quad a_3 = 0, \quad 5!\,a_5 + 2a_1 p_1 = 0, \quad 6!\,a_6 - 2p_0 p_1 = 0, \text{ etc.} \tag{6}$$

Only a_1, a_4, a_7,... are at our disposal. When the conditions are broken, the solution has an algebraic singularity at $x = 0$ (**1**).†

At the position of separation $\partial u/\partial y = 0$ at $y = 0$, i.e. $a_1 = 0$. The conditions for the absence of singularities when $a_1 = 0$ are considerably more complicated than those above.‡ If we suppose u expanded in a power series in x (we drop the dashes for the present)

$$u = u_0 + u_1 x + u_2 x^2 + \dots, \tag{7}$$

where u_0, u_1, u_2 are functions of y, expressible as power series,

$$\left.\begin{aligned} u_0 &= a_2 y^2 + a_3 y^3 + \dots \\ u_1 &= b_1 y + b_2 y^2 + \dots \\ u_2 &= c_1 y + c_2 y^2 + \dots \end{aligned}\right\}, \tag{8}$$

the conditions are

$$\left.\begin{aligned} 2a_2 + p_0 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 0, \quad 6!\,a_6 = 2p_0 p_1, \\ a_7 = 0, \quad a_9 = 0, \text{ etc.} \end{aligned}\right\}. \tag{9}$$

Only a_8, a_{12}, a_{16}, a_{20},... are at our disposal. In addition b_1, c_1, d_1,... are determined, not from the equations for u_1, u_2, u_3,... respectively, but from

† In the last term in equation (5) on p. 4 of ref. (**1**), the denominator should be $8a_1^3$, not $4a_1^3$; I am indebted to Prof. Hartree for this correction. See also ref. (**2**) at end.

‡ Goldstein, loc. cit., pp. 17, 18. [In the last line of p. 17 of ref. (**1**), in the equations of the footnote, for $6!\,a_6 = 2p_0 p_2$, read $6!\,a_6 = 2p_0 p_1$.]

the conditions for the absence of singularities in u_2, u_3, u_4,... respectively. There is also an ambiguity of sign, which can be determined only from physical considerations. If the conditions are broken there is a formal solution for the flow downstream of the form

$$\left.\begin{aligned} \psi &= \xi^3[f_0(\eta)+\xi f_1(\eta)+\ldots] \\ u &= \tfrac{1}{4}\xi^2[f_0'(\eta)+\xi f_1'(\eta)+\ldots] \end{aligned}\right\}, \tag{10}$$

where $$\xi = x^{\frac{1}{4}}, \qquad \eta = y/4x^{\frac{1}{4}}. \tag{11}$$

This formal solution fails, however, in certain circumstances, one of which is that the condition $$2a_2+p_0 = 0 \tag{12}$$

is satisfied, while the other conditions are not satisfied.

No other work has been reported on possible singularities at separation.

No analytical solution is known for a boundary-layer flow involving separation, and the methods used are approximate and numerical. The published methods of computation are rather rough, but recently more exact methods have been suggested and tried. The work described here arose out of an unpublished communication from Professor Hartree, in which he repeated Dr. Howarth's computation (3) for a linearly decreasing velocity distribution, $U = \beta_0-\beta_1 x$, with $u = U$ at $x = 0$. Professor Hartree replaces the partial derivatives with respect to x by finite differences, and retains the y-derivatives, so the partial differential equation is replaced approximately by a sequence of ordinary differential equations, each of which relates the velocity distribution through the boundary layer at one section to that at another section a short distance upstream, where it is known. The ordinary differential equations were solved laboriously on hand calculating machines rather than on the Differential Analyser in order that more significant figures might be retained.

Now all computations in which any attempt was made to obtain real accuracy at and near separation seem to have met with considerable difficulty. As a result of his computations, Professor Hartree was convinced that there was a singularity in the solution at the position of separation, and I undertook to try to find some formulae that would hold near this singularity and would help in finishing the computation.

To study the singularity near separation, the equations are put into non-dimensional form in a special way. Let x_s, U_s, U_s' be the values of x, U, dU/dx at separation, so that $U_s > 0$, $U_s' < 0$. We are not interested in any other properties or dimensions of the system, so as representative length l and Reynolds number R we take

$$l = -U_s/U_s', \qquad R = U_s l/\nu. \tag{13}$$

We are also concerned with the flow upstream of separation, so for our non-dimensional distances we write

$$x_1 = (x_s - x)/l, \qquad y_1 = R^{1/2} y/l. \tag{14}$$

Also put

$$u_1 = u/U_s, \quad v_1 = R^{1/2} v/U_s, \quad U_1 = U/U_s, \quad p_1 = p/\rho U_s^2, \quad \psi_1 = R^{1/2}\psi/lU_s. \tag{15}$$

The equations become

$$\left.\begin{aligned} -u_1\frac{\partial u_1}{\partial x_1}+v_1\frac{\partial u_1}{\partial y_1} &= \frac{\partial p_1}{\partial x_1}+\frac{\partial^2 u_1}{\partial y_1^2} \\ u_1 = \frac{\partial \psi_1}{\partial y_1}, &\qquad v_1 = \frac{\partial \psi_1}{\partial x_1} \\ \frac{\partial p_1}{\partial x_1} &= -U_1\frac{dU_1}{dx_1} \end{aligned}\right\}. \tag{16}$$

It is easy to see that $U_1 = 1$ and $dU_1/dx_1 = 1$ at $x_1 = 0$, so we may write

$$\frac{\partial p_1}{\partial x_1} = -(1+P_1 x_1+P_2 x_1^2+\ldots), \tag{17}$$

i.e. p_0 (in our previous notation) $= -1$. For the linear velocity distribution $U = \beta_0 - \beta_1 x$,

$$\frac{\partial p_1}{\partial x_1} = -(1+x_1), \qquad P_1 = 1, \qquad P_2 = P_3 = \ldots = 0. \tag{18}$$

Formulae for the P_1, P_2,... are easily found in the general case, and if U is taken as a given function of x, it is easily found that the P's are independent of the position of separation if

$$U = (\text{constant})e^{-\beta x}, \quad \text{or} \quad U = (\beta_0 - \beta_1 x)^m \quad \text{or} \quad (\beta_0+\beta_1 x)^{-m}, \tag{19}$$

the constants β, β_0, β_1, m being positive. (However, since $l = -U_s/U'_s$, the scale varies as x_s varies in the last two cases.) For other values of U, the P's depend on x_s, the position of separation.

The boundary conditions are $\psi_1 = 0$, $u_1 = 0$ at $y_1 = 0$, and $u_1 \to U_1$, etc., as $y_1 \to \infty$. Since $x_1 = 0$ is a position of separation, $(\partial u_1/\partial y_1)_{y_1=0} = 0$ at $x_1 = 0$, so

$$u_1 = a_2 y_1^2 + a_3 y_1^3 + \ldots \quad \text{at} \quad x_1 = 0. \tag{20}$$

Singularities in the solution for the corresponding system of equations for the motion downstream have been considered (see equations (10) and (11)); near $x_1 = 0$, $y_1 = 0$, ψ_1 is a function of $x_1^{1/4}$ and $y_1/x_1^{1/4}$. The skin-friction is $\mu(\partial u/\partial y)_{y=0}$ and the determining quantity is $(\partial u_1/\partial y_1)_{y_1=0}$, which is an ascending series of powers of $x_1^{1/4}$, beginning with a multiple of $x_1^{1/2}$. If, however, $2a_2+p_0 = 0$, which corresponds to $a_2 = \frac{1}{2}$ in the new notation, there are special features in the solution; in particular the series for $(\partial u_1/\partial y_1)_{y_1=0}$ now begins with a term in $x_1^{3/4}$. Now Professor Hartree was quite certain that this particular feature was present in his computed solution. The calculations reported here therefore rest on three assumptions, all of which were satisfied in Hartree's numerical solution: (i) there is a singularity at separation; (ii) there is a finite value of u_1 at separation for $y_1 \neq 0$; (iii) $a_2 = \frac{1}{2}$. Related to (iii) Professor Hartree found (empirically) that in his solution $(\partial u_1/\partial y_1)_{y_1=0}$ behaved near $x_1 = 0$ like a multiple of x_1^r, where r is certainly less than 1 and greater than $\frac{1}{4}$. Thus, we must take

$$u_1 = \tfrac{1}{2}y^2 + a_3 y^3 + \ldots \quad \text{at} \quad x_1 = 0, \tag{21}$$

and as a result we find that

$$(\partial u_1/\partial y_1)_{y_1=0} = 2^{\frac{3}{2}}(\alpha_1 x_1^{\frac{1}{2}} + \alpha_2 x_1^{\frac{3}{4}} + \alpha_3 x_1 + \alpha_4 x_1^{\frac{5}{4}} + \ldots), \tag{22}$$

where the α's are constants. (The factor $2^{\frac{3}{2}}$ is inserted to conform to the notation in § 2.)

The first purpose of the calculations was to find the connexions between $a_3, a_4, a_5, a_6, \ldots$ and $\alpha_1, \alpha_2, \alpha_3, \ldots$ and other formulae for u at and near $x_1 = 0$ to see if the results fitted the numerical values for the special solution. There is no mathematical proof that a solution exists with singularities of the type considered near separation, but with the above assumptions it is difficult to see how the solution could be of a different type.

We may remark that in assuming that $a_2 = \frac{1}{2}$, we are in effect assuming that $(\partial^2 u_1/\partial y_1^2)_{y_1=0}$ is continuous at $x_1 = 0$, and then $(\partial^n u_1/\partial y_1^n)_{y_1=0}$ is found to be continuous at $x_1 = 0$ for $n = 1, 2, 3, 4$ and discontinuous for $n = 5$ and 6 and probably for all $n \geqslant 5$, though $\partial^n u_1/\partial y_1^n$ is continuous for $y_1 \neq 0$. More important, it is found that at separation v_1 and $\partial u_1/\partial x_1$ become infinite in such a way that $x_1^{\frac{1}{4}} v_1$ and $x_1^{\frac{1}{2}} \partial u_1/\partial x_1$ have finite non-zero limits as $x_1 \to 0$ for all non-zero y_1. The basic assumptions of boundary-layer theory therefore do not hold at and near separation. Nevertheless, large cross-velocities are to be expected at separation, otherwise the assumptions of boundary-layer theory would not break down.

