EXPLICIT STABILITY CONDITIONS FOR INTEGRO-DIFFERENTIAL EQUATIONS WITH OPERATOR COEFFICIENTS

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SUMMARY

New explicit stability conditions are derived for integro-differential equations with operator coefficients which arise in viscoelasticity. Unlike previous studies, the coefficients are not assumed to be self-adjoint and commuting operators. Our stability conditions are formulated in terms of norms of the operator coefficients and some auxiliary operators, as well as two specific characteristics of kernels of the integral operators. The conditions developed are applied to determine the ultimate flow velocity in the flutter problem for a viscoelastic panel in supersonic gas flow. The effect of the material viscosity on the critical gas velocity is analysed numerically.

1. Introduction

The paper is concerned with the study of stability of the linear Volterra integro-differential equation

\[ \frac{d^2z}{dt^2}(t) + AZ(t) - \int_0^t R(t-s)Bz(s)\,ds = 0 \]  

with the initial conditions

\[ z(0) = z_1, \quad dz(0)/dt = z_2. \]  

Here \( z(t) \) is an unknown function which is an element of a Hilbert space \( \mathcal{H} \), \( A \) and \( B \) are linear (unbounded) operators which are densely defined in \( \mathcal{H} \) and which map \( \mathcal{H} \) into itself, \( R(t) = -Q(t) \) is a given kernel of the integral operator, where the superscript dot denotes differentiation with respect to time. The function \( Q(t) \) is assumed to be twice continuously differentiable and to satisfy the conditions

\[ -1 < Q(t) \leq Q(0) = 0, \quad Q'(t) \leq Q'() = 0, \quad Q(t) \geq Q() = 0. \]

The physical meaning of conditions (3) has been discussed in a number of works; see, for example, (1). For example, inequalities (3) together with the assumption that the function \( \dot{Q}(t) \) is not equal to zero on a set of positive measure imply that the function

\[ H(t) = Q(t) - Q() \]
is of strong positive type; see Gripenberg et al. (2). In applications to viscoelasticity, the latter assertion is equivalent to the energy dissipation in a viscoelastic medium.

We assume additionally that there are positive constants $K_1$ and $K_2$ such that for any $t \geq 0$

$$\frac{1}{K_2} \leq \frac{Q(t)}{|Q(t)|} \leq \frac{1}{K_1}.$$  

(5)

Inequality (5) has also been used by a number of authors to study the asymptotic stability of linear viscoelastic media; see Dafermos (3) and Engler (4). As common practice, in viscoelasticity the function $Q(t)$ is presented by means of a truncated Prony series

$$Q(t) = - \sum_{n=1}^{N} \mu_n [1 - \exp(-\gamma_n t)],$$

where $\gamma_n$ and $\mu_n$ are positive constants; see, for example, (5). In this case, (5) is satisfied. In the sequel we employ the following inequality, which follows from condition (5); see (1):

$$\frac{1}{K_2^2} H(t) \leq Q(t) \leq \frac{1}{K_1^2} H(t).$$  

(6)

We do not intend to discuss the existence of solutions to the problem (1) and (2) and assume that for any initial conditions (2) a unique classical solution of (1) exists. Our objective is to derive explicit restrictions on the operators $A$ and $B$ which would ensure the stability of the zero solution of (1). We employ the standard definition of the Lyapunov stability: the zero solution of (1) is stable if for any $\epsilon > 0$ there is a $\delta > 0$ such that $\|z(t)\| < \epsilon$ $(t > 0)$ provided that $\|z_1\| + \|z_2\| < \delta$, where $\|\cdot\|$ is a norm in $H$.

To formulate assumptions regarding the operators $A$ and $B$, we first recall some well-known facts in the theory of elastic stability. Oscillations in an elastic structural member can be described by (1) without the integral term: see, for example, (6). The stability problem is called conservative if the operator $A$ is selfadjoint, and it is called non-conservative if $A$ is a scalar-type operator. To the best of our knowledge, other types of non-selfadjoint operators have not been considered in applications. It is easy to check that our definitions of conservative and non-conservative problems coincide with the classical definitions formulated in engineering mechanics; see, for example, Bolotin (7) and Dowell et al. (8).

Typical examples of conservative problems are beams, plates and shells under the action of 'dead' forces which do not change direction during the deformation process. By expanding the unknown function $z(t)$ into a series
in the eigenfunctions of the operator $A$, one can show that the zero solution of (7) is stable provided all the eigenvalues $\lambda_k(A)$ of $A$ are positive. The conditions

$$\lambda_k(A) > 0 \quad (k = 1, 2, \ldots)$$

(8)
determine the critical 'conservative' loads applied to a structural member.

Typical examples of non-conservative problems are a cantilevered beam under a follower force, a pipe filled with a moving fluid, and a panel in a supersonic gas flow. In these problems, it is supposed that $A$ has a purely discrete spectrum, and the set of its functions is dense in $\mathcal{H}$. By assuming additionally that all the values of $A$ are simple, that is, $A$ is a scalar-type operator, we obtain that there exists a bounded linear operator $U$ with a bounded inverse $U^{-1}$ such that $A_1 = U^{-1}AU$ is selfadjoint; see (9). To derive stability conditions, it suffices to introduce the new unknown function $u = U^{-1}z$ which satisfies (7) with the selfadjoint operator $A_1$ and to refer to inequalities (8). As a result, we arrive at two restrictions on the operator $A$: the first requires its eigenvalues to be simple, while the other demands that the eigenvalues be positive.