The formal solution for the motion upstream is found by writing

$$\xi = x_1^{\frac{1}{4}}, \qquad \eta = y_1/2^{\frac{1}{2}} x_1^{\frac{1}{4}}, \tag{23}$$

$$\psi_1 = 2^{\frac{3}{2}}\xi^3[f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \ldots], \tag{24}$$

$$u_1 = 2\xi^2[f_0'(\eta) + \xi f_1'(\eta) + \xi^2 f_2'(\eta) + \ldots] \tag{25}$$

in (16), and equating powers of ξ. Since $\psi_1 = 0$ and $u_1 = 0$ at $y_1 = 0$, $f_r(0) = f_r'(0) = 0$, and from the value (21) of u_1 at $x_1 = 0$ we find the condition

$$\lim_{\eta \to \infty} \frac{f_r'}{\eta^{r+2}} = 2^{\frac{1}{2}r} a_{r+2} \quad (r = 0, 1, 2, \ldots). \tag{26}$$

The solution for f_r must have a double zero at the origin, and must not involve exponentially large terms as $\eta \to \infty$.

The condition $u_1 \to U_1$ as $y_1 \to \infty$ is satisfied for $x_1 > 0$ if it is satisfied at $x_1 = 0$, i.e. if $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$.

With $a_2 = \frac{1}{2}$, the solution for f_0 is found to be $f_0 = \eta^3/6$. The solution for f_1 is $f_1 = \alpha_1 \eta^2$, and

$$a_3 = \frac{1}{\sqrt{2}} \lim \frac{f_1'}{\eta^3} = 0. \tag{27}$$

Then

$$f_2 = \alpha_2 \eta^2 - \frac{\alpha_1^2}{15} \eta^5, \tag{28}$$

and

$$a_4 = -\tfrac{1}{6}\alpha_1^2. \tag{29}$$

[The a's and α's are as in (21) and (22).] The equation for f_3 becomes

$$f_3'''' - \tfrac{1}{2}\eta^3 f_3'' + \tfrac{7}{2}\eta^2 f_3' - 6\eta f_3 = 5f_1'' f_2 - 7f_1' f_2 + 4f_1 f_2''. \tag{30}$$

The equations for all succeeding f's are non-homogeneous linear equations, with the right-hand side rapidly becoming more and more complicated; thus for f_4 it involves f_1, f_2, f_3, P_1, for f_5 the first four f's, and so on. The complementary functions involve integrals of confluent hypergeometric functions; the particular integrals are very involved. The condition for the absence of exponentially large terms in f_3 is†

$$\alpha_2 = \frac{2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^2, \tag{31}$$

and then from (26)

$$a_5 = -\frac{2^{\frac{1}{4}}\pi}{40(\frac{1}{4}!)^2}\alpha_1^3. \tag{32}$$

The condition for the absence of exponentially large terms in f_4 is

$$\alpha_3 = \frac{\pi^3}{400(\frac{1}{4}!)^6}(35 - 8\sqrt{2})\alpha_1^3, \tag{33}$$

and from (26)

$$a_6 = \left(\frac{1}{9} - \frac{7\pi^2}{600(\frac{1}{4}!)^4}\right)\alpha_1^4 - \frac{P_1}{360}. \tag{34}$$

The condition for the absence of exponentially large terms in f_5 gives α_4 as a multiple of α_1^4, though the constant must be found numerically, and then a_7 is found; but the condition for the absence of exponentially large terms in f_6 does *not* give α_5; it requires that

$$\int_0^\infty H_6\left(\eta^2 - \frac{\eta^6}{5} + \frac{\eta^{10}}{180}\right)\exp\left(-\frac{\eta^4}{8}\right) d\eta = 0, \tag{35}$$

where H_6 is a complicated function of η, involving f_5.

Again, α_6 is determined from f_7 in terms of α_1, α_5, and P_1, α_7 from f_8 in terms of α_1, α_5, and P_1, and so on until we come to f_{10}. It is possible, though it has not been proved, that α_5 is determined from f_{10}, α_9 from f_{14},

† $x!$ is written for $\Gamma(x+1)$.

and so on. If so, then only α_1 remains to be determined, and a_4 (and therefore α_1) is probably determined by the condition $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$. If so the whole solution is determined at separation. In fact, if it is true that all the other constants are determinate in terms of a_4 and the P's, there is a solution only if it is possible to choose a_4 so that the condition $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$ is satisfied. Unless this condition is satisfied for every value of a_4, it will presumably fix a_4 in terms of the P's. If a_4 is so fixed, the non-dimensional velocity distribution at separation, $x_1 = 0$, and just upstream of separation, for small positive values of x_1, is fixed in terms of the P's. Suppose now we have a problem in which U is a given decreasing function of x, and u a given function of y for some $x < x_s$. There are some U's for which the P's are independent of x_s; otherwise they vary with x_s. When separation takes place, the non-dimensional velocity distribution at and near separation is independent of the initial distribution of u for the former values of U, and for the others it is the same for all initial distributions of u for which separation takes place at the same value of x. This suggests that what has been found is an *asymptotic* solution at and near separation, and that the full non-dimensional solutions in the above cases all behave asymptotically in the same way near separation.

It appears that the singularity at separation is due to the non-linear character of the equations. It is possible to simulate the phenomenon of separation by a linear system of equations, and there is then no singularity at separation. For example, the solution of

$$\frac{\partial u}{\partial t} = -1+\frac{\partial^2 u}{\partial y^2}, \quad u = 1 \text{ at } t = 0, \quad u = 0 \text{ at } y = 0, \quad u \text{ finite as } y \to \infty, \tag{36}$$

is

$$u = \tfrac{1}{2}y^2+(1-t-\tfrac{1}{2}y^2)\operatorname{erf}\frac{y}{2\sqrt{t}}-y\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2}{4t}\right), \tag{37}$$

where

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-w^2}\,dw, \tag{38}$$

so

$$(\partial u/\partial y)_{y=0} = 0 \quad \text{at} \quad t = \tfrac{1}{2}. \tag{39}$$

We should remark also that in equations that are linear but are otherwise similar to the equations here considered (the equation for the *temperature in the theory of the conduction of heat,† for example*), if we attempt to work backwards (i.e. to solve for negative time) from a

† The relationship of the boundary-layer equations to the equation of heat conduction has been stressed by Prandtl (2) (loc. cit., pp. 79, 80) in connexion with difficulties to be expected when $u \leqslant 0$.

singularity we encounter exponentially large terms. With given initial and boundary conditions, however, the solution for such a linear equation is free from singularities for positive non-zero time, whereas the basis of the present discussion is the assumption that singularities may occur at separation in the solution of the non-linear equations considered.

The special case considered by Professor Hartree is one in which by a correct choice of scale the P's may be made independent of x_s. There is always the possibility, therefore, that the occurrence of a singularity at separation is restricted to such cases. Another possibility is that a singularity will always occur except for certain special pressure variations in the neighbourhood of separation, and that, experimentally, whatever we may do, the pressure variations near separation will always be such that no singularity will occur.

It is a necessary consequence of the discussion of the motion upstream of separation that a_4 is negative or zero. Professor Hartree finds a negative a_4 from his special numerical solution. When we consider the motion *downstream* of separation in a similar way, we find that when a_4 is negative the solution downstream is not *real*. When there is a singularity at separation there is no real solution at all farther downstream. When $a_4 = 0$ there is a solution downstream, but then we have a case in which the whole solution is free from singularities. These cases include that in which $(\partial u/\partial y)_{y=0} = 0$ for all x. There must, of course, be restrictions on the pressure distributions in order that this should happen, and these conditions arise from the condition that $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$. This our method does not permit us to discuss, but one solution (due to Falkner and Skan, 4†) is known in which $U = cx^m$, $m = -0{\cdot}0904$ approximately.

As far as numerical values are concerned, and comparison with the computed values, Professor Hartree fitted his solution to the formulae here obtained, and considered that he had obtained a reasonable fit. The matter has been recently reconsidered by Mr. C. W. Jones, who has tabulated f_3, f_4, f_5 and has found that within the accuracy of his computation, the integral condition (35) for the absence of exponentially large terms in f_6 is satisfied.

Mr. Jones has compared the skin-friction, the velocity distributions at separation not far from the wall, the transition to main-stream conditions at separation, and the velocity distribution just downstream of separation (at $(8\beta_1/\beta_0)x = 0{\cdot}956$, where $U = \beta_0 - \beta_1 x$ and $(8\beta_1/\beta_0)x_s = 0{\cdot}959$). A satisfactory fit is obtained with α_1 about 0·47 or 0·48. A satisfactory transition to main-stream conditions seems to be obtained, but it is not sensitive to changes in α_1.

† See also (5).

If we assume that Mr. Jones's numerical work is sufficient to answer certain questions, and to make it plausible that our formulae fit the solution in the case considered, we still do not know for certain that α_5, α_9,... are determined from the equations for f_9, f_{13},..., and, if they are, that a_4 (and therefore α_1) can be determined from the condition $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$. It is clear that an adequate discussion is not possible by the method used here. Three more important questions also remain:

(1) Is it correct that the formulae represent an asymptotic solution at and near separation?

(2) What are the most general restrictions on the pressure distribution in order that solutions should exist for which $(\partial u/\partial y)_{y=0} = 0$ for all x?

(3) A singularity when $U = \beta_0 - \beta_1 x$ being assumed, is the occurrence of a singularity restricted to cases in which the P's are independent of x_s? Or does a singularity always occur unless $(\partial u/\partial y)_{y=0} = 0$ for all x? Or does a singularity always occur except for certain special pressure distributions near separation, and are experimental pressure distributions always of the special type?

It should be remarked that although there is a certain physical plausibility in the notion that large cross-velocities should occur at separation, the existing experimental information is insufficient to settle the question.

The work described may be summed up by saying that it throws doubt on the validity of the boundary-layer equations at and near separation on the upstream side, and also downstream of separation; inferences from these equations in these regions, which are fairly common in the literature, are therefore also in doubt; mathematically the work suggests that singularities may arise in the solution of non-linear parabolic equations, due to their non-linearity; and formulae have been found which may help computers of laminar boundary layers, who desire more than a rough solution, to have an end-point at which to aim.

2. The solution upstream

Substitute (23), (24), and (17) into (16), and equate coefficients of powers of ξ. The equation for f_0, obtained from the coefficient of ξ^0, is

$$f_0''' - 3f_0 f_0'' + 2f_0'^2 = 1. \tag{40}$$

Since $\psi_1 = 0$ and $u_1 = 0$ at $y_1 = 0$,

$$f_r(0) = f_r'(0) = 0 \quad (r = 0, 1, 2,...). \tag{41}$$

When $\xi \to 0$, $\eta \to \infty$ if $y_1 \neq 0$, and since $2^{\frac{1}{2}}\xi\eta = y_1$, $\lim_{\xi \to 0} u_1$ is given by (21) if

$$\lim_{\eta \to \infty} \frac{f_r'}{\eta^{r+2}} = 2^{\frac{1}{2}r} a_{r+2} \quad (r = 0, 1, 2,...). \tag{42}$$

The condition that the velocity should pass over smoothly into the velocity of the main stream will be considered in § 3, when the solution for large values of $y_1/x^{\frac{1}{4}}$ is considered.

Since $a_2 = \frac{1}{2}$, the solution for f_0 is

$$f_0 = \eta^3/6. \tag{43}$$

The equation for f_r is then found to be

$$f_r''' - \tfrac{1}{2}\eta^3 f_r'' + \tfrac{1}{2}(r+4)\eta^2 f_r' - (r+3)\eta f_r = G_r, \tag{44}$$

where
$$G_1 = 0, \qquad G_2 = 4f_1 f_1'' - 3f_1'^2, \tag{45}$$

and for $r \geqslant 2$,

$$G_r = \sum_{s=1}^{r-1} [(r-s+3)f_s'' f_{r-s} - (r-s+2)f_s' f_{r-s}'] + P_{r/4}, \tag{46}$$

$P_{r/4}$ being put equal to zero except when $\frac{1}{4}r$ is integral. The solution for f_1 with a double zero at the origin is

$$f_1 = \alpha_1 \eta^2, \tag{47}$$

where α_1 is a constant; hence from (42),

$$a_3 = \frac{1}{\surd 2} \lim_{\eta \to \infty} \frac{f_1'}{\eta^3} = 0. \tag{48}$$

The solution for f_2 with a double zero at the origin is now found to be

$$f_2 = \alpha_2 \eta^2 - \frac{\alpha_1^2}{15}\eta^5, \tag{49}$$

where α_2 is a constant; hence from (42)

$$a_4 = -\tfrac{1}{6}\alpha_1^2. \tag{50}$$

In order to write down the general solution for f_3 with a double zero at the origin, and to consider its behaviour as $\eta \to \infty$, some discussion is necessary of the complementary functions, and it is advisable to break off and discuss generally the complementary functions of the equation for f_r.

Three independent complementary functions are η^2, g_r, and h_r, where, if the function ${}_1F_1(a,b,x)$ is defined by

$${}_1F_1(a,b,x) = 1 + \frac{a}{1.b}x + \frac{a(a+1)}{2!\,b(b+1)}x^2 + \frac{a(a+1)(a+2)}{3!\,b(b+1)(b+2)}x^3 + \ldots, \tag{51}$$

then†
$$g_r = -\sum_{m=0}^{\infty} \frac{(m-\frac{3}{2}-\frac{1}{4}r)!\,\frac{1}{4}!\,\eta^{4m+1}}{m!\,(-\frac{3}{2}-\frac{1}{4}r)!\,(m+\frac{1}{4})!\,8^m(4m-1)}$$

$$= \eta - \eta^2 \int_0^{\eta} \eta^{-2}\{{}_1F_1(-\tfrac{3}{2}-\tfrac{1}{4}r, \tfrac{5}{4}, \eta^4/8) - 1\}\, d\eta, \tag{52}$$

† $x!$ is written for $\Gamma(x+1)$, as before.

and $$h_r = -\sum_{m=0}^{\infty} \frac{(m-\frac{7}{4}-\frac{1}{4}r)!\,(-\frac{1}{4})!\,\eta^{4m}}{m!\,(-\frac{7}{4}-\frac{1}{4}r)!\,(m-\frac{1}{4})!\,8^m(2m-1)}$$

$$= 1-2\eta^2\int_0^{\eta} \eta^{-3}\{{}_1F_1(-\tfrac{3}{4}-\tfrac{1}{4}r,\tfrac{3}{4},\eta^4/8)-1\}\,d\eta. \tag{53}$$

The series for g_r terminates when $r = 4m+2$, and that for h_r terminates when $r = 4m+1$, m being a positive integer or zero.

As regards asymptotic expansions, in addition to the solution η^2 the equation (44) with G_r put equal to zero has two solutions whose asymptotic expansions for large η commence with multiples of

$$\eta^{r+3} \quad \text{and of} \quad \eta^{-(r+10)}\exp(\eta^4/8)$$

respectively.

When x is large and positive (6),†

$${}_1F_1(a,b,x) \sim \frac{(b-1)!}{(a-1)!}e^x x^{a-b} \times$$

$$\times\left\{1+\frac{(b-a)(1-a)}{x}+\frac{(b-a)(b-a+1)(1-a)(2-a)}{2!\,x^2}+\ldots\right\} \tag{54}$$

so

$${}_1F_1(-\tfrac{1}{2}-\tfrac{1}{4}r,\tfrac{5}{4},\eta^4/8) \sim \frac{2^{(3r+13)/4}(-\frac{3}{4})!}{(-\frac{3}{2}-\frac{1}{4}r)!}\exp(\eta^4/8)\eta^{-(r+7)}\{1+\ldots\} \quad (r \neq 4m+2) \tag{55}$$

and

$${}_1F_1(-\tfrac{3}{4}-\tfrac{1}{4}r,\tfrac{3}{4},\eta^4/8) \sim -\frac{2^{(3r+10)/4}(-\frac{5}{4})!}{(-\frac{7}{4}-\frac{1}{4}r)!}\exp(\eta^4/8)\eta^{-(r+6)}\{1+\ldots\} \quad (r \neq 4m+1). \tag{56}$$

Hence

$$g_r \sim -\frac{2^{(3r+17)/4}(-\frac{3}{4})!}{(-\frac{3}{2}-\frac{1}{4}r)!}\exp(\eta^4/8)\eta^{-(r+10)}\{1+\ldots\} \quad (r \neq 4m+2) \tag{57}$$

and $$h_r \sim \frac{2^{(3r+18)/4}(-\frac{5}{4})!}{(-\frac{7}{4}-\frac{1}{4}r)!}\exp(\eta^4/8)\eta^{-(r+10)}\{1+\ldots\} \quad (r \neq 4m+1). \tag{58}$$

Exponentially large terms must not occur in the solution for f_r, so when $r \neq 4m+1$ or $4m+2$, g_r and h_r must occur in the combination

$$(-\tfrac{5}{4})!\,(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,g_r+2^{-\frac{1}{4}}(-\tfrac{3}{4})!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!\,h_r. \tag{59}$$

But‡

$$(a-1)!\,(-b)!\,{}_1F_1(a,b,x)+(a-b)!\,(b-2)!\,x^{1-b}{}_1F_1(a+1-b,2-b,x)$$

$$\sim (a-1)!\,(a-b)!\,x^{-a}\left\{1-\frac{a(a+1-b)}{x}+\frac{a(a+1)(a+1-b)(a+2-b)}{2!\,x^2}-\ldots\right\}; \tag{60}$$

† The formula is on p. 258 of ref. (6).
‡ Barnes (5), op. cit., p. 259.

hence

$$(-\tfrac{5}{4})!\,(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,{}_1F_1(-\tfrac{1}{2}-\tfrac{1}{4}r,\tfrac{5}{4},\eta^4/8)+$$
$$+8^{\frac{1}{4}}(-\tfrac{3}{4})!(-\tfrac{7}{4}-\tfrac{1}{4}r)!\,\eta^{-1}{}_1F_1(-\tfrac{3}{4}-\tfrac{1}{4}r,\tfrac{3}{4},\eta^4/8)$$
$$\sim 8^{-(r+2)/4}(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!\,\eta^{r+2}\Big\{1-\frac{(2+r)(3+r)}{2\eta^4}+$$
$$+\frac{(2+r)(2-r)(3+r)(1-r)}{2.4.\eta^8}-\frac{(2+r)(2-r)(6-r)(3+r)(1-r)(5-r)}{2.4.6.\eta^{12}}+\dots\Big\}$$
$$(r\neq 4m+1 \text{ or } 4m+2) \quad (61)$$

and

$$(-\tfrac{5}{4})!\,(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,g_r+2^{-\frac{1}{4}}(-\tfrac{3}{4})!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!\,h_r$$
$$=(-\tfrac{5}{4})!(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,\eta+2^{-\frac{1}{4}}(-\tfrac{3}{4})!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!-$$
$$-\eta^2\int\limits_0^\eta \eta^{-2}\{(-\tfrac{5}{4})!\,(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,[{}_1F_1(-\tfrac{1}{2}-\tfrac{1}{4}r,\tfrac{5}{4},\eta^4/8)-1]+$$
$$+8^{\frac{1}{4}}(-\tfrac{3}{4})!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!\,\eta^{-1}[{}_1F_1(-\tfrac{3}{4}-\tfrac{1}{4}r,\tfrac{3}{4},\eta^4/8)-1]\}\,d\eta$$
$$\sim -8^{-(r+2)/4}(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!\Big\{\frac{\eta^{r+3}}{r+1}-\frac{(2+r)(3+r)\eta^{r-1}}{2(r-3)}+$$
$$+\frac{(2+r)(2-r)(3+r)(1-r)\eta^{r-5}}{2.4.(r-7)}-$$
$$-\frac{(2+r)(2-r)(6-r)(3+r)(1-r)(5-r)\eta^{r-9}}{2.4.6.(r-11)}+\dots\Big\}+$$
$$+\text{constant}.\,\eta^2 \qquad (r=4m). \quad (62)$$