Due to the wide spread of polymeric materials demonstrating viscoelastic properties, it is of essential importance for applications to extend the above stability conditions to viscoelastic structural members whose behaviour obeys the integro-differential equation (1). After introducing constitutive inequalities (3) and (5) in the early seventies, stability conditions for (1) with selfadjoint operators $A$ and $B$ have been easily derived by using either a method of the Lyapunov functionals, see, for example, Dafermos (3), or the Laplace-transform technique, see, for example, Fabrizio and Morro (10) and Pruss (11). As a result, we arrive at the condition which states that the operator

$$\Delta = A + Q(\infty)B$$

(9)

should be positive definite; see (1).

The situation with the non-selfadjoint operators $A$ and $B$ is not so simple. To the best of our knowledge, there are no explicit stability conditions similar to (8) which turn into an appropriate stability condition for (7) when the integral term in (1) vanishes. Moreover, the study of some simple examples shows that stability conditions for a viscoelastic solid under non-conservative load should differ from that for the corresponding elastic solid due to the so-called destabilization paradox, see, for example, Herrmann and Jong (12) and Nemat-Nasser et al. (13). Some sufficient stability conditions for specific operators $A$ and $B$ have been developed in (14, 15). An attempt to study the stability of (1) for arbitrary operators $A$ and $B$ has been undertaken in (16). Stability conditions developed in that paper are (i) rather complicated for application to engineering problems and (ii) not presentable in terms of an analog of the operator $\Delta$. In this work we
intend to derive new explicit stability conditions which are formulated directly in terms of norms of the operators $A$ and $B$ (and some auxiliary operators associated with them) and are acceptable for applications.

The exposition is as follows. In section 2 we develop stability conditions for the zero solution of (1) by constructing Lyapunov functionals. In section 3 these conditions are applied to study flutter of a viscoelastic panel in a supersonic gas flow. We analyse numerically the effect of viscoelastic properties of the panel on the critical flow velocity. Finally, some concluding remarks are formulated in section 4.

2. Stability conditions

To derive stability conditions for (1) we employ the direct Lyapunov method and construct Lyapunov functionals which are positive definite and whose derivatives with respect to time are non-positive. We divide the construction into four steps.

Step 1. At this step we treat the operator

$$\frac{d^2 z(t)}{dt^2} + Az(t)$$

as basic, and the integral term in (1) as a perturbation of this operator. We assume that $A$ is a scalar-type operator, which implies that a bounded operator $U$ exists with a bounded inverse $U^{-1}$ such that the operator $A_1 = U^{-1}AU$ is selfadjoint. Introducing the new function

$$u = U^{-1}z,$$ (10)

we rewrite (1) as follows:

$$\frac{d^2 u}{dt^2}(t) + A_1u(t) + \int_0^t \dot{Q}(t-s)B_1u(s) \, ds = 0,$$ (11)

where

$$A_1 = U^{-1}AU, \quad B_1 = U^{-1}BU.$$ (12)

We begin with the functional

$$W_1(t) = \left(\frac{du}{dt}(t), \frac{du}{dt}(t)\right) + \langle u(t), A_1u(t)\rangle,$$ (13)

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $H$. The functional $W_1$ determines the total energy for the corresponding 'elastic' system. Calculating the derivative of $W_1$ with respect to time and using (11) and (13), we find that

$$\frac{dW_1}{dt}(t) = -2\left(\frac{du}{dt}(t), \int_0^t \dot{Q}(t-s)B_1u(s) \, ds\right).$$ (14)

Let

$$S_1 = \frac{1}{2}(B_1 + B_1^*), \quad \Omega_1 = \frac{1}{2}(B_1 - B_1^*)$$ (15)
be symmetrical and skew-symmetrical parts of the operator \( B_1 \). Here the asterisk denotes the adjoint operator, and we assume that the operators \( B_1 \) and \( B_1^* \) have the same domain. We now introduce the functional

\[
W_2(t) = W_1(t) - \int_0^t \langle \hat{Q}(t - s)(u(t) - u(s), S_1(u(t) - u(s))) \rangle \, ds. \tag{16}
\]

It follows from (14) to (16) that

\[
\frac{dW_2}{dt}(t) = -2\langle \frac{du}{dt}(t), \int_0^t \hat{Q}(t - s)\Omega_1 u(s) \, ds \rangle - 2\langle \frac{du}{dt}(t), S_1 u(t) \rangle \int_0^t \hat{Q}(t - s) \, ds - \int_0^t \hat{Q}(t - s)(u(t) - u(s), S_1(u(t) - u(s))) \, ds. \tag{17}
\]

Calculating the integral in the second term in the right-hand side, using (3), we obtain

\[
\frac{dW_2}{dt}(t) = -2\langle \frac{du}{dt}(t), \int_0^t \hat{Q}(t - s)\Omega_1 u(s) \, ds \rangle - 2\hat{Q}(t)\langle \frac{du}{dt}(t), S_1 u(t) \rangle - \int_0^t \hat{Q}(t - s)(u(t) - u(s), S_1(u(t) - u(s))) \, ds. \tag{17}
\]