When $r = 4m+3$ the term with a zero denominator must be replaced by

$$(-1)^m 8^{-m-5/4}(-m-\tfrac{9}{4})!\,(-m-\tfrac{5}{2})!\times$$
$$\times\frac{5.9.13\dots(4m+5).3.5.7\dots(2m+3)}{(m+1)!}\eta^2\log\eta,$$

which reduces to
$$\frac{(-1)^{m+1}.2^{\frac{1}{4}}.\pi^{\frac{1}{2}}.(-\tfrac{5}{4})!}{(m+1)!}\eta^2\log\eta. \quad (63)$$

(62) may be verified and the constant multiplier of η^2 found by considering

$$\frac{1}{2\pi i}\int(-s-1)!\,(-\tfrac{5}{4}-s)!\,(s-\tfrac{3}{2}-\tfrac{1}{4}r)!\,\frac{\eta^{4s+1}}{8^s(4s-1)} \quad (64)$$

taken round a contour consisting of the straight line from $-N-\frac{1}{2}-\infty i$ to $-N-\frac{1}{2}+\infty i$ and the part of a circle of infinite radius to the right of the line. When $r\neq 4m+3$ the constant multiple of η^2 in (62) is found to be

$$2^{-7/4}\pi^{\frac{1}{2}}(-\tfrac{5}{4})!\,(-\tfrac{5}{4}-\tfrac{1}{4}r)!\,\eta^2. \quad (65)$$

When $r = 4m+3$ the term with a zero denominator in (62) must be omitted, and (65) replaced by

$$\frac{(-1)^{m+1}\pi^{\frac{1}{2}}(-\frac{5}{4})!}{2^{7/4}(m+1)!}\eta^2\left[\log\frac{\eta^4}{8}-\mathfrak{F}(-\tfrac{5}{4})-\mathfrak{F}(-\tfrac{3}{2})+\mathfrak{F}(m+1)\right], \tag{66}$$

where

$$\mathfrak{F}(z) = \frac{d}{dz}\log z!. \tag{67}$$

The multiplier of $\eta^2\log\eta$ agrees with that in (63). If γ is Euler's constant (equal to 0·5772...)

$$\left.\begin{aligned}\mathfrak{F}(m+1) &= 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{m+1}-\gamma,\\ \mathfrak{F}(-\tfrac{5}{4}) &= \tfrac{1}{2}\pi+4-3\log 2-\gamma,\\ \mathfrak{F}(-\tfrac{3}{2}) &= 2-2\log 2-\gamma,\end{aligned}\right\} \tag{68}$$

so (66) is equal to

$$\frac{(-1)^{m+1}\pi^{\frac{1}{2}}(-\frac{5}{4})!}{2^{7/4}(m+1)!}\eta^2\left[4\log\eta+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{m+1}+2\log 2+\gamma-\tfrac{1}{2}\pi-5\right]. \tag{69}$$

In particular, since

$$(-\tfrac{5}{4})! = -\frac{2^{\frac{1}{2}}.\pi}{\frac{1}{4}!},\quad (-\tfrac{5}{2})! = \frac{4\pi^{\frac{1}{2}}}{3},\quad (-\tfrac{9}{4})! = \frac{4.2^{\frac{1}{2}}.\pi}{5.\frac{1}{4}!},\quad (-\tfrac{11}{4})! = \frac{64.\frac{1}{4}!}{21}, \tag{70}$$

it follows from (62), (65), and (69) that

$$h_3-\frac{3.2^{\frac{1}{2}}.\pi^{\frac{3}{2}}}{10.(\frac{1}{4}!)^3}g_3 \sim -\frac{\pi}{160(\frac{1}{4}!)^2}\left\{\eta^6-\frac{15}{2!}\frac{1}{\eta^2}+\frac{15.1.3}{2.3!}\frac{1}{\eta^6}-\frac{15.1.3.3.7}{3.4!}\frac{1}{\eta^{10}}+\dots\right\}+\frac{3\pi}{32(\frac{1}{4}!)^2}\eta^2[4\log\eta+2\log 2+\gamma-\tfrac{1}{2}\pi-5] \tag{71}$$

and

$$h_4-\frac{7.2^{\frac{3}{4}}.\pi^{\frac{3}{2}}}{64(\frac{1}{4}!)^3}g_4 \sim -\frac{\pi^{\frac{1}{2}}}{48.2^{\frac{1}{4}}.\frac{1}{4}!}\left\{\frac{\eta^7}{5}-\frac{3.7}{1!}\eta^3-\frac{3.1.7.3}{3.2!}\frac{1}{\eta}+\frac{3.1.1.7.3.1}{7.3!}\frac{1}{\eta^5}-\frac{3.1.1.3.7.3.1.5}{11.4!}\frac{1}{\eta^9}+\frac{3.1.1.3.5.7.3.1.5.9}{15.5!}\frac{1}{\eta^{13}}-\dots\right\}-\frac{21.2^{\frac{3}{4}}.\pi^{\frac{3}{2}}}{640(\frac{1}{4}!)^4}\eta^2. \tag{72}$$

We now return to the equation for f_3, which is

$$f_3'''-\tfrac{1}{2}\eta^3f_3''+\tfrac{7}{2}\eta^2f_3'-6\eta f_3 = -10\alpha_1\alpha_2\eta^2-\tfrac{4}{3}\alpha_1^3\eta^5. \tag{73}$$

The general integral with a double zero at the origin is

$$f_3 = \alpha_3\eta^2+4\alpha_1\alpha_2(\eta-g_3)-\tfrac{8}{3}\alpha_1^3(1+\tfrac{1}{4}\eta^4-h_3), \tag{74}$$

where α_3 is a constant. In order that exponentially large terms should not occur in f_3, g_3 and h_3 must appear in (74) in the same combination as in (71). Hence we must have

$$\alpha_2 = \frac{2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^2 \tag{75}$$

and

$$f_3 = \alpha_3\eta^2 - \tfrac{8}{3}\alpha_1^3\left\{1+\tfrac{1}{4}\eta^4 - h_3 - \frac{3.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{10(\frac{1}{4}!)^3}(\eta - g_3)\right\}$$

$$= \alpha_3\eta^2 - \tfrac{8}{3}\alpha_1^3\left\{\sum_{m=2}^{\infty}\frac{(m-\frac{5}{2})!\,(-\frac{1}{4}!)\,\eta^{4m}}{m!\,(-\frac{5}{2})!\,(m-\frac{1}{4})!\,8^m(2m-1)} - \right.$$

$$\left. - \frac{3.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{10(\frac{1}{4}!)^3}\sum_{m=1}^{\infty}\frac{(m-\frac{9}{4})!\,\frac{1}{4}!\,\eta^{4m+1}}{m!\,(-\frac{9}{4})!\,(m+\frac{1}{4})!\,8^m(4m-1)}\right\}. \tag{76}$$

For large values of η,

$$f_3 \sim -\frac{\pi}{60(\frac{1}{4}!)^2}\alpha_1^3\left\{\eta^6 - \frac{15}{2!}\frac{1}{\eta^2} + \frac{15.1.3}{2.3!}\frac{1}{\eta^6} - \frac{15.1.3.3.7}{3.4!}\frac{1}{\eta^{10}} + \dots\right\} +$$

$$+ \frac{\pi}{4(\frac{1}{4}!)^2}\alpha_1^3\eta^2[4\log\eta + 2\log 2 + \gamma - \tfrac{1}{2}\pi - 5] +$$

$$+ \alpha_3\eta^2 - \tfrac{8}{3}\alpha_1^3\left[1+\tfrac{1}{4}\eta^4 - \frac{3.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{10(\frac{1}{4}!)^3}\eta\right]. \tag{77}$$

Hence, from (42),

$$a_5 = -\frac{2^{\frac{1}{4}}.\pi}{40(\frac{1}{4}!)^2}\alpha_1^3. \tag{78}$$

We now turn to the equation for f_4, which on substituting in (44) we find to be

$$f_4''' - \tfrac{1}{2}\eta^3 f_4'' + 4\eta^2 f_4' - 7\eta f_4 = P_1 - 6(\alpha_2^2 + 2\alpha_1\alpha_3)\eta^2 - 2\alpha_1^2\alpha_2\eta^5 -$$

$$- \tfrac{32}{3}\alpha_1^4\left\{\sum_{m=2}^{\infty}\frac{(16m^2-20m+3)(m-\frac{5}{2})!\,(-\frac{1}{4})!\,\eta^{4m}}{m!\,(-\frac{5}{2})!\,(m-\frac{1}{4})!\,8^m(2m-1)} - \right.$$

$$\left. - \frac{3.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{10(\frac{1}{4}!)^3}\sum_{m=1}^{\infty}\frac{(16m^2-12m-1)(m-\frac{9}{4})!\,\frac{1}{4}!\,\eta^{4m+1}}{m!\,(-\frac{9}{4})!\,(m+\frac{1}{4})!\,8^m(4m-1)}\right\}. \tag{79}$$

The general solution of (79) with a double zero at the origin is (with α_4 denoting a constant)

$$f_4 = \alpha_4\eta^2 + \frac{P_1}{6}\left(\eta^3 - \frac{\eta^7}{105}\right) + 2(\alpha_2^2 + 2\alpha_1\alpha_3)(\eta - g_4) - \tfrac{16}{7}\alpha_1^2\alpha_2\left(1+\frac{7\eta^4}{24} - h_4\right) -$$

$$- \tfrac{32}{3}\alpha_1^4\left\{L + \frac{3.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{28.(\frac{1}{4}!)^3}\left(1+\frac{7\eta^4}{24} - h_4\right)\right\}, \tag{80}$$

where

$$L = \sum_{m=2}^{\infty} \frac{(m-\frac{5}{2})!\,(-\frac{1}{4})!\,(2m-3)\eta^{4m+3}}{m!\,(-\frac{5}{2})!(m-\frac{1}{4})!\,8^m(4m+2)(4m+3)} - $$

$$-\frac{3.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{40(\frac{1}{4}!)^3} \sum_{m=1}^{\infty} \frac{(m-\frac{9}{4})!\,\frac{1}{4}!\,(4m-5)\eta^{4m+4}}{m!\,(-\frac{9}{4})!\,(m+\frac{1}{4})!\,8^m(4m+3)(4m+4)}. \tag{81}$$

Now

$$\frac{d^2L}{d\eta^2} = \eta^8 \frac{d}{d\eta}\left\{\frac{1}{2\eta^6}[{}_1F_1(-\tfrac{3}{2},\tfrac{3}{4},\eta^4/8)-1+\tfrac{1}{4}\eta^4]- \right.$$

$$\left. -\frac{3.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{40(\frac{1}{4}!)^3}\frac{1}{\eta^5}[{}_1F_1(-\tfrac{5}{4},\tfrac{5}{4},\eta^4/8)-1]\right\}, \tag{82}$$

and since, from (61) and (70)

$$\frac{1}{2\eta^6}{}_1F_1(-\tfrac{3}{2},\tfrac{3}{4},\eta^4/8)-\frac{3.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{40(\frac{1}{4}!)^3}\frac{1}{\eta^5}{}_1F_1(-\tfrac{5}{4},\tfrac{5}{4},\eta^4/8)$$

$$\sim \frac{3\pi}{32(\frac{1}{4}!)^2}\left\{\frac{1}{15}-\frac{1}{\eta^4}+\frac{1}{2!\,\eta^8}-\frac{3}{3!\,\eta^{12}}+\frac{3.7.1.3}{4!\,\eta^{16}}-\frac{3.7.11.1.3.5}{5!\,\eta^{20}}+\dots\right\}, \tag{83}$$

the asymptotic expansion of L may be deduced apart from an additive constant and an additive multiple of η. The resulting expression may be checked, and the constant and the multiple of η determined, by considering

$$\frac{1}{2\pi i}\int (-s-1)!\,(-s-\tfrac{5}{4})!\,(s-\tfrac{9}{4})!\,\frac{(4s-5)\eta^{4s+4}}{8^s(4s+3)(4s+4)}\,ds \tag{84}$$

taken round a contour consisting of the straight line from $-N-\frac{1}{2}-\infty i$ to $-N-\frac{1}{2}+\infty i$ and the part of a circle of infinite radius to the right of the line. After some calculation it is found that†

$$L \sim -\frac{\eta^7}{168}-\frac{2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{32(\frac{1}{4}!)^3}\eta^4+\frac{\eta^3}{2}-\frac{3\pi}{32(\frac{1}{4}!)^2}\eta\{4\log\eta+2\log 2+\gamma-\tfrac{1}{2}\pi-3\}-$$

$$-\frac{3.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{20(\frac{1}{4}!)^3}+\frac{3\pi}{32(\frac{1}{4}!)^2}\left\{\frac{\eta^5}{5}+\frac{1}{1.2!}\frac{1}{\eta^3}-\frac{1.3.3}{2.3!}\frac{1}{\eta^7}+\frac{1.3.5.3.7}{3.4!}\frac{1}{\eta^{11}}-\dots\right\}. \tag{85}$$

As before γ denotes Euler's constant, and use has been made of (70), of the first equation of (68) with $m = 1$, and of the formulae

$$\mathfrak{F}(-\tfrac{1}{4}) = \tfrac{1}{2}\pi-3\log 2-\gamma, \qquad \mathfrak{F}(-\tfrac{1}{2}) = -2\log 2-\gamma. \tag{86}$$

The asymptotic expansions of all the terms in the expression for f_4 in (80) are now known. In order that exponentially large terms should be absent from the asymptotic expansion of f_4, g_4 and h_4 must occur in (80) in the same combination as in (72). The terms containing g_4 and h_4 in (80) are

$$-2(\alpha_2^2+2\alpha_1\alpha_3)g_4+\left(\tfrac{16}{7}\alpha_1^2\alpha_2+\frac{8.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{7(\frac{1}{4}!)^3}\alpha_1^4\right)h_4. \tag{87}$$

† I am indebted to Mr. C. W. Jones for the correction of three errors in (85) as first written.

Compare with (72), and use the value of α_2 from (75). We must have

$$2(\alpha_2^2+2\alpha_1\alpha_3) = \frac{7\pi^3}{20(\frac{1}{4}!)^6}\alpha_1^4 \tag{88}$$

and

$$\alpha_3 = \frac{\pi^3}{400(\frac{1}{4}!)^6}(35-8.2^{\frac{1}{2}})\alpha_1^2. \tag{89}$$

Substituting for the coefficients of g_4 and h_4 in (80) from (88) and (75), we find that

$$f_4 = \alpha_4\eta^2+\frac{P_1}{6}\left(\eta^3-\frac{\eta^7}{105}\right)-\tfrac{32}{3}\alpha_1^4 L+$$
$$+\frac{8.2^{\frac{1}{2}}.\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^4\left\{\left(h_4-1-\frac{7\eta^4}{24}\right)-\frac{7.2^{\frac{1}{2}}.\pi^{\frac{3}{2}}}{64.(\frac{1}{4}!)^3}(g_4-\eta)\right\}. \tag{90}$$

The series expansions of L, g_4, and h_4 are given by (81), (52), and (53), so that the series expansion of f_4 is easily written down. Its asymptotic expansion is found from (72) and (85) to be

$$f_4 \sim \left(\frac{4\alpha_1^4}{63}-\frac{P_1}{630}\right)\eta^7-\frac{2^{\frac{1}{2}}.\pi^{\frac{3}{2}}}{15(\frac{1}{4}!)^3}\alpha_1^4\eta^4+\left(\frac{P_1}{6}-\frac{16\alpha_1^4}{3}\right)\eta^3+$$
$$+\left(\alpha_4-\frac{21.2^{\frac{1}{2}}.\pi^4}{400(\frac{1}{4}!)^7}\alpha_1^4\right)\eta^2+\frac{4\pi}{(\frac{1}{4}!)^2}\alpha_1^4\eta\log\eta+$$
$$+\frac{\pi}{(\frac{1}{4}!)^2}\alpha_1^4\eta\left[2\log 2+\gamma-\tfrac{1}{2}\pi-3+\frac{7\pi^2}{20(\frac{1}{4}!)^4}\right]-$$
$$-\frac{\pi^2}{30(\frac{1}{4}!)^4}\alpha_1^4\left\{\frac{\eta^7}{5}-\frac{3.7}{1!}\eta^3-\frac{3.1.7.3}{1!}\frac{1}{\eta}+\frac{3.1.1.7.3.1}{7.3!}\frac{1}{\eta^5}-\right.$$
$$\left.-\frac{3.1.1.3.7.3.1.5}{11.4!}\frac{1}{\eta^9}+\ldots\right\}-$$
$$-\frac{\pi}{(\frac{1}{4}!)^2}\alpha_1^4\left\{\frac{\eta^5}{5}+\frac{1}{1.2!}\frac{1}{\eta^3}-\frac{1.3.3}{2.3!}\frac{1}{\eta^7}+\frac{1.3.5.3.7}{3.4!}\frac{1}{\eta^{11}}-\ldots\right\}. \tag{91}$$

It follows from (42) that

$$a_6 = \alpha_1^4\left[\frac{1}{9}-\frac{7\pi^2}{600(\frac{1}{4}!)^4}\right]-\frac{P_1}{360}. \tag{92}$$

The analytical discussion of f_4 is now complete. The equations for f_5, f_6, etc., have not been completely solved, but some discussion of these equations will now be given.

In addition to the complementary function $\alpha_r\eta^2$, the equation for f_r has one complementary function whose asymptotic expansion for large positive values of η commences with a multiple of $\eta^{-(r+10)}\exp(\eta^4/8)$, and another complementary function whose asymptotic expansion commences with a multiple of η^{r+3} followed by a multiple of η^{r-1}. For the solution to be successful, exponentially large terms must not occur in the asymptotic expansion of f_r, which must begin with a multiple of η^{r+3}. For $r \leqslant 4$ this condition is satisfied, and, moreover, the next term in the asymptotic expansion is a multiple of η^{r+1}. Assume for the moment that these statements are correct for $r \leqslant n-1$. Consider the asymptotic expansion of G_n, the left-hand side of the equation for f_n in (44). The terms of highest degree occurring could be multiples of η^{n+4}, but it is not difficult to prove that the terms in η^{n+4} always cancel, and that in fact the asymptotic expansion of G_n begins with a multiple of η^{n+2}. It follows that the equation for f_n has a particular integral, which we denote by I, whose asymptotic expansion begins with a multiple of η^{n+1}. Any particular integral may be expressed as the sum of I and multiples of the complementary functions. The particular integral with a double zero at the origin is indeterminate only to the extent of an additive multiple of η^2, and what is required is that it should not involve the complementary function that is exponentially large at infinity. If the presence of the exponentially large complementary function can be avoided, the asymptotic expansion of the solution with a double zero at the origin will begin with a multiple of η^{n+3} (from the other complementary function), followed by a multiple of η^{n+1} (from I). Hence by induction it will be true generally that the asymptotic expansion of f_r will begin with a multiple of η^{r+3}, followed by a multiple of η^{r+1}.

Now f_{n-1} contains a term $\alpha_{n-1}\eta^2$, where α_{n-1} is a constant which is undetermined at that stage of the solution at which we solve for f_{n-1}. The only terms in G_n containing α_{n-1} arise from the terms in the expression for G_n which contain f_{n-1} or its derivatives, and it is easy to see that the sum of the terms in G_n containing α_{n-1} is $-(2n+4)\alpha_1\alpha_{n-1}\eta^2$. The corresponding term in the solution for f_n with a double zero at the origin is $4\alpha_1\alpha_{n-1}(\eta-g_n)$. Unless $n = 4m+2$, where m is a positive integer or zero, the asymptotic expansion of g_n involves exponentially large terms. Other exponentially large terms occurring in the solution for f_n can arise only from multiples of g_n or (when $n \neq 4m+1$) from multiples of h_n. Since a suitable combination of h_n and g_n has an asymptotic expansion devoid of exponentially large terms, and this combination involves h_n unless $n = 4m+2$, it follows that when $n \neq 4m+2$ the presence of exponentially large terms in the solution for f_n can always be avoided by a suitable choice of α_{n-1}. In this way, when $n \neq 4m+2$, α_{n-1} is determined.