Finally, we introduce the functional

\[
W_3(t) = W_2(t) + Q(t)(u(t), S_1 u(t)). \tag{18}
\]

Differentiation of formula (18) with the use of (17) implies that

\[
\frac{dW_3}{dt}(t) = -2\langle \frac{du}{dt}(t), \int_0^t \hat{Q}(t - s)\Omega_1 u(s) \, ds \rangle + \hat{Q}(t)(u(t), S_1 u(t) + \int_0^t \hat{Q}(t - s)(u(t) - u(s), S_1(u(t) - u(s))) \, ds. \tag{19}
\]

Let us assume for an instant that the operator \( B_1 \) is selfadjoint. It follows from (15) that in this case the first term in the right-hand side of (19) vanishes. According to (3), the second and third terms in the right-hand side of (19) are non-positive provided the operator \( S_1 = B_1 \) is positive definite. Therefore, the derivative of the functional

\[
W_3(t) = \langle \frac{du}{dt}(t), \frac{du}{dt}(t) \rangle + (u(t), (A_1 + Q(t)S_1)u(t)) - \int_0^t \hat{Q}(t - s)(u(t) - u(s), S_1(u(t) - u(s))) \, ds \tag{20}
\]
is non-positive. It follows from (3) that the functional (20) is positive
definite (that is, $W_3$ is a Lyapunov functional) if and only if the operator
$\Delta_1 = A_1 + Q(\infty)B_1$ is positive definite. As a result, we arrive at the following
assertion.

**Proposition 1** Suppose that (i) $A$ is a scalar-type operator, (ii) $B$ commutes
with $A$, (iii) the operators $B_1$ and $\Delta_1$ are positive definite, and (iv) inequalities
(3) are satisfied. Then the zero solution of (1) is stable.

In order to prove Proposition 1 it suffices to note that commutativity of $A$
and $B$ implies commutativity of $A_1$ and $B_1$. This, together with the
assumption that $A_1$ is selfadjoint, implies that $B_1$ is selfadjoint as well, and
therefore $W_3$ is the Lyapunov functional.

When the operators $A$ and $B$ coincide, Proposition 1 is proved in (3),
where Dafermos shows that conditions (i), (iii), and (iv) imply the
asymptotic stability of the zero solution of (1). For commutative operators
$A$ and $B$, conditions (i) and (iii) of Proposition 1 provide a criterion of the
asymptotic stability for (1); see (1, 11) for a discussion of this question and
bibliographic references.

The most burdensome assumption of Proposition 1 is in regard to the
commutativity of $A$ and $B$, since this hypothesis is not satisfied in
‘non-conservative’ stability problems for viscoelastic structural members.
Thus, the above assertion is mainly of theoretical interest. To derive
stability conditions which can be applied to engineering problems, we
proceed with constructing the Lyapunov functionals and introduce the
functional

$$W_4(t) = W_3(t) + 2\int_{0}^{t} Q(t-s)(u(t), \Omega_1 u(s)) \, ds. \quad (21)$$

Differentiating (21) with respect to time and combining the obtained
equality with (19), we find that

$$\frac{dW_4}{dt}(t) = 2\int_{0}^{t} \dot{Q}(t-s)(u(t), \Omega_1 u(s)) \, ds + \dot{Q}(t)(u(t), S_1 u(t))$$

$$- \int_{0}^{t} \dot{Q}(t-s)(u(t) - u(s), S_1(u(t) - u(s))) \, ds. \quad (22)$$

**Step 2.** At this step we suppose that the operator

$$\frac{d^2 z}{dt^2}(t) + \int_{0}^{t} \dot{Q}(t-s)Bz(s) \, ds$$

is ‘basic’, and assume that $\Delta = A + Q(\infty)B$ is a scalar-type operator.
Therefore, a bounded linear operator $V$ exists with a bounded inverse $V^{-1}$ such that the operator

$$\Delta_2 = V^{-1}(A + Q(\infty)B)V$$

is selfadjoint. By introducing the new function

$$v = V^{-1}z$$

we rewrite (1) as follows:

$$\frac{d^2v}{dt^2}(t) + A_2v(t) + \int_0^t Q(t-s)B_2v(s)\,ds = 0, \quad (24)$$

where

$$A_2 = V^{-1}AV, \quad B_2 = V^{-1}BV. \quad (25)$$

We begin constructing Lyapunov's functionals with the functional

$$F_1(t) = \frac{dv}{dt}(t) + \int_0^t H(t-s)B_2v(s)\,ds, \quad (26)$$

where the function $H(t)$ has the form (4). Differentiating (26) and using (3) and (24) we obtain

$$\frac{dF_1(t)}{dt} = -\Delta_2v(t). \quad (27)$$

Let us assume that $\Delta_2$ is a positive-definite operator and set

$$F_2(t) = \langle F_1(t), \Delta_2^{-1}F_1(t) \rangle. \quad (28)$$

It follows from (27) and (28) that

$$\frac{dF_2(t)}{dt} = -2\langle v(t), \frac{dv}{dt}(t) + \int_0^t H(t-s)B_2v(s)\,ds \rangle. \quad (29)$$