On the other hand, when $n = 4m+2$ the series for g_n in (52) terminates, the term of highest degree in g_n being a multiple of η^{n+3}. In such cases it is not possible to arrange for the absence of exponentially large terms in f_n by a suitable choice of α_{n-1}, and some other condition must be satisfied.

At each stage, in solving for f_n, the value of a_{n+2} is fixed by (42) in terms of such of the P_r as have occurred in the equations up to and including the equation for f_n and of such of the α_r, for $r \leqslant n-1$, as have not been determined by the conditions for the absence of exponentially large terms.

In order to proceed farther, we consider in some detail the equations for f_5 and f_6. The equation for f_5 may be written

$$f_5'''-\tfrac{1}{2}\eta^3 f_5''+\tfrac{9}{2}\eta^2 f_5'-8\eta f_5 = -14\alpha_1\alpha_4\eta^2-\tfrac{8}{3}\alpha_1 P_1\left(\eta^3+\frac{\eta^7}{30}\right)+\alpha_1^5 H_5(\eta), \quad (93)$$

where $H_5(\eta)$ is a function of η independent of α_1, α_4, and P_1, and the solution with a double zero at the origin is

$$f_5 = \alpha_5\eta^2+4\alpha_1\alpha_4(\eta-g_5)-\frac{\alpha_1 P_1}{45}\eta^6+\alpha_1^5 k_5(\eta), \quad (94)$$

where $k_5(\eta)$ is a function of η independent of α_1, α_4, α_5, and P_1. The presence of exponentially large terms in the asymptotic expansion of f_5 can be avoided by a suitable choice of α_4, of the form

$$\alpha_4 = (\text{constant})\alpha_1^4. \quad (95)$$

The equation for f_6 is then found to be of the form

$$f_6'''-\tfrac{1}{2}\eta^3 f_6''+5\eta^2 f_6'-9\eta f_6$$
$$= -16\alpha_1\alpha_5\eta^2-\tfrac{26}{45}\alpha_1^2 P_1\eta^6-\tfrac{8}{3}\alpha_2 P_1\left(\eta^3+\frac{\eta^7}{20}\right)+\alpha_1^6 H_6(\eta), \quad (96)$$

where $H_6(\eta)$ is independent of α_1, α_4, α_5, and P_1, and the solution with a double zero at the origin is

$$f_6 = \alpha_6\eta^2+4\alpha_1\alpha_5(\eta-g_6)-\frac{13\alpha_1^2 P_1}{11340}\eta^9-\frac{\alpha_2 P_1}{45}\eta^6+\alpha_1^6 k_6(\eta), \quad (97)$$

where $k_6(\eta)$ is independent of α_1, α_5, α_6, and P_1. From (52)

$$g_6 = \eta+\frac{\eta^5}{15}-\frac{\eta^9}{1260}. \quad (98)$$

Hence

$$f_6 = \alpha_6\eta^2+\alpha_1\alpha_5\left(\frac{\eta^9}{315}-\frac{4\eta^5}{15}\right)-\frac{13\alpha_1^2 P_1}{11340}\eta^9-\frac{\alpha_2 P_1}{45}\eta^6+\alpha_1^6 k_6(\eta). \quad (99)$$

In order that the asymptotic expansion of f_6 should contain no exponentially large terms it is necessary that the asymptotic expansion of k_6 should contain no exponentially large terms, i.e. that the particular integral of

$$f_6'''-\tfrac{1}{2}\eta^3 f_6''+5\eta^2 f_6'-9\eta f_6 = H_6(\eta) \quad (100)$$

with a double zero at the origin should contain no exponentially large terms. This particular integral is indeterminate only to the extent of an additive multiple of η^2, and since the expansion of H_6 in ascending powers of η begins with a multiple of η^2, we may consider the condition to refer to the particular integral of which the expansion in ascending powers of η begins with a multiple of η^5. The condition may be given a more definite form. Put

$$f_6 = \eta^2 y_6, \quad \text{and} \quad y_6' = z_6. \tag{101}$$

Then (100) becomes

$$\eta^2 z_6'' + (6\eta - \tfrac{1}{2}\eta^5)z_6' + (6+3\eta^4)z_6 = H_6(\eta). \tag{102}$$

A complementary function of (102) (corresponding with the complementary function $g_6(\eta)$ of the equation for f_6) is

$$u_6 = -\frac{1}{\eta^2} + \frac{\eta^2}{5} - \frac{\eta^6}{180}. \tag{103}$$

If we put

$$z_6 = u_6 v_6 \quad \text{and} \quad v_6' = w_6, \tag{104}$$

(102) becomes

$$\frac{d}{d\eta}[u_6^2 \eta^6 w_6 \exp(-\eta^4/8)] = H_6 u_6 \eta^4 \exp(-\eta^4/8). \tag{105}$$

The expansion of the right-hand side of (105) in ascending powers of η begins with a multiple of η^4. As explained, the expansion of the required particular integral for f_6 begins with a multiple of η^5, and hence the expansion of $u_6^2 \eta^6 w_6 \exp(-\eta^4/8)$ begins with a multiple of η^5. The required solution of (105) is therefore

$$u_6^2 \eta^6 w_6 \exp(-\eta^4/8) = \int\limits_0^\eta H_6 u_6 \eta^4 \exp(-\eta^4/8)\, d\eta. \tag{106}$$

If the asymptotic expansion of f_6 contains no exponentially large terms, the asymptotic expansion of w_6 contains no exponentially large terms, and the left-hand side of (106) $\to 0$ when $\eta \to \infty$. Hence the required condition is

$$\int\limits_0^\infty H_6 u_6 \eta^4 \exp(-\eta^4/8)\, d\eta = 0, \tag{107}$$

i.e.

$$\int\limits_0^\infty H_6 \left(\eta^2 - \frac{\eta^6}{5} + \frac{\eta^{10}}{180}\right) \exp(-\eta^4/8)\, d\eta = 0. \tag{108}$$

The expression for H_6 is easily found from the expression for G_6 in (46), but I have not been able to evaluate analytically the integral on the right of (108). The matter has been further considered by Mr. C. W. Jones, who finds numerically that the integral condition is satisfied to the accuracy of his computations.

If we assume for the present that the condition (108) is satisfied, we may proceed to consider the equations for f_7, f_8, etc., in the same way as we considered the equations for f_5 and f_6. From the condition for the absence of exponentially large terms in the asymptotic expansion of f_7, α_6 is determined in terms of α_1, α_5, and P_1. Similarly α_7 is determined from the equation for f_8 in terms of α_1, α_5, and P_1,† and so on until we come to the equation for f_{10}. In the right-hand side, G_{10}, of the equation for f_{10}, α_9 occurs only in the term $-24\alpha_1\alpha_9\eta^2$, and the corresponding part of the particular integral with a double zero at the origin is $4\alpha_1\alpha_9(\eta-g_{10})$. Since the series for g_{10} terminates, the term of highest degree being a multiple of η^{13}, this part of the particular integral contains no exponentially large terms, and α_9 will not be determined from the condition that the asymptotic expansion of f_{10} should contain no exponentially large terms. G_{10} also contains a multiple of $\alpha_5^2\eta^2$ and certain very complicated terms linear in α_5, in particular a term in $\alpha_1^5\alpha_5$. (It further contains in particular complicated terms that are multiples of α_1^{10} and $\alpha_1^6 P_1$.) The part of the particular integral that corresponds with the multiple of $\alpha_5^2\eta^2$ in G_{10} will be a multiple of $\alpha_5^2(\eta-g_{10})$, and will not be exponentially large at infinity. Unless, therefore, the part of the particular integral corresponding with those complicated terms in G_{10} which are linear in α_5 also fails to become exponentially large at infinity, α_5 will be determined from the condition that f_{10} as a whole should not be exponentially large, which condition it will thus always be possible to fulfil. In other words, it seems probable (though I have not proved it) that α_5 is determined from the equation for f_{10}, α_9 from the equation for f_{14}, and so on. If this is correct, then only α_1 remains undetermined among the α's, and therefore only a_4 among the a's. As explained in the introduction, a_4 is then probably determined by the condition $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$, in which case the whole solution is determined at separation, and probably what we have considered is an asymptotic solution at and near separation, applying (in the non-dimensional form here considered) to all cases in which the P's are the same.

3. The solution upstream for large values of $y_1/x_1^{\frac{1}{4}}$

Our first aim in this section is to exhibit the *form* of the solution for large values of $y_1/x_1^{\frac{1}{4}}$. By carrying through the suggested calculation in some detail we also provide a check on part of the work in the preceding section.

† The right-hand side, G_8, of the equation for f_8 contains P_2 and a term, $-\frac{P_1^2}{3}\left(\eta^4+\frac{\eta^8}{15}\right)$, in P_1^2. The corresponding parts of the particular integral with a double zero at the origin are, however,

$$P_2\left(\frac{\eta^3}{6}-\frac{\eta^7}{315}+\frac{\eta^{11}}{31185}\right) \quad \text{and} \quad -P_1^2\left(\frac{\eta^7}{630}+\frac{\eta^{11}}{155925}\right)$$

and contain no exponentially large terms, so that the expression for α_7 will involve neither P_2 nor P_1^2.

According to the solution in the preceding section, ψ_1 is found in the form of the series (24), where

$$f_0 = \frac{\eta^3}{6}, \qquad f_1 = \alpha_1 \eta^2, \qquad f_2 = \alpha_2 \eta^2 - \frac{\alpha_1^2}{15}\eta^5, \tag{109}$$

α_2 is given by (75), and for large positive values of η

$$f_3 \sim A_3 \eta^6 + C_3 \eta^4 + E_3' \eta^2 \log \eta + E_3 \eta^2 + F_3 \eta + G_3 + K_3 \eta^{-2} + \ldots, \tag{110}$$

where

$$\left.\begin{aligned} A_3 &= -\frac{\pi}{60(\frac{1}{4}!)^2}\alpha_1^3, \qquad C_3 = -\tfrac{2}{3}\alpha_1^3, \qquad E_3' = \frac{\pi}{(\frac{1}{4}!)^2}\alpha_1^3, \\ E_3 &= \frac{\pi}{4(\frac{1}{4}!)^2}\alpha_1^3\left\{2\log 2 + \gamma - \tfrac{1}{2}\pi - 5 + \frac{\pi^2}{100(\frac{1}{4}!)^4}(35 - 8.2^{\frac{1}{2}})\right\}, \\ F_3 &= \frac{4.2^{\frac{1}{2}}.\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^3, \qquad G_3 = -\tfrac{8}{3}\alpha_1^3, \qquad K_3 = \frac{\pi}{8(\frac{1}{4}!)^2}\alpha_1^3, \end{aligned}\right\} \tag{111}$$

and

$$f_4 \sim A_4 \eta^7 + C_4 \eta^5 + D_4 \eta^4 + E_4 \eta^3 + F_4 \eta^2 + G_4' \eta \log \eta + G_4 \eta + K_4 \eta^{-1} + \ldots, \tag{112}$$

where

$$\left.\begin{aligned} A_4 &= \left[\frac{4}{63} - \frac{\pi^2}{150(\frac{1}{4}!)^4}\right]\alpha_1^4 - \frac{P_1}{630}, \qquad C_4 = -\frac{\pi}{5(\frac{1}{4}!)^2}\alpha_1^4, \\ D_4 &= -\frac{2^{\frac{3}{2}}.\pi^{\frac{3}{2}}}{15(\frac{1}{4}!)^3}\alpha_1^4, \qquad E_4 = \frac{P_1}{6} - \left[\frac{16}{3} - \frac{7\pi^2}{10(\frac{1}{4}!)^4}\right]\alpha_1^4, \\ F_4 &= \alpha_4 - \frac{21.2^{\frac{3}{2}}.\pi^4}{400(\frac{1}{4}!)^7}\alpha_1^4 \quad (\alpha_4 \text{ has not been calculated}), \\ G_4' &= \frac{4\pi}{(\frac{1}{4}!)^2}\alpha_1^4, \qquad G_4 \frac{\pi}{(\frac{1}{4}!)^2}\alpha_1^4\left[2\log 2 + \gamma - \tfrac{1}{2}\pi - 3 + \frac{7\pi^2}{20(\frac{1}{4}!)^4}\right], \\ K_4 &= \frac{7\pi^2}{20(\frac{1}{4}!)^4}\alpha_1^4. \end{aligned}\right\} \tag{113}$$

Now substitute the expressions for f_0, f_1, f_2 and the asymptotic formulae for f_3 and f_4 into the series (24) for ψ_1, substitute for η from $2^{\frac{1}{2}}\xi\eta = y_1$, and rearrange in powers of ξ and $\log \xi$. The work is purely formal, and no justification is attempted. The result, which can hold only for large values of $y_1/x_1^{\frac{1}{4}}$, is

$$\begin{aligned} \psi_1 = \frac{y_1^3}{6} - \frac{\alpha_1^2}{30}y_1^5 + \frac{A_3}{2^{\frac{1}{2}}}y_1^6 + \frac{A_4}{4}y_1^7 + \ldots \\ + \xi^2(2^{\frac{1}{2}}\alpha_1 y_1^2 + 2^{-\frac{1}{2}}C_3 y_1^4 + \tfrac{1}{2}C_4 y_1^5 + \ldots) + \\ + \xi^3(2^{\frac{1}{2}}\alpha_2 y_1^2 + 2^{-\frac{1}{2}}D_4 y_1^4 + \ldots) + \\ + \xi^4\{[2^{\frac{1}{2}}E_3 - 2^{-\frac{1}{2}}E_3' \log 2]y_1^2 + 2^{\frac{1}{2}}E_3' y_1^2 \log y_1 + E_4 y_1^3 + \ldots\} + \\ + \xi^4 \log \xi(-2^{\frac{1}{2}}E_3' y_1^2 + \ldots) + \\ + \xi^5(2F_3 y_1 + 2^{\frac{1}{2}}F_4 y_1^2 + \ldots) + \ldots. \end{aligned} \tag{114}$$

We are therefore led to assume, as a form valid for sufficiently large values of $y_1/x_1^{\frac{1}{4}}$,

$$\psi_1 = \chi_0(y_1)+\xi^2\chi_2(y_1)+\xi^3\chi_3(y_1)+\xi^4\chi_4(y_1)+(\xi^4\log\xi)\bar{\chi}_4(y_1)+\xi^5\chi_5(y_1)+\ldots, \tag{115}$$

and the initial terms of the expansions of the χ in powers of y_1 are found by comparing with (114). In fact, since the form (115) is not valid unless $y_1/x_1^{\frac{1}{4}}$ is sufficiently large, the boundary conditions at $y_1 = 0$ cannot be applied, and the solution found must be joined to the solution in the preceding section by using the first few terms of the expansions of the χ's as given by (114). On the other hand, the solution in this section should satisfy the condition $u_1 \to U_1$ as $y_1 \to \infty$.

From (115)

$$u_1 = \frac{\partial\psi_1}{\partial y_1} = \chi_0'+\xi^2\chi_2'+\xi^3\chi_3'+\xi^4\chi_4'+(\xi^4\log\xi)\bar{\chi}_4'+\xi^5\chi_5'+\ldots, \tag{116}$$

$$v_1 = \frac{\partial\psi_1}{\partial x_1} = \frac{1}{4\xi^2}\{2\chi_2+3\xi\chi_3+\xi^2(4\chi_4+\bar{\chi}_4)+4(\xi^2\log\xi)\bar{\chi}_4+5\xi^3\chi_5+\ldots\}. \tag{117}$$

We substitute in the reduced equation of motion (16) (with $\partial p_1/\partial x_1$ as in (17)), multiply by $4\xi^2$, equate coefficients of ξ^0, ξ, ξ^2, $\xi^2\log\xi$, and ξ^3, and obtain the following equations:

$$\chi_0''\chi_2-\chi_0'\chi_2' = 0, \tag{118}$$

$$\chi_0''\chi_3-\chi_0'\chi_3' = 0, \tag{119}$$

$$\chi_0''(4\chi_4+\bar{\chi}_4)-\chi_0'(4\chi_4'+\bar{\chi}_4') = 4+4\chi_0'''+2(\chi_2'^2-\chi_2\chi_2''), \tag{120}$$

$$\chi_0''\bar{\chi}_4-\chi_0'\bar{\chi}_4' = 0, \tag{121}$$

$$5(\chi_0''\chi_5-\chi_0'\chi_5') = 5\chi_2'\chi_3'-3\chi_2''\chi_3-2\chi_2\chi_3''. \tag{122}$$

Now

$$\chi_0' = \tfrac{1}{2}y_1^2-\tfrac{1}{6}\alpha_1^2y_1^4+\frac{3A_3}{2^{\frac{1}{4}}}y_1^5+\frac{7A_4}{4}y_1^6+\ldots, \tag{123}$$

and is, of course, the value of u_1 at $\xi = 0$, namely,

$$\tfrac{1}{2}y_1^2+a_3y_1^3+a_4y_1^4+\ldots,$$

as may be verified from the values of A_3, A_4, a_3, a_4, a_5, a_6.

The solution of (118) is

$$\chi_2 = (\text{constant})\chi_0'. \tag{124}$$

The constant multiplier is determined by making the coefficient of y_1^2 in the expansion of χ_2 equal to $2^{\frac{1}{4}}\alpha_1$. Hence

$$\chi_2 = 2^{\frac{1}{4}}\alpha_1\chi_0'. \tag{125}$$

(It may now be verified that the coefficients of y_1^4 and y_1^5 in the expansion of χ_2 are the same as in (114), and the values of C_3 and C_4 are thus checked.)

Similarly from (119) and (121) it follows that χ_3 and $\bar{\chi}_4$ are constant multiples of χ_0', and the multipliers are determined from the coefficients of y_1^2 by comparison with (114). In these multipliers the values of α_2 and E_3' are substituted from (75) and (111), and it is found that

$$\chi_3 = \frac{2^{\frac{1}{4}}.\pi^{\frac{1}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^2\chi_0', \tag{126}$$

$$\bar{\chi}_4 = -\frac{2^{\frac{1}{2}}\pi}{(\frac{1}{4}!)^2}\alpha_1^3\chi_0'. \tag{127}$$

(The coefficient of y_1^4 in the expansion of χ_3 may now be verified, and the value of D_4 so checked.)

In (120) the terms in $\bar{\chi}_4$ may be omitted, since they cancel by (121), and the value of χ_2 may be substituted from (125). The equation for χ_4 is thus seen to be

$$\frac{d}{dy_1}\left(\frac{\chi_4}{\chi_0'}\right) = \chi_0'^{-2}\{1-\chi_0'''-4\alpha_1^2(\chi_0''^2-\chi_0'\chi_0''')\}. \tag{128}$$

The expression on the right of (128) may be expanded in a series of ascending powers of y_1; the first terms are

$$-\frac{120.2^{\frac{1}{4}}.A_3}{y_1}+(\tfrac{8}{3}\alpha_1^4-210A_4)+\dots. \tag{129}$$

Hence the solution of (128) is

$$\chi_4 = \chi_0'\int\limits_0^{y_1}\left\{\frac{1-\chi_0'''-4\alpha_1^2(\chi_0''^2-\chi_0'\chi_0''')}{\chi_0'^2}+\frac{120.2^{\frac{1}{4}}.A_3}{y_1}\right\}dy_1-$$
$$-120.2^{\frac{1}{4}}.A_3\chi_0'\log y_1+(\text{constant})\chi_0'. \tag{130}$$

The constant multiplier of χ_0' is found from the coefficient of y_1^2 to be equal to

$$2^{\frac{1}{2}}E_3-2^{\frac{1}{2}}E_3'\log 2 = \frac{2^{\frac{1}{2}}\pi}{2(\frac{1}{4}!)^2}\alpha_1^3\left\{\gamma-\tfrac{1}{2}\pi-5+\frac{\pi^2}{100(\frac{1}{4}!)^4}(35-8.2^{\frac{1}{2}})\right\}. \tag{131}$$

(The coefficients of $y_1^2\log y_1$ and y_1^3 may now be verified, and the values of E_3' and E_4' so checked.)