According to (29) the functional

$$F_3(t) = F_2(t) + \langle v(t), v(t) \rangle$$

has the derivative

$$\frac{dF_3(t)}{dt} = -2\langle v(t), \int_0^t H(t-s)B_2v(s)\,ds \rangle. \quad (31)$$

We assume that the adjoint operator $B_2^\ast$ has the same domain as the
operator $B_2$ and introduce the symmetrical and skew-symmetrical parts of the operator $B_2$ according to the formulae

$$S_2 = \frac{1}{2}(B_2 + B_2^2), \quad \Omega_2 = \frac{1}{2}(B_2 - B_2^2).$$

(32)

Since

$$-2(v(t), S_2 v(s)) = \langle v(t) - v(s), S_2(v(t) - v(s)) \rangle$$

$$- \langle v(t), S_2 v(t) \rangle - \langle v(s), S_2 v(s) \rangle,$$

(31) can be written as

$$\frac{dF_3}{dt}(t) = -2\left< v(t), \int_0^t H(t-s)\Omega_2 v(s) \, ds \right>$$

$$- \Gamma(t)\langle v(t), S_2 v(t) \rangle - \int_0^t H(t-s)\langle v(s), S_2 v(s) \rangle \, ds$$

$$+ \int_0^t H(t-s)\langle v(t) - v(s), S_2(v(t) - v(s)) \rangle \, ds,$$

(33)

where

$$\Gamma(t) = \int_0^t H(s) \, ds.$$  

(34)

Step 3. We now combine functionals $W_4(t)$ and $F_3(t)$ and introduce the functional

$$F(t) = W_4(t) + \alpha F_3(t),$$

(35)

where $\alpha$ is a positive constant which will be chosen below. It follows from (22), (33) and (35) that

$$dF(t)/dt = I_1(t) - I_2(t) - I_3(t) - I_4(t),$$

(36)

where

$$I_1(t) = 2\int_0^t \left[ Q(t-s)\langle u(t), \Omega_1 u(s) \rangle - \alpha H(t-s)\langle v(t), \Omega_2 v(s) \rangle \right] \, ds,$$

$$I_2(t) = \int_0^t \left[ Q(t-s)\langle u(t)-u(s), S_1(u(t)-u(s)) \rangle \right.$$

$$\left. - \alpha H(t-s)\langle v(t)-v(s), S_2(v(t)-v(s)) \rangle \right] \, ds,$$

$$I_3(t) = \alpha \Gamma(t)\langle v(t), S_2 v(t) \rangle - \bar{Q}(t)\langle u(t), S_1 u(t) \rangle,$$

$$I_4(t) = \alpha \int_0^t H(t-s)\langle v(s), S_2 v(s) \rangle \, ds.$$  

(37)
According to the polar-decomposition formula, linear bounded operators $U$ and $V$ can be presented in the form

$$U = T_U O_U, \quad V = T_V O_V,$$

(38)

where $T_U$ and $T_V$ are selfadjoint operators, while $O_U$ and $O_V$ are orthogonal operators:

$$T_U^* = T_U, \quad T_V^* = T_V, \quad O_U^* = O_U^{-1}, \quad O_V^* = O_V^{-1}.$$

It follows from (10) and (23) that

$$u(t) = O_U^* T_U^{-1} z(t), \quad u(t) = O_V^* T_V^{-1} z(t).$$

(39)

Substitution of expressions (38) into (12), (15), (25) and (32) implies that

$$S_1 = \frac{1}{2} O_U^*(T_U^{-1} B T_U + T_U B^* T_U^{-1}) O_U, \quad \Omega_1 = \frac{1}{2} O_V^*(T_V^{-1} B T_V - T_V B^* T_V^{-1}) O_V,$$

$$S_2 = \frac{1}{2} O_V^*(T_V^{-1} B T_V + T_V B^* T_V^{-1}) O_V, \quad \Omega_2 = \frac{1}{2} O_V^*(T_V^{-1} B T_V - T_V B^* T_V^{-1}) O_V.$$

(40)

It follows from (39) and (40) that

$$\langle u(t), \Omega_1 u(s) \rangle = \frac{1}{2} (O_U^* T_U^{-1} z(t), O_U^*(T_U^{-1} B T_U - T_U B^* T_U^{-1}) O_U O_U^* T_U^{-1} z(s)),$$

$$= \frac{1}{2} (z(t), (T_U^{-2} B - B^* T_U^{-2}) z(s)).$$

Similarly,

$$\langle u(t), \Omega_2 u(s) \rangle = \frac{1}{2} (z(t), (T_V^{-2} B - B^* T_V^{-2}) z(s)),$$

$$\langle u(t), S_1 u(t) \rangle = \frac{1}{2} (z(t), (T_U^{-2} B + B^* T_U^{-2}) z(t)) = \frac{1}{2} \| (T_U^{-2} B + B^* T_U^{-2})^{1/2} z(t) \|^2,$$

$$\langle u(t), S_2 u(t) \rangle = \frac{1}{2} (z(t), (T_V^{-2} B + B^* T_V^{-2}) z(t)) = \frac{1}{2} \| (T_V^{-2} B + B^* T_V^{-2})^{1/2} z(t) \|^2.$$

We assume here that $S_1$ and $S_2$ are positive-definite operators, which guarantees the existence of the square roots of operators in the right-hand sides of these equalities. Clearly

$$\langle u(t), S_2 u(t) \rangle = \frac{1}{2} \| (T_V^{-2} B + B^* T_V^{-2})^{1/2} (T_U^{-2} B + B^* T_U^{-2})^{-1/2}$$

$$\times (T_U^{-2} B + B^* T_U^{-2})^{1/2} z(t) \|^2$$

$$\leq \theta(u(t), S_1 u(t)),$$

(41)

where

$$\theta = \| (T_V^{-2} B + B^* T_V^{-2})^{1/2} (T_U^{-2} B + B^* T_U^{-2})^{-1/2} \|^2.$$

(42)
Our purpose now is to estimate the functionals \( I_k(t) \). We begin with \( I_2(t) \).

It follows from (6), (37) and (41) that

\[
I_2(t) \geq \int_0^t \left[ \dot{Q}(t-s) - \alpha \theta H(t-s) \right] (u(t) - u(s), S_1(u(t) - u(s))) \, ds
\]

\[
\geq \left( \frac{1}{K^2_2} - \alpha \theta \right) \int_0^t H(t-s) (u(t) - u(s), S_1(u(t) - u(s))) \, ds.
\]

Setting

\[
\alpha = 1/(\theta K^2_2),
\]

we obtain that

\[
I_2(t) \geq 0.
\] (44)

Let us now estimate the functional \( I_3(t) \). According to (3), (37) and (41), we have

\[
I_3(t) \geq \Gamma_1(t)(v(t), S_2 v(t)),
\] (45)

where

\[
\Gamma_1(t) = \alpha \Gamma(t) - \frac{1}{\theta} \frac{dQ}{dt}(t).
\]

This equality together with (3), (6), (34) and (43) implies that

\[
\frac{d\Gamma_1}{dt}(t) = \alpha H(t) - \frac{1}{\theta} \frac{d^2Q}{dt^2}(t) = \frac{1}{\theta} \left[ \frac{1}{K^2_2} H(t) - \frac{d^2Q}{dt^2}(t) \right] \leq 0.
\]

Therefore, for any \( t \geq 0 \),

\[
\Gamma_1(t) \geq \Gamma_1(\infty) = \alpha \Gamma(\infty).
\]

Substitution of this expression into (45) implies that

\[
I_3(t) \geq \alpha \Gamma(\infty)(v(t), S_2 v(t)).
\] (46)

Let us estimate the inner products \( (u(t), \Omega_1 u(s)) \) and \( (v(t), \Omega_2 v(s)) \) by using the Cauchy inequality. For \( (u(t), \Omega_1 u(s)) \) we have

\[
|(u(t), \Omega_1 u(s))| = \frac{1}{2} \|(z(t), (T_v^{-2}B - B^* T_v^{-2})z(s))|
\]

\[
= \frac{1}{2} \|((T_v^{-2}B + B^* T_v^{-2})z(t), C_U(T_v^{-2}B + B^* T_v^{-2})z(s))|
\]

\[
\leq \frac{1}{2} \|C_U\| \|((T_v^{-2}B + B^* T_v^{-2})z(t))\| \|((T_v^{-2}B + B^* T_v^{-2})z(s))\|
\]

\[
\leq \frac{1}{2} \|C_U\| \left[ \|(T_v^{-2}B + B^* T_v^{-2})z(t))\|^2 + \|(T_v^{-2}B + B^* T_v^{-2})z(s))\|^2 \right]
\]

\[
= \frac{1}{2} \|C_U\| [(u(t), S_2 u(t)) + (v(s), S_2 v(s))],
\] (47)

where

\[
C_U = (T_v^{-2}B + B^* T_v^{-2})^{-1}(T_v^{-2}B - B^* T_v^{-2})(T_v^{-2}B + B^* T_v^{-2})^{-1}.
\] (48)

Similarly,

\[
|(u(t), \Omega_1 u(s))| \leq \frac{1}{2} \|C_U\| [(u(t), S_2 u(t)) + (v(s), S_2 v(s))],
\] (49)
where
\[ C_v = (T_v^2 B + B^* T_v^2)^{-\frac{1}{2}}(T_v^2 B - B^* T_v^2)(T_v^2 B + B^* T_v^2)^{-\frac{1}{2}}. \] (50)

Substituting expressions (47) and (49) into (37), we find that
\[
|I_1(t)| \leq \int_0^t \left[ \hat{Q}(t-s) \left\| C_v \right\| + \alpha H(t-s) \left\| C_v \right\| \right] \times \left[ \langle v(t), S_2 v(t) \rangle + \langle v(s), S_2 v(s) \rangle \right] ds.
\]

This inequality together with (3), (6) and (34) implies that
\[
|I_1(t)| \leq \left( \frac{1}{K_1^2} \left\| C_v \right\| + \alpha \left\| C_v \right\| \right) \int_0^t H(t-s)[\langle v(t), S_2 v(t) \rangle + \langle v(s), S_2 v(s) \rangle] ds
\times \left[ \Gamma(\infty)(v(t), S_2 v(t)) + \int_0^t H(t-s)\langle v(s), S_2 v(s) \rangle ds \right].
\] (51)

Combining (36), (37), (44), (46) and (51), we obtain
\[
\frac{dF}{dt}(t) \leq - \left[ \alpha - \left( \frac{1}{K_1^2} \left\| C_v \right\| + \alpha \left\| C_v \right\| \right) \right] \left[ \Gamma(\infty)(v(t), S_2 v(t)) \right.
+ \int_0^t H(t-s)\langle v(s), S_2 v(s) \rangle ds \left. \right].
\]

It follows from this inequality that
\[
\frac{dF}{dt}(t) \leq 0
\] (52)
provided
\[
\alpha(1 - \left\| C_v \right\|) \geq \frac{1}{K_1^2} \left\| C_v \right\|.
\] (53)

Step 4. Substituting expressions (20), (21), (28) and (30) into (35), we find that
\[
F(t) = \left\langle \frac{du}{dt}(t), \frac{du}{dt}(t) \right\rangle + \langle u(t), (A_1 + Q(t)S_1) u(t) \rangle
- \int_0^t \hat{Q}(t-s)[u(t) - u(s), S_1(u(t) - u(s))] ds
+ 2 \int_0^t \hat{Q}(t-s)[u(t), \Omega_1 u(s)] ds + \alpha[(F_1(t), \Delta_2^{-1} F_1(t)) + \langle v(t), v(t) \rangle].
\]

This equality together with (3), (26) and (52) implies that
\[
\langle u(t), (A_1 + Q(t)S_1) u(t) \rangle \leq -2 \int_0^t \hat{Q}(t-s)[u(t), \Omega_1 u(s)] ds + \Sigma,
\] (54)
\[ \Sigma = \langle u(0), A_1 u(0) \rangle + \left\langle \frac{du}{dt}(0), \frac{du}{dt}(0) \right\rangle + \alpha \left[ \langle v(0), v(0) \rangle + \left\langle \frac{dv}{dt}(0), \Delta^{-1} \frac{dv}{dt}(0) \right\rangle \right]. \] (55)

Introduce the notation

\[ \Xi = A_1 + Q(\infty) S_1. \]

We assume that the operator \( \Xi \) is positive-definite. Since

\[ A_1 + Q(t) S_1 = \Xi + H(t) S_1 \]

and \( S_1 \) is a positive-definite operator, (54) can be rewritten as

\[ \langle u(t), \Xi u(t) \rangle \leq -2 \int_0^t \dot{Q}(t - s) \langle u(t), \Omega_1 u(s) \rangle \, ds + \Sigma. \] (56)

The left-hand side of (56) is equal to \( \| \Xi u(t) \|^2 \). The first term in the right-hand side is estimated with the use of (3) and the Cauchy inequality

\[
\left| -2 \int_0^t \dot{Q}(t - s) \langle u(t), \Omega_1 u(s) \rangle \, ds \right| \leq -2 \int_0^t \dot{Q}(t - s) \| u(t), \Omega_1 u(s) \| \, ds \\
\leq -2 \int_0^t \dot{Q}(t - s) \langle \Xi u(t), C \Xi u(s) \rangle \, ds \\
\leq -2 \| C \| \| \Xi u(t) \| \int_0^t \dot{Q}(t - s) \| \Xi u(s) \| \, ds,
\]

where

\[ C = \Xi^{-\frac{1}{2}} \Omega_1 \Xi^{-\frac{1}{2}}. \] (57)

Substitution of these expressions into (56) yields

\[ Y^2(t) \leq -2 \| C \| Y(t) \int_0^t \dot{Q}(t - s) Y(s) \, ds + \Sigma, \] (58)

where

\[ Y(t) = \| \Xi u(t) \|. \]

Let

\[ Y_0(t) = \sup_{0 \leq s \leq t} Y(s). \]

Then (3) and (58) imply that for any \( t \geq 0 \)

\[ Y^2(t) \leq -2 \| C \| Q(t) Y_0^2(t) + \Sigma \leq -2Q(\infty) \| C \| Y_0^2(t) + \Sigma. \]
It follows from this inequality that
\[
1 + 2Q(\infty) \| C \| \| \Xi^1 u(t) \|^2 \leq \Sigma. \tag{59}
\]
Suppose that the unbounded operators \( A_1 \) and \( \Xi \) are equivalent, that is, there exist positive constants \( \kappa_1 \) and \( \kappa_2 \) such that for any vector \( u \)
\[
\kappa_1 \| A_1 u \| \leq \| \Xi u \| \leq \kappa_2 \| A_1 u \|. \tag{60}
\]
Then, by using standard reasoning, we obtain from (55) and (59) that the zero solution of (1) is stable provided
\[
1 + 2Q(\infty) \| C \| > 0. \tag{61}
\]
We arrive at the following assertion.

**Proposition 2** Suppose that (i) \( A \) and \( A + Q(\infty)B \) are scalar-type operators and that the operators \( A_1 \) and \( \Xi \) are equivalent, (ii) \( S_1, S_2 \) and \( \Xi \) are positive-definite operators, (iii) conditions (3) and (5) are satisfied, (iv) operators \( C_U \) and \( C_V \) satisfy the inequality
\[
\| C_V \| + \theta(K_2/K_1)^2 \| C_U \| < 1, \tag{62}
\]
and (v) inequality (60) holds. Then the zero solution of (1) is stable.

It is of interest to consider the case of vanishing viscosity \( Q(\infty) \to 0 \). This assumption implies that conditions (iii) and (v) are satisfied identically, conditions (i) and (ii) turn into the stability conditions for 'elastic' equation (7), whereas condition (iv) provides an additional inequality which may lead to the destabilization paradox. Since \( \| C_V \| = \| C_U \| \) and \( \theta = 1 \) for a system with vanishing viscosity, this additional condition can be rewritten as
\[
\| C_U \| \leq [1 + (K_2/K_1)^2]^{-1}. \tag{63}
\]
Finally, by assuming that \( K_2 = K_1 \) in (62), we arrive at the following formula convenient for applications:
\[
\| C_U \| \leq \frac{1}{2}. \tag{64}
\]

3. **Flutter of a viscoelastic panel**

In this section we apply the above conditions to the stability problem for a viscoelastic panel in supersonic gas flow. For an elastic panel, this problem provides a typical example of non-conservative stability problems, see, for example, Bolotin (7). The flutter-type instability has attracted significant attention in the past four decades owing to its aeronautical applications. For the earliest bibliography on the dynamic stability of elastic plates under aerodynamic forces see (7, 17). A brief survey of recent studies is presented in (1).

Let us consider a viscoelastic rectangular plate of length \( l \), thickness \( h \)
Supersonic gas flow with gas density $\rho_m$, flow velocity $v_m$ and the Mach number $M_m$ passes over the top surface of the plate along the positive $x$-direction.

It is assumed that

(i) the behavior of the plate material is governed by the constitutive equation, see, for example, (5),

$$\sigma(t) = E\left[\epsilon(t) + \int_0^t \dot{Q}(t - s)\epsilon(s) \, ds\right],$$

where $\sigma$ is the stress, $\epsilon$ is the strain, $E$ is a constant Young modulus, and $Q(t)$ is a relaxation measure which obeys the constitutive restrictions (3) and (5);

(ii) the plate deflection is so small that the nonlinear terms can be neglected in formulae for the curvatures of the middle surface;

(iii) the Kirchhoff hypotheses hold;

(iv) the gas response is governed by the quasistatic supersonic aerodynamic theory;

(v) the speed of the plate deflection is essentially less than the gas velocity.

Under these hypotheses, the panel deflection $y(t, x)$ satisfies the following integro-differential equation:

$$\rho_h \frac{\partial^2 y}{\partial t^2}(t, x) + \frac{Eh^3}{12} \left[\frac{\partial^4 y}{\partial x^4}(t, x) + \int_0^t \frac{\partial}{\partial s} \frac{\partial^4 y}{\partial x^4}(s, x) \, ds\right]$$

$$+ \frac{\rho_m v_m^2}{(M_m^2 - 1)^2} \frac{\partial y}{\partial x}(t, x) = 0. \quad (65)$$

Let $y^o$ be the characteristic deflection of the panel. Introduce dimensionless variables and constants:

$$x_\circ = x/l, \quad t_\circ = t/T, \quad y_\circ(t_\circ, x_\circ) = y(t, x)/y^o,$$

$$Q_\circ(t_\circ) = Q(t), \quad T = \left(\frac{12\rho/l^4}{Eh^2}\right)^{\frac{1}{4}}, \quad \beta = \frac{32\rho_m v_m^2 l^3}{Eh^3(M_m^2 - 1)^{\frac{3}{2}}}.$$  

In the new notation, (65) is written as follows (for simplicity the asterisks are omitted):

$$\frac{\partial^2 y}{\partial t^2}(t, x) + \left[\frac{\partial^4 y}{\partial x^4} + \frac{3\beta}{8} \frac{\partial}{\partial x}\right] y(t, x) + \int_0^t \frac{\partial}{\partial s} \frac{\partial^4 y}{\partial x^4}(s, x) \, ds = 0. \quad (66)$$

We confine ourselves to the panel with simply-supported edges

$$y(t, 0) = y(t, 1) = 0, \quad \frac{\partial^2 y}{\partial x^2}(t, 0) = \frac{\partial^2 y}{\partial x^2}(t, 1) = 0. \quad (67)$$

Other boundary conditions can be studied similarly.
Equation (66) coincides with (1) provided that

\[ A = \frac{\partial^4}{\partial x^4} + \frac{3\beta}{8} \frac{\partial}{\partial x}, \quad B = \frac{\partial^4}{\partial x^4}, \]

where the operators \( A \) and \( B \) are defined on the set of functions which are four times continuously differentiable and satisfy conditions (67).

The main difficulty in applying the derived stability conditions consists in calculating the operators \( U \) and \( V \), which transform the differential operators \( A \) and \( A + Q(\infty)B \) into selfadjoint operators. To obviate it, the integro-differential equation (1) with operator coefficients is reduced to a matrix integro-differential equation by employing Galerkin's approach. We seek solutions of (66) in the form

\[ y(t, x) = \sum_{m=1}^{M} y_m(t) \sin \pi mx. \]  

(68)

Since the eigenvalues of the operators \( A \) and \( A + Q(\infty)B \) tend to zero very fast, only a few terms (less than 10) in expression (68) are necessary to ensure an appropriate accuracy of numerical results.

To find the critical \( \beta \) value, we employ the following recurrent algorithm. At the \( k \)th step, \( k = 1, 2, \ldots \), we set \( \beta_k = 0.01 \). Since for a fixed integer \( M \), the operators \( A(\beta_k) \) and \( B \) are equivalent to \((M \times M)\)-matrices \( A(\beta_k) \) and \( B \), the standard numerical procedure is employed to find the matrices \( U(\beta_k) \) and \( V(\beta_k) \), which transform the matrices \( A(\beta_k) \) and \( B \) to the diagonal form, see, for example, (18). We substitute \( U(\beta_k) \) into (12) and (15) and determine the matrices \( A_1(\beta_k), S_1(\beta_k) \), and \( \Omega_1(\beta_k) \). Using these matrices, we find \( \Xi(\beta_k) = A_1(\beta_k) + Q(\infty)S_1(\beta_k) \). The same procedure is carried out with \( V(\beta_k) \) to determine the matrix \( S_2(\beta_k) \) with the use of (25) and (32). Afterwards, we calculate eigenvalues of the matrices \( S_1(\beta_k), S_2(\beta_k) \), and \( \Xi(\beta_k) \) and check whether all the eigenvalues are positive; see condition (ii) in Proposition 2.

The square roots \( T_U(\beta_k) \) and \( T_V(\beta_k) \) of the matrices \( U(\beta_k) \) and \( V(\beta_k) \) are determined by using standard numerical procedures (18). These matrices are substituted into (48), (50), and (57) to calculate the matrices \( C_U(\beta_k), C_V(\beta_k), \) and \( C(\beta_k) \). Their norms and the constant \( \theta(\beta_k) \) in (42) are found as the maximal eigenvalues of the corresponding matrices. Finally, we check whether inequalities (60) and (61) are satisfied. The critical parameter \( \beta \) is determined as the first value either when one of the matrices \( S_1(\beta_k), S_2(\beta_k) \) and \( \Xi(\beta_k) \) becomes negative-definite, or when one of the inequalities (60) and (61) fails.

The algorithm is tested for \( Q(\infty) = 0.2, Q(\infty) = 0.5, \) and \( Q(\infty) = 0.8, \) and for the values \( K = 1 \) and \( K = 3, \) where \( K = K_2/K_1 \). The results of numerical analysis demonstrate that the difference between the critical \( \beta \) values
obtained at $M = 8$ and at $M = 10$ is less than one per cent. Further calculations are carried out for $M = 8$.

Our numerical analysis shows that for any ratio $K = K_2 / K_1$ and for viscosities $-Q(\infty) \leq 0.9$, only inequality (61) determines the dimensional critical flow velocity $\beta$. For materials with extremely high viscosity, when $0.9 < -Q(\infty) < 1$, the critical flow velocity is described by inequality (60).

The critical flow velocity $\beta$ is plotted versus $-Q(\infty)$ in Fig. 1.

The maximal critical velocity corresponds to an elastic panel and is equal to 169.22. This value is less than the critical flow velocity $\beta = 729.09$ found from the condition that all the eigenvalues of the operator $A$ are simple and positive; see (17). Thus the destabilization paradox is possible in the flutter problem.

For a fixed ratio $K$, the critical flow velocity $\beta$ decreases with the growth of $|Q(\infty)|$ and vanishes when $Q(\infty) = -1$ (unbounded creep). For a fixed material viscosity, the critical velocity decreases significantly in the ratio $K$.

4. Concluding remarks

The paper is concerned with sufficient stability conditions for integro-differential equations with operator coefficients. Such equations describe the dynamics of viscoelastic structural members. Our objective is to derive stability conditions when the coefficients are not selfadjoint and do not commute with each other. From the mechanical standpoint, these assumptions correspond to the so-called non-conservative loading.

Using a modification of the Lyapunov direct method, explicit stability
conditions are developed (Proposition 2). There conditions are formulated in terms of the operator coefficients and some auxiliary operators and are characterized by two parameters \( Q(\infty) \) and \( K \) of the kernel of the Volterra operator.

The stability conditions are applied to the analysis of the flutter problem for a viscoelastic panel in a supersonic gas flow. Restrictions on the dimensionless critical flow velocity are found numerically for arbitrary material parameters of the panel. It is demonstrated that the critical velocity decreases with the growth of \( |Q(\infty)| \) and vanishes when \( Q(\infty) = -1 \) (unbounded creep). For a fixed material viscosity \( -Q(\infty) \), the critical velocity decreases significantly in \( K \).

Proposition 2 ensures sufficient stability conditions. Numerical analysis shows that even for an 'elastic' problem described by ordinary differential equations, when necessary stability conditions are known our conditions are not far from the criterion of stability. For 'viscoelastic' problems described by integro-differential equations, when necessary stability conditions are absent Propositions 1 and 2 can serve as a tool for estimating critical loads.

REFERENCES