When we substitute for χ_2 and χ_3 from (125) and (126), the equation (122) for χ_5 reduces to

$$\chi_0''\chi_5-\chi_0'\chi_5' = \frac{8.2^{\frac{1}{4}}.\pi^{\frac{1}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^3(\chi_0''^2-\chi_0'\chi_0''') \tag{132}$$

with the solution

$$\chi_5 = \frac{8.2^{\frac{1}{4}}.\pi^{\frac{1}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^3\chi_0''+(\text{constant})\chi_0'. \tag{133}$$

From the coefficient of y_1^2 the constant multiplier is seen to be $2^{\frac{1}{2}}F_4$, but it cannot be fully determined from results already found, since the expression for F_4 in (113) contains α_4, which has not been calculated.

(The coefficient of y_1 in the expansion of χ_5 is determined without a knowledge of the constant multiplier, and is easily seen to be $2F_3$, as in (114). The value of F_3 is thus checked.)

It remains to consider the condition that the velocity should pass smoothly over into the velocity of the main stream. As explained, we must suppose that this condition is satisfied at $x_1 = 0$; as far as our formulae go, u_1 is given by (116), so if the condition is satisfied at $x_1 = 0$, $\chi_0' \to 1$, and the second and higher derivatives $\to 0$, as $y_1 \to \infty$. From (125), (126), (127), and (133) it follows at once that χ_2', χ_3', $\bar{\chi}_4'$, χ_5' and their derivatives all $\to 0$ as $y_1 \to \infty$. As regards χ_4, we suppose that, as $y_1 \to \infty$, $\chi_0'' \to 0$ more rapidly than y_1^{-1}; it may then be proved that as $y_1 \to \infty$, χ_4 is asymptotically equal to a multiple of y_1, plus a constant, plus terms which $\to 0$, so that $\chi_4 \to \infty$ as $y_1 \to \infty$; but $\chi_0'' \chi_4 \to 0$, and we may then show from (118) that $\chi_4' \to 1$ and the second and higher derivatives $\to 0$, as $y_1 \to \infty$. We thus check that, as far as (116) goes, $u_1 \to U_1$ (since $\xi^4 = x_1$ and $U_1 = 1$, $dU_1/dx_1 = 1$ at $x_1 = 0$), while $\partial u_1/\partial y_1$ and higher derivatives $\to 0$.

4. Discussion of the solution upstream

The solution in the preceding section holds only for large values of η. The solution in § 2 applies for small or moderate values of η; it is valid also for large values of η (the asymptotic expansions of f_3 and f_4 being used), but is then useful only for small values of y_1, since in such a case it is essentially equivalent to the solution in § 3 with the χ's expanded in series and with only a few terms in each expansion known. The results that can be obtained from § 2 when η is large may therefore be more advantageously obtained from § 3 after we have obtained the formulae for the coefficients a_n in the expansion, in powers of y_1, of the value of u_1 at $x_1 = 0$. (It should also be remembered that the asymptotic formulae of § 2 were necessary for the completion of the solution in § 3.)

Thus the formulae in §§ 2 and 3 are useful in different regions. In particular, the formulae of § 2 will be useful for studying the values at $y_1 = 0$, $x_1 \neq 0$ of the derivatives of u_1 with respect to y_1, etc., whereas the formulae of § 3 will be useful for studying the nature of the solution at $x_1 = 0$, $y_1 \neq 0$. The formulae of § 2 show that $\lim_{x_1 \to 0} (\partial^n u_1/\partial y_1^n)_{y_1=0} \neq n!\,a_n$ for $n = 5$ and 6. For any non-zero value of y_1, however, $\partial^n u_1/\partial y_1^n$ is continuous at $x_1 = 0$. On the other hand, the formulae of § 3 show that v_1 (and $\partial u_1/\partial x_1$) are infinite at $x_1 = 0$.

Because of the singularity at $x_1 = 0$ the usual assumptions of boundary-layer theory are invalid at $x_1 = 0$ and in the immediate neighbourhood. Nevertheless, the mathematical result that v_1 is infinite may be taken to indicate that large cross-velocities are to be expected at separation;

otherwise the assumptions of boundary-layer theory would not break down.

5. The solution downstream

In considering the motion downstream from separation we reduce the equations of motion and continuity to non-dimensional form by the same substitutions as before, except that in place of x_1 we use

$$x_1' = -x_1 = (x-x_s)/l, \tag{134}$$

so that x_1' is positive downstream. The governing equations are obtained by replacing x_1 by $-x_1'$ in (16) and (17), and in place of (23) and (24) we write

$$\xi' = x_1'^{\frac{1}{4}}, \qquad \eta' = y_1/2^{\frac{1}{2}}x'^{\frac{1}{4}}, \tag{135}$$

$$\psi_1 = 2^{\frac{3}{2}}\xi'^3[F_0(\eta')+\xi' F_1(\eta')+\xi'^2 F_2(\eta')+...]. \tag{136}$$

We obtain differential equations for the F in the same way as before; moreover, the boundary conditions are the same, for the conditions $\psi_1 = 0$ and $u_1 = 0$ at $y_1 = 0$ lead to

$$F_r(0) = F_r'(0) = 0 \quad (r = 0, 1, 2,...) \tag{137}$$

and the condition that $\lim_{\xi'\to 0} u_1$ is given by (21) leads to

$$\lim_{\eta'\to\infty} \frac{F_r'}{\eta'^{r+2}} = 2^{\frac{1}{2}r}a_{r+2} \quad (r = 0, 1, 2,...). \tag{138}$$

The equation for F_0 is

$$F_0'''+3F_0 F_0''-2F_0'^2 = 1, \tag{139}$$

and, since $a_2 = \frac{1}{2}$, the solution is the same as for f_0, namely,

$$F_0 = \eta'^3/6. \tag{140}$$

The equation for F_1 is

$$F_1'''+\tfrac{1}{2}\eta'^3 F_1'' -\tfrac{5}{2}\eta'^2 F_1'+4\eta' F_1 = 0, \tag{141}$$

and, since $a_3 = 0$, the solution is

$$F_1 = \beta_1 \eta'^2, \tag{142}$$

where β_1 is a constant. The equation for F_2 now becomes

$$F_2'''+\tfrac{1}{2}\eta'^3 F_2''-3\eta'^2 F_2'+5\eta' F_2 = 4\beta_1^2 \eta'^2, \tag{143}$$

and the general solution for F_2 with a double zero at the origin is

$$F_2 = \beta_2 \eta'^2+\tfrac{1}{15}\beta_1^2 \eta'^5, \tag{144}$$

where β_2 is a constant. Hence, from (138),

$$a_4 = \tfrac{1}{6}\beta_1^2. \tag{145}$$

But, from (50), a_4 is zero (in which case α_1 is zero) or negative. If a_4

is negative, there is no real solution downstream of separation. Hence there is no real solution downstream of separation unless $a_4 = 0$.

6. The special case $a_4 = 0$. The solution without singularities

If we return to § 2, and consider the motion upstream of separation when $\alpha_1 = 0$, the equations for the f are easily integrated, and we may show that $\alpha_2 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_8 = \alpha_9 = \text{etc.} = 0$; the whole solution is free from singularities and may be verified by expanding ψ_1 in a double power series in x_1 and y_1. Moreover, since the solution is free from singularities it will hold also downstream. It is not possible, however, by the methods used, to consider if there are pressure distributions for which $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$.

The most interesting special case of this solution is that in which $\alpha_3 = \alpha_7 = ... = 0$, when it reduces to one that is easily found independently, namely (with $x_1' = -x_1$),

$$\psi_1 = \frac{1}{2}\frac{\partial p_1}{\partial x_1'}\frac{y_1^3}{3} + \frac{1}{360}\frac{\partial^2 p_1}{\partial x_1'^2}\frac{y_1^7}{7} + \frac{5\left(\frac{\partial p_1}{\partial x_1'}\right)^2\frac{\partial^3 p_1}{\partial x_1'^3} - 2\frac{\partial p_1}{\partial x_1'}\left(\frac{\partial^2 p_1}{\partial x_1'^2}\right)^2}{453600}\frac{y_1^{11}}{11} + ..., \tag{146}$$

so the expansion of u_1 in powers of y_1 contains only multiples of y_1^{4m+2}. This is a solution in which $(\partial u/\partial y)_{y=0} = 0$ for all x, but in order that it should be valid it is necessary that $\partial p_1/\partial x_1'$ should be chosen so that $u_1 \to 1$ when $y_1 \to \infty$ at $x_1' = 0$. Included in this solution is that special case of the solution discovered by Falkner and Skan (4) for which $(\partial u/\partial y)_{y=0} = 0$ at all values of x. In the case considered by Falkner and Skan the velocity distributions at different values of x are similar; if more general solutions of the type shown in (146) exist, the velocity distributions at different values of x will not be similar in the general case.

It is a fairly straightforward matter to check that the known solution for $U = cx^m$, when m has the appropriate value, agrees with (146) as far as that equation goes; but the value of m is determined from the condition $u \to U$ as $y \to \infty$ and cannot be found by the methods used here. Since the velocity distributions at different values of x are similar, the appropriate value of m may be found from the solutions of an ordinary differential equation, and has been so found by Hartree (5).† No such method is available in the general case. Meanwhile the formulae of Falkner and Skan, when $(\partial u/\partial y)_{y=0} = 0$, have been fitted as a very special case into the formulae of this section, so far as those formulae go.

† For negative values of m the solution of the equation with the conditions $\psi = 0$, $u = 0$ at $y = 0$ and $u/U \to 1$ as $y \to \infty$ is not unique, but may be made unique by requiring that $1-u/U$ shall be positive and shall $\to 0$ exponentially as $y \to \infty$.

